Physics 227, Winter 2001 **Problem set #5** Due: Wed. 2/7/01

Example problems. *Not to be turned in — solutions given in the companion volume of solved problems:*

§3.3: 4, 12
§3.6: 2, 10, 27

Assigned problems. *To be turned in. Problem A below will be graded; four additional problems chosen at random from those in Boas will be graded. All graded problems are worth 20 points each. Problem B is extra credit: doing extra credit problems do not change your score, but at the end of the term can make the difference in borderline grade assignments.*

§3.3: 5, 20 (using matrix methods)
§3.6: 3, 22, 24
§3.9: 1, 2
§3.10: 19

A. (20 points) The three (complex) Pauli matrices are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad (1)$$

(a.) The trace of an $n \times n$ square matrix M is defined as the sum of the n diagonal elements, or $\mathrm{Tr}M = \sum_{i=1}^{n} M_{ii}$. Compute the three traces $\mathrm{Tr}\sigma_a$ for $a = 1, 2, 3$.

(b.) Compute the 9 products $\sigma_a \sigma_b$ for all possible values $a = 1, 2, 3$ and $b = 1, 2, 3$. Express the results in terms of Pauli matrices. For example,

$$\sigma_1 \sigma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2. \qquad (2)$$

Do the matrices commute? (That is, does $\sigma_a \sigma_b = \sigma_b \sigma_a$?)

(c.) Compute all traces $\text{Tr}\sigma_a\sigma_b$ for $a = 1, 2, 3$ and $b = 1, 2, 3$.

(d.) We have seen that we can write e^x as the series $1+x+x^2/2+x^3/3!+\ldots$. We can similarly define the exponential of a square $n \times n$ matrix M as

$$e^M = I + M + M^2/2 + M^3/3! + \ldots = \sum_{p=0}^{\infty} \frac{M^p}{p!} \tag{3}$$

where $M^0 = I$ is the $n \times n$ unit matrix (the matrix with ones along the diagonal and zeros everywhere else). Use this definition to compute the 2×2 matrix $e^{i\theta\sigma_1}$ where θ is a number and σ_1 is the first Pauli matrix given above.

Hint: Find a simple formula for $(i\theta\sigma_1)^p$ for even power p, another simple formula for $(i\theta\sigma_1)^p$ with odd p, and then split the sum $\sum_{p=0}^{\infty} \frac{M^p}{p!}$ into a sum over odd p and a sum over even p. You ought to be able to express those sums in terms of simple functions of θ.

B. (Extra credit)

Define $\vec{\theta} \cdot \vec{\sigma} \equiv \theta_1\sigma_1 + \theta_2\sigma_2 + \theta_3\sigma_3$, where θ_1, θ_2 and θ_3 are 3 different numbers.

(a.) Use results from problem A to derive a simple expression for $(\vec{\theta} \cdot \vec{\sigma})^p$ for p odd and for p even.

(b) Find an expression for the 2×2 matrix $e^{i\vec{\theta}\cdot\vec{\sigma}}$. This matrix is important in quantum mechanics, and is related to how electrons behave under rotations.