Physics 228, Spring 2001 **Problem set #5** Due: Wed. 5/9/01

1. (25 points) Consider the function $f(x,y) = xe^{-(x^2+y^2)}$ in two dimensions.

(a) Compute $\vec{\nabla} f$ (the gradient of f) in Cartesian coordinates.

(b) Compute $\nabla^2 f$ (the Laplacian of f) in Cartesian coordinates.

(c) Write the function f in polar coordinates $\{r,\, \theta\}$.

(d) Compute $\vec{\nabla} f$ in polar coordinates. Note: your answer should be in the form $a(r,\theta)\hat{r} + b(r,\theta)\hat{\theta}$, where a and b are functions you are to determine.

(e) Compute $\nabla^2 f$ in polar coordinates.

2. (25 points) Consider the function $f(r,\theta) = r^2$.

(a) Compute $\vec{\nabla} f$ and $\nabla^2 f$.

(b) Compute the two dimensional integral $\int_S \nabla^2 f \, dS$ over the region S which is a disk of radius R centered at the origin. Remember that in Cartesian coordinates, the infinitesimal area element dS equals $dx\, dy$, while in polar coordinates it equals $r\, dr\, d\theta$.

(c) Compute the line integral $\int_C \hat{r} \cdot \vec{\nabla} f \, d\ell$, where C is the circle of radius R centered at the origin. You should write the infinitesimal line element $d\ell$ as $R\, d\theta$. Note that C is the edge of the disk S, and that $\hat{r}$ is the unit vector pointing normal (perpendicular) to the circle C. Therefore you should get the same answer as in part (b), by the divergence theorem.

3. (15 points) Consider the function $f(\vec{r},t) = e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ where $\vec{r} = (x,y,z)$, and $\vec{k} = (k_x, k_y, k_z)$ is a constant vector. Find the relation between ω, $\vec{k}$ and c which allows f to satisfy the wave equation in three dimensions,

$$\frac{\partial^2 f}{\partial t^2} - c^2 \nabla^2 f = 0 \; .$$

The function f is called a "plane wave", $\vec{k}$ is the "wave number vector", ω is the frequency, and c is the phase velocity of the wave.

4. (35 points) One often needs to express the 3-dimensional Laplacian in spherical coordinates (for example, when solving the Schrödinger equation for the hydrogen atom). This is a somewhat messy exercise, but very useful. Spherical coordinates r, θ, ϕ are defined by

$$x = r\sin\theta\cos\phi\ , \qquad y = r\sin\theta\sin\phi\ , \qquad z = r\cos\theta\ . \tag{1}$$

(a) Compute the partial derivatives ∂_x, ∂_y, ∂_z in terms of r, θ, ϕ and the partial derivatives ∂_r, ∂_θ and ∂_ϕ. (I am using the shorthand $\partial_x \equiv \frac{\partial}{\partial x}$, etc.).

(b) Use the above results to show that in spherical coordinates, the Laplacian $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is given by

$$\nabla^2 = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left(\partial_\theta^2 + \cot\theta\,\partial_\theta\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 \tag{2}$$

(c) Verify that you get the same answer for $\nabla^2 x$ in both Cartesian coordinates (where it is trivial) and in spherical coordinates.

5. Extra credit (a) Consider the function $\frac{1}{2}\ln(x^2+y^2) = \ln r$ in 2 dimensions. Compute $\vec{\nabla} f$ and $\nabla^2 f$ away from the origin (Note that they are ill-defined at the origin, $x = y = r = 0$.). It is easiest if you use the expressions derived in class for $\vec{\nabla}$ and ∇^2 in polar coordinates.

(b) By the divergence theorem,

$$\int_S \nabla^2 f\, dS = \int_C \hat{r}\cdot\vec{\nabla} f\, d\ell$$

where S which is a disk of radius R centered at the origin and C is its perimeter, the circle of radius R centered at the origin. Compute the second integral.

(c) Show that your results make sense if

$$\nabla^2 \ln r = a\delta^2(\vec{r}) \equiv a\delta(x)\delta(y)$$

where $\delta(x)$, $\delta(y)$ are Dirac delta functions, and find the constant a. (Note that the 2-dimensional Dirac delta function $\delta^2(\vec{r})$ has the properties $\delta^2(\vec{r}) = 0$ for $\vec{r} \neq (0,0)$, and

$$\int_S \delta^2(\vec{r})\, dS = \begin{cases} 1 & \text{if the region } S \text{ contains the origin,} \\ 0 & \text{otherwise} \end{cases}$$

If you want, you can prove a similar relation in 3 dimensions:

$$\nabla^2 \frac{1}{r} = b\delta^3(\vec{r})$$

and find the number b.