1 Solving the damped harmonic oscillator using Green functions

We wish to solve the equation

$$\ddot{y} + 2b\dot{y} + \omega_0^2 y = f(t) \ . \tag{1}$$

1.1 The case $f(t) \propto e^{i\omega t}$

First we solve the equation for $f(t) = \frac{F(\omega)}{2\pi} e^{i\omega t}$, where $F(\omega)$ is some number:

$$\ddot{y} + 2b\dot{y} + \omega_0^2 y = \frac{F(\omega)}{2\pi} e^{i\omega t} \tag{2}$$

We guess a particular solution of the form $y_p = ce^{i\omega t}$. Plugging in this guess in order to determine c, we find

$$ce^{i\omega t}\left(-\omega^2 + 2ib\omega + \omega_0^2\right) = \frac{F(\omega)}{2\pi} e^{i\omega t} \ , \tag{3}$$

which implies a solution for c

$$c = \frac{F(\omega)/2\pi}{-\omega^2 + 2ib\omega + \omega_0^2} \ . \tag{4}$$

Therefore the general solution for y with $f(t) = \frac{F(\omega)}{2\pi} e^{i\omega t}$ is

$$y = y_h + \frac{F(\omega)/2\pi}{-\omega^2 + 2ib\omega + \omega_0^2} e^{i\omega t} \ , \tag{5}$$

and y_h is the solution to the homogeneous equation (ie, eq. (1) with $f(t) = 0$):

$$y_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \ , \tag{6}$$

where $C_{1,2}$ are arbitrary constants and $r_{1,2}$ are the roots of the quadratic equation

$$x^2 + 2bx + \omega_0^2 = 0 \ . \tag{7}$$

In this note I am assuming $r_1 \neq r_2$, so that the two exponentials in y_h represent independent solutions.

1.2 The case $f(t) = \int F(\omega) e^{i\omega t} \frac{d\omega}{2\pi}$

We now consider a general forcing function written as a Fourier transform: $f(t) = \int F(\omega) e^{i\omega t} \frac{d\omega}{2\pi}$. The particular solution now is just the one we found above, similarly integrated over ω. So the general solution is

$$y = y_h + \int \frac{F(\omega)}{-\omega^2 + 2ib\omega + \omega_0^2} e^{i\omega t} \frac{d\omega}{2\pi} \tag{8}$$

1.3 The case $f(t) = \delta(t - t_0)$

The Green function for this problem is the function $G(t, t_0)$ which satisfies

$$\left[\frac{d^2}{d_t^2} + 2b\frac{d}{d_t} + \omega_0^2 \right] G(t, t_0) = \delta(t - t_0) \ , \qquad G(t, t_0) = 0 \text{ for } t < t_0 \ . \tag{9}$$

We can solve for G by writing the Dirac δ-function $\delta(t - t_0)$ in terms of its Fourier transform, and then by using the results of the above section:

$$\int \delta(t - t_0) e^{-i\omega t} = e^{-i\omega t_0} \quad \Longrightarrow \quad \delta(t - t_0) = \int e^{-i\omega t_0} e^{i\omega t} \frac{d\omega}{2\pi} \ . \tag{10}$$

Therefore, from eq. (8), with $e^{-i\omega t_0}$ replacing $F(\omega)$, we get

$$\begin{aligned} G(t, t_0) &= y_h + \int \frac{e^{i\omega(t-t_0)}}{-\omega^2 + 2ib\omega + \omega_0^2} \frac{d\omega}{2\pi} \\ &= y_h + \begin{cases} 0 & t < t_0 \\ \frac{1}{r_1 - r_2} \left[e^{r_1(t-t_0)} - e^{r_2(t-t_0)} \right] & t > t_0 \end{cases} \end{aligned} \tag{11}$$

The integral I performed above in going from the first line to the second is easy using complex analysis, hard without it. If we apply the condition that $G(t, t_0) = 0$ for $t < t_0$, then we can set the constants in y_h above to zero. Therefore our Green function for this problem is:

$$G(t, t_0) = \begin{cases} 0 & t < t_0 \\ \frac{1}{r_1 - r_2} \left[e^{r_1(t-t_0)} - e^{r_2(t-t_0)} \right] & t > t_0 \end{cases} \ . \tag{12}$$

1.4 Solving the general problem using Green function techniques

Now we return to the general problem of eq. (1). We can write any function $f(t)$ as a sum (integral) of delta functions $\delta(t - t_0)$ for different values of t_0

with different strengths:

$$f(t) = \int \delta(t - t_0) f(t_0)\, dt_0 \tag{13}$$

Thus, since $G(t, t_0)$ is the particular solution to our differential equation with $f(t) = \delta(t - t_0)$, we can construct a particular solution for the general $f(t)$ by summing up (integrating) $G(t, t_0)$ for differnt t_0 with weight $f(t_0)$:

$$y_p(t) = \int G(t, t_0) f(t_0)\, dt_0 \ . \tag{14}$$

Note that if $f(t)$ vanishes for $t < t_1$ (ie, the force turns on at time $t = t_1$), then $y_p(t)$ similarly vanishes for $t < t_1$ since $G(t, t_0)$ vanishes for $t < t_0$ and therefore $G(t, t_0) f(t_0)$ vanishes for $t < t_1$. The most general solution then is

$$y = y_h + \int G(t, t_0) f(t_0)\, dt_0 \ . \tag{15}$$

The nice thing about this expression is that $G(t, t_0)$ and y_h are known, so that one need only perform the integral of your forcing function $f(t_0)$ times the known Green function.

1.5 When is this technique useful

Note that we can solve this problem is various ways. However, when trying to solve linear, second order, inhomogenous *partial* differential equations, this method is the easiest. One can also use the method, along with something called "perturbation theory" to solve partial differential equations which are "almost" linear. Green functions are used extensively in many branches of physics; we will reexamine them when we do partial differential equations later this quarter.