The bulk limit of pairing energy in SU(4)

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Pairing Hamiltonians

The exact solutions of the pairing Hamiltonian with complete orbital degeneracy are available as simple analytic formulas not only for ordinary $S=0$ pairing of identical particles[1] but also for systems of particles having an additional isospin degeneracy, i.e. neutrons and protons [2, 3]. With isospin included, we can consider a isospin-invariant pairing Hamiltonian made of the three $S=0$ pair creation and annilation operators transforming under isospin as $T=1$. We call this the isovector pairing Hamiltonian. We can also consider the $SU(4)$-symmetric Hamiltonian made of the six pair operators that carry $S=0, T=1$ and $S=1, T=0$. We show that the bulk limit of the pair correlation energy in the systems with neutrons and protons is equal to sum of the pairing energies of the two kinds of nucleons in isolation. Also, the gap in quasiparticle excitation energies is the same. This exercise supports the use of the HFB theory to calculate pairing energies for the bulk limit of systems of neutrons and protons together.

The pairing Hamiltonian with degenerate orbitals is written

$$H = -G \sum_{i,j}^{\Omega} \sum_{\alpha} P_i^\dagger(\alpha) P_j(\alpha) \tag{1}$$

Here Ω is the orbital degeneracy and α labels the spin-isospin structure of the pair creation and annilation operators $P^\dagger$ and P. The number of terms in the sum over α is 2,3, and 6 for identical particle pairing, isovector pairing, and $SU(4)$-symmetric pairing, respectively. The general formula for the energy of the identical particle pairing Hamiltonian is [1]

$$E_v(N_\tau) = -\frac{1}{4} G (N_\tau - v)(2\Omega - N_\tau - v + 2) \tag{2}$$

where v is the seniority and N_τ is the number of nucleons of isospin τ. We only consider the case of equal numbers of neutrons and protons,

$$N_p = N_n = \frac{N}{2}. \tag{3}$$

Bulk limit

We shall define the bulk limit by letting $\Omega \to \infty$ and letting N increase so that the orbitals have a fixed occupancy of 1/2. Thus in eq. (2) we set $N_\tau = \Omega$. The bulk limit also requires that the energy scale with the size of the system. This requires that G vary inversely to Ω for large values of Ω, i.e. $G = g/\Omega$, with g a constant. With these substitutions the energy per particle of the $v = 0$ ground state is

$$\frac{E_0}{N} = -\frac{1}{4}g\left(1 + \frac{2}{\Omega}\right). \tag{4}$$

The first term is the bulk limit and the second term is the finite-size correction. The HFB theory for the identical-identical Hamiltonian is equivalent to the variational BCS theory. Its energies differ from eq. (4) only by a term of order $\mathcal{O}(g/\Omega)$. Thus, the HFB for identical particles is exact in the bulk limit.

The corresponding formulas for isovector pairing is given in ref. [3, eq. (9)]. The eigenvalues are

$$E_{v_1,T}(N) = -\frac{1}{8}G(N-v)(4\Omega - N - v + 6) + \frac{G}{2}T(T+1); \quad \text{isovector} \tag{5}$$

where T is the isospin of the eigenfunction. Let us assume that N_n and N_p are even. Then the ground state has seniority $v = 0$ and isospin $T = 0$. The energy per particle is

$$\frac{E_0}{N} = -\frac{1}{4}g\left(1 + \frac{3}{\Omega}\right). \tag{6}$$

One sees that the leading term is the same as in the identical-particle Hamiltonian. The finite-size correction is different, giving a lower energy for isovector pairing.

The general formula for the energies of the $SU(4)$-symmetric Hamiltonian is derived in ref. [2]. It is rather complicated for general seniority, but for $v = 0$ it reduces to a form very similar to eq. (5) [3, 4],

$$E_{0,\lambda_2}(N) = -\frac{1}{8}GN(4\Omega - N + 12) + \frac{G}{2}\lambda_2(\lambda_2 + 4); \quad SU(4). \tag{7}$$

Here λ_2 is a quantum number for $SU(4)$. In the ground state $\lambda_2 = 0$ and the energy per particle is

$$\frac{E_0}{N} = -\frac{1}{4}g\left(1 + \frac{6}{\Omega}\right). \tag{8}$$

Again the bulk limit is the same as in identical-particle pairing. Note also that the finite-size correction is larger than either of the two other Hamiltonians.

HFB with higher symmetry

In HFB, the pairing condensation as defined by the Δ submatrix of the HFB matrix may offer a choice of orbitals to be paired. When the Hamiltonian has a high symmetry, the HFB then permits degenerate ground states. Each of these states is equivalent to the identical pairing ground state under some unitary transformation. Thus, the bulk condensation energy in HFB does not increase with additional degrees of freedom. The exact results for the degenerate-orbit model confirms the corresponding HFB ground state has the correct bulk limit.

Quasiparticle excitations

In the HFB theory of the identical-particle Hamiltonian, all excited states have the character of quasiparticle excitations and there is a finite gap in the bulk limit. This is very clear in the eigenergies eq. (2) of the degenerate model, where the seniority v quantum number corresponds to the number of quasiparticles. The first excited state in a system with N_τ even has two quasiparticles, and the excitation energy is

$$E_2 - E_0 = G\Omega = g. \tag{9}$$

Thus, the gap in the excitation spectrum is given by g. The solution of the isovector pairing Hamiltonian also has excitations labelled by seniority. From eq. (5) the excitation energy of the $v_1 = 2, T = 0$ state is

$$E_{2,0} - E_{0,0} = g, \tag{10}$$

the same as in the identical-particle pairing. The $SU(4)$-invariant Hamiltonian also depends on seniority in the same way as in the isospin pairing Hamiltonian[2]. Thus, in both extended Hamiltonians the bulk spectrum contains quasiparticles states with the same excitation energies as in HFB.

However, these Hamiltonians have additional excitations corresponding to the other quantum numbers. For nonzero values for T in the isovector pairing Hamiltonian, the lowest excitation has $v = 0, T = 2$ with an excitation energy of

$$E_{0,2} - E_{0,0} = \frac{3g}{\Omega}. \tag{11}$$

This is inside the quasiparticle gap and goes to zero in the bulk limit. The excitation corresponds directly to the degeneracy of the HFB solutions under transformations of the Δ matrix that rotate the orbitals in isospin.

For the $SU(4)$-invariant Hamiltonian, the excitations with $v = 0$ and finite λ_2 are inside the quasiparticle gap and have vanishing excitation energy in the bulk limit. There quantum numbers corresponds exactly to the $SU(4)$ degeneracy of the Δ matrix of the HFB ground state.

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[1] "From Nucleons to Nucleus," J. Suhonen, (Springer, 2007), eq. (12.42).

[2] B.H. Flowers and S. Szikowski, Proc. Phys. Soc. 84 673 (1964).

[3] J.A. Evans, et al. Nucl. Phys. A367 77 (1981).

[4] J. Engel, et al., Phys. Rev. C **55** 1781 (1997).