The Nuclear Response Function

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(Received July 16, 1982)

These lectures present the theory of the nuclear response in the Random Phase Approximation (RPA). In the first lecture, various relations are derived between densities and currents which give rise to the well-known sum rules. Then RPA is derived via the time-dependent Hartree theory. The various formulations of RPA are shown: the configuration space representation, the coordinate space representation, the Landau theory of infinite systems and the RPA for separable interactions constrained by consistency. The remarkable success of RPA in describing the collective density oscillations of closed shell nuclei is illustrated with a few examples. In the final lecture, the $\sigma\tau$ response is discussed with the application of simple theoretical considerations to the empirical data. Finally, we point out several problems which remain in the response theory.

§ 1. Introduction

These lectures will present the theory of the nuclear response function, aimed toward students learning the techniques for calculating the vibrational properties of nuclei. The response function measures how the one-body observables for the nucleus change in the presence of external fields. We define the response function in terms of the external field V and the one-body operator O to be measured as

$$R(V, O, \omega) = -\sum_i \left(\frac{\langle 0|O|i\rangle\langle i|V|0\rangle}{E_i - \omega} + \frac{\langle 0|O|i\rangle\langle i|V|0\rangle}{E_i + \omega} \right). \tag{1}$$

Here we label the eigenstates of the nucleus by i, with energies E_i. The energy transferred by the external field is ω. In principle, $R(\omega)$ contains all the information one needs about the vibrational energies and the transition strengths. The theory which has proven to be the most useful for calculational purposes is RPA.*) This has many formulations, which are more or less convenient depending on the application. In the general theory of the response, sum rules play an important role, because they provide a firm point of knowledge about the response function. In the first lecture I will discuss

*) The name Random Phase Approximation (RPA) was coined by Pines and Bohm.[1)]

the single particle operators whose matrix elements are the object of the theory and the sum rules they obey. In the second lecture, I will derive the RPA response from the point of view of time-dependent Hartree-Fock theory, and make the connection with other formulations of RPA. In the final lecture, I will show some of the simplifications that have been applied to extract the gross features of the response by analytic methods. Several of the lecturers and participants in this summer school have made important contributions in this regard. I will concentrate on the role of velocity dependent interactions, since this has not been much emphasized in previous lecture series. Otherwise, much of this lecture material may also be found in my Gull Lake lecture notes.[2)]

§ 2. Operators and sum rules

The most important operators we deal with are the density and current. There are four kinds of these operators, depending on their spin and isospin character. For the moment I will concentrate on the ordinary density, which may be expressed in terms of particle creation and annihilation operators as

$$\hat{\rho}(\boldsymbol{r}) = \sum_{\pm 1/2} a^{\dagger}_{\boldsymbol{r},s_z,t_z} a_{\boldsymbol{r},s_z,t_z} \tag{2}$$

in a position space representation, or as

$$\hat{\rho} = \sum_{\kappa,\lambda} \phi_{\kappa}^{*}(\boldsymbol{r})\,\phi_{\lambda}(\boldsymbol{r})\,a_{\kappa}^{\dagger} a_{\lambda} \tag{3}$$

in a shell model representation, where κ, λ label complete sets of shell model orbits. The first representation will be useful for deriving commutator relations and sum rules. For detailed numerical calculations, the shell model representation is indispensable. These matrix elements may be derived using the helicity formalism, as discussed in Ref. 2) or Ref. 3).

We will abbreviate the off-diagonal density matrix elements that enter Eq. (1) by $\delta\rho$,

$$\delta\rho = \langle i|\hat{\rho}|0\rangle\,. \tag{4}$$

This is a transition density. There are two simple models for the transition density that are useful for comparison purposes and simplified calculations. The first is the deformed model of Bohr and Mottelson, in which it is imagined that the surface moves a slight amount without changing the intrinsic density. The transition density is then related to the derivative of the ground state density with some proportionality constant. The conventional definition for a transition from a spherical ground state to an excited state of angular momentum L, M is

$$\delta\rho_{LM} \equiv \langle LM|\hat{\rho}(\boldsymbol{r})|0\rangle = \frac{d}{\sqrt{2L+1}}\frac{d\rho_0}{dr}Y^{*}_{LM}(\hat{r}) = \frac{\beta R}{\sqrt{2L+1}}\frac{d\rho_0}{dr}Y^{*}_{LM}(\hat{r}), \tag{5}$$

where ρ_0 is the ground state density. The parameter d is the deformation length, and is the product of the Bohr-Mottelson β and the nuclear radius R. Such properties as the electromagnetic transition strength can be related to the β-moment between the states as follows:

$$\begin{aligned} B(EL\downarrow) &= \left(\int r^L Y^L \langle L|\hat{\rho}|0\rangle d^3r \right)^2 \\ &\approx \frac{(\beta R)^2}{2L+1} \left(\int r^{L+2} \frac{d\rho_0}{dr} dr \right)^2 = \frac{(\beta R)^2}{2L+1} (L+2)^2 \left(\int r^{L+1} \rho_0 dr \right)^2 \\ &\approx \frac{9(\beta R)^2}{2L+1} R^{2L-2} \frac{Z^2}{(4\pi)^2}, \end{aligned} \tag{6}$$

where in the last step a uniform charge density was assumed.

Another macroscopic model, proposed by Tassie,[4] has a superior functional form for the transition density. The model is

$$\delta\rho_L \sim Y^L r^{L-1} \frac{d\rho_0}{dr}, \quad L \neq 0, \tag{7}$$

$$\sim 3\rho_0 + r \frac{d\rho_0}{dr}, \quad L = 0. \tag{8}$$

We will see in detail later how this model can be justified by sum rules and the assumption that the smoothest motions remain most coherent. It turns out that the Tassie model is remarkably accurate in describing the radial form of the transition densities associated with the strongest states.

The spin- and isospin-densities are also important observables, the matrix elements of the operators

$$\hat{\rho}_\sigma(\boldsymbol{r}) = \sum_{s_z, s_z', t_z} \langle s_z|\sigma|s_z'\rangle a^\dagger_{\boldsymbol{r}, s_z, t_z} a_{\boldsymbol{r}, s_z', t_z}, \tag{9}$$

$$\hat{\rho}_\tau(\boldsymbol{r}) = \sum_{t_z, s_z, t_z'} \langle t_z|\tau|t_z'\rangle a^\dagger_{\boldsymbol{r}, s_z, t_z} a_{\boldsymbol{r}, s_z, t_z'}, \tag{10}$$

$$\hat{\rho}_{\sigma\tau}(\boldsymbol{r}) = \sum_{s_z, s_z', t_z, t_z'} \langle s_z|\sigma|s_z'\rangle \langle t_z|\tau|t_z'\rangle a^\dagger_{\boldsymbol{r}, s_z, t_z} a_{\boldsymbol{r}, s_z', t_z'}. \tag{11}$$

Since $\boldsymbol{\sigma}$ has three components, there will be in general three distinct matrix elements of σ. It is convenient to specify these either by coupling with an orbital angular momentum to a total J, or by the spin helicity.

We also consider the current operator,

$$\boldsymbol{j} = \frac{\vec{\nabla} - \overleftarrow{\nabla}}{2i}. \tag{12}$$

This can be expressed in terms of the coordinate space creation operators as the following limit,

$$j_x(\boldsymbol{r}) = \lim_{x\to 0} \frac{a^\dagger_{\boldsymbol{r}-x/2} a_{\boldsymbol{r}+x/2} - a^\dagger_{\boldsymbol{r}+x/2} a_{\boldsymbol{r}-x/2}}{2i|x|} . \tag{13}$$

There is a relation between density and current which must be satisfied in any reasonable theory, which is the quantum version of the equation of continuity. This is derived by evaluating the commutator of the Hamiltonian H and the density $\hat{\rho}(\boldsymbol{r})$. If the interaction is a function of the local densities $\hat{\rho}(\boldsymbol{r})$, $\hat{\rho}_\sigma(\boldsymbol{r})$, etc., as is the case for interactions based on meson exchange, then the interaction in H commutes with $\hat{\rho}(\boldsymbol{r})$. We need only consider the kinetic energy in the Hamiltonian,

$$T = \sum_{xyz} \lim_{x\to 0} \int \frac{a^\dagger_{\boldsymbol{r}} a_{\boldsymbol{r}+x} - 2a^\dagger_{\boldsymbol{r}} a_{\boldsymbol{r}} + a^\dagger_{\boldsymbol{r}+x} a_{\boldsymbol{r}}}{2mx} d^3r . \tag{14}$$

We now use the coordinate Fock space representations of $\hat{\rho}(\boldsymbol{r})$ and $\boldsymbol{j}(\boldsymbol{r})$, Eqs. (2) and (13), and the commutation relations

$$a_{\boldsymbol{r}}^\dagger a_{\boldsymbol{r}'} = -a_{\boldsymbol{r}'} a_{\boldsymbol{r}}^\dagger + \delta_{\boldsymbol{r},\boldsymbol{r}'} \tag{15}$$

to evaluate $[H, \hat{\rho}] = [T, \hat{\rho}]$. The result is

$$[H, \hat{\rho}(\boldsymbol{r})] = \frac{i\vec{\nabla}\cdot\boldsymbol{j}(\boldsymbol{r})}{m} . \tag{16}$$

We see this relation as the equation of continuity by taking its expectation value in a time-varying wave function,

$$\langle\psi|[H, \hat{\rho}]|\psi\rangle = -i\frac{\partial}{\partial t}\langle\hat{\rho}(\boldsymbol{r})\rangle = -i\dot{\rho} \tag{17}$$

$$= i\frac{\vec{\nabla}\cdot\langle\psi|\boldsymbol{j}|\psi\rangle}{m} = i\frac{\vec{\nabla}\cdot\langle\boldsymbol{j}\rangle}{m} = i\vec{\nabla}\cdot\boldsymbol{v}\rho ,$$

$$\text{i.e.,}\quad \dot{\rho} + \vec{\nabla}\cdot\boldsymbol{v}\rho = 0 . \tag{18}$$

Since Eq. (16) is an operator relation, all its matrix elements must obey the relation, and we can write down relations between transition densities and transition currents,

$$(E_i - E_0)\langle i|\hat{\rho}(\boldsymbol{r})|0\rangle = \frac{i\vec{\nabla}\cdot\langle i|\boldsymbol{j}(\boldsymbol{r})|0\rangle}{m} . \tag{19}$$

The left-hand side is obviously closely related to experiment, but what about the current? We can get a further relation by taking the commutator of (16) with an arbitrary external potential field,

$$[V, [H, \hat{\rho}(\boldsymbol{r})]] = \frac{+i\vec{\nabla}\cdot[V, \boldsymbol{j}(\boldsymbol{r})]}{m} , \tag{20}$$

where

$$V=\int d^3rV(\boldsymbol{r})\hat{\rho}(\boldsymbol{r}).$$

The right-hand side is evaluated using the commutator relation,

$$[\boldsymbol{j}(\boldsymbol{r}),\hat{\rho}(\boldsymbol{r}')]=\frac{\hat{\rho}(\boldsymbol{r})}{i}\vec{\nabla}_{\boldsymbol{r}}\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{21}$$

We now take the expectation value of both sides of (20) in a state i, and write out explicitly the sum over intermediate states on the left-hand side.

$$\sum_f(E_f-E_i)\langle f|\hat{\rho}(\boldsymbol{r})|i\rangle\langle i|V|f\rangle=-\frac{1}{2m}\vec{\nabla}\cdot\langle i|\hat{\rho}(\boldsymbol{r})|i\rangle\vec{\nabla}V(\boldsymbol{r}). \tag{22}$$

This is the sum rule first utilized by Fallieros[5] and Noble[6] (see also Ref. 6a)). If the potential field V should happen to connect only a single eigenstate, then there is only one term in the sum and it can be solved for $\langle f|\hat{\rho}(\boldsymbol{r})|i\rangle$,

$$\langle f|\hat{\rho}(\boldsymbol{r})|i\rangle=\frac{-\vec{\nabla}\cdot\rho_0\vec{\nabla}V}{2m(E_f-E_i)\langle i|V|f\rangle}. \tag{23}$$

We can now "derive" the Tassie model by demanding that (23) be satisfied for the smoothest potential fields. These fields are the harmonic polynomials for $L\neq 0$,

$$V(\boldsymbol{r})=r^LY^L(\hat{r}). \tag{24}$$

For $L=0$, the simplest field, a constant, gives no transitions and we take the next smoothest,

$$V(\boldsymbol{r})=r^2 \quad \text{for} \quad L=0. \tag{25}$$

It is easy to see in physical terms what the sum rule is telling us. Let us imagine a projectile passing quickly by a nucleus, so that its potential field acts for only a very short time,

$$V(\boldsymbol{r},t)=V(\boldsymbol{r})\delta(t).$$

A nucleon at position $\boldsymbol{r}$ will receive an impulse, changing its momentum by $\Delta\boldsymbol{p}=\vec{\nabla}V$. The nucleon started with zero velocity expectation value so its final velocity is

$$\boldsymbol{v}=\frac{\vec{\nabla}V}{m}. \tag{26}$$

This is the velocity field created by the projectile. The equation of continuity,

(18), relates this to the rate of change of density at $t=0_+$. Now if the field V happens to excite only a single normal mode, the time varying density has to have the same r-dependence at all times.[41] This gives the Tassie model. Of course, the assumption of coherence of a single normal mode is a very strong one, and it is not obvious that it should be a reasonable approximation in the many-body system.

The conventional energy-weighted sum rule is derived by multiplying (22) by $V(\boldsymbol{r})$ and integrating over $\boldsymbol{r}$. The result is

$$\sum_i (E_i - E_0)\langle i|V|0\rangle^2 = \int d^3r \rho_0 \frac{(\vec{\nabla} V)^2}{2m}. \tag{27}$$

This sum rule also has a nice physical interpretation. Again we consider a fast-moving projectile with its perturbative field $V(\boldsymbol{r})\delta(t)$. The energy transferred to a nucleon at some position $\boldsymbol{r}$ is $(\Delta \boldsymbol{p})^2/2m$, so the right-hand side of (27) is the total energy transferred to the nucleus. When the quantum mechanics of a sudden perturbation is calculated, one finds that the probability of exciting a state f is given by $\langle i|V|0\rangle^2/\hbar^2$. Thus the left-hand side also measures the average energy absorbed by the nucleus. This is independent of the dynamics because the energy is absorbed before the nucleus is much disturbed from equilibrium.

When the specific field $r^L Y^L$ are inserted in (27), the right-hand side can be evaluated in terms of the expectation value of a power of r. This is Lane's sum rule,[7]

$$\sum_i (E_i - E_0)\langle i|r^L Y^L|0\rangle^2 = \frac{AL(2L+1)\langle r^{2L-2}\rangle}{8\pi m}. \tag{28}$$

This sum rule is a valuable tool for both experimenters and theoreticians. When results for transition strengths are quoted as a fraction of the sum rule, there is no ambiguity as to the definition of the quoted matrix elements.

Another useful sum rule, derived by Satchler,[8] is based on a mixed use of the macroscopic model and Lane's sum rule. We earlier evaluated the matrix element of $r^L Y^L$ in the macroscopic model, assuming a uniform matter distribution,

$$\langle i|r^L Y^L|0\rangle = \frac{\beta_{i0} R}{\sqrt{2L+1}} \frac{3}{4\pi} A R^{L-1}. \tag{29}$$

We now use the uniform model to evaluate $\langle r^{2L-2}\rangle$ on the right-hand side of (28),

$$\langle r^{2L-2}\rangle_{\text{uniform}} = \frac{3}{2L+1} R^{2L-2}. \tag{30}$$

Inserting the above into (29), we obtain Satchler's sum rule,

$$\sum_i (E_i - E_0)\,(\beta_{i0}R)^2 = \frac{2\pi L\,(2L+1)}{3Am}\,. \tag{31}$$

This sum rule is not exact, but it has proved very convenient for discussing transition strengths associated with nuclear projectiles. This is because it is very easy to extract from experimental data a value for the deformation length associated with the strong inelastic transitions.

I will now illustrate Eqs. (28) and (31) with a typical application. The giant quadrupole vibration is located at $63/A^{1/3}$ MeV; it is at 10.8 MeV in ^{208}Pb. If it were the only state with quadrupole transition strength, it would exhaust the sum rule and we calculate the transition strength as*)

$$\begin{aligned}\langle GQ|r^2Y^2|0\rangle^2 &= \frac{A^{4/3}(2)\,(5)\langle r^2\rangle\hbar^2}{8\pi m\,(63\,\mathrm{MeV})}\\ &\approx \frac{(208)^{4/3}(10)\,(3/5)\,(1.2\mathrm{fm})^2(208)^{2/3}(197.3\mathrm{MeV\text{-}fm})^2}{8\pi\,(938\mathrm{MeV})\,(63\mathrm{MeV})}\\ &\approx 0.98\times 10^4\mathrm{fm}^4 \quad \text{for} \quad {}^{208}\mathrm{Pb}\,.\end{aligned} \tag{32}$$

Similarly, the deformation length is computed as

$$(\beta R)_{GQ} \simeq \sqrt{\frac{2\pi\cdot 10}{3A^{2/3}m63\mathrm{MeV}}} \approx 0.63\mathrm{fm} \quad \text{for} \quad {}^{208}\mathrm{Pb}\,. \tag{33}$$

Since (βR) is small compared to the nuclear radius, we conclude that heavy nuclei are quite stiff with respect to these vibrations. The fact that (βR) is of the same size as the surface thickness means that strongly absorbed projectiles will excite the giant quadrupole with a high probability if they pass by the nucleus at an appropriate impact parameter. There are of course lower frequency vibrations which also have deformation lengths of the order of 1 fm. These low-frequency vibrations contribute only 10$\sim$15% of the energy weighted sum rule, and so can be neglected for the estimates (32) and (33).

There is another important energy-weighted sum rule in the theory of isoscalar vibrations, involving the gradient operator. The gradient commutes with a translationally invariant Hamiltonian, so the right-hand side is zero. The only way this condition can be met on the left-hand side is for all of the strength of the operator to be concentrated at zero energy. This is of course the spurious state. We shall see that it is helpful in dealing with low-lying collective states to have this translation invariance built into the theory.

Energy-weighted sum rules for spin excitations and isovector excitations are necessarily more complicated, because the interaction does not commute with the corresponding densities. Stated differently, the mesons carry spin

*) I have left $\hbar$ out of all the equations, but it must be put back in the numerical formulas to express results in the usual units of MeV and fm.

and isospin, and must therefore be considered explicitly when discussing spin- and isospin-densities. They do not carry baryon number, and so their effect can be ignored on the isoscalar densities. There is however a non-energy-weighted sum rule, due to Gaarde,[9] which has proven to be very useful in the discussion of the (p, n) reaction. This starts with the operator identity*)

$$\left[\int d^3r\hat{\rho}_{\sigma_z\tau_+}(\boldsymbol{r}),\ \int d^3r'\hat{\rho}_{\sigma_z\tau_-}(\boldsymbol{r}')\right]=2\int d^3r\hat{\rho}_{\tau_z}(\boldsymbol{r}). \tag{34}$$

We evaluate the expectation value in the nuclear ground state, putting an intermediate set of states between the two operators on the left-hand side. This yields

$$\begin{aligned}&\sum_i(\langle 0|\sigma_z\tau_+|i\rangle\langle i|\sigma_z\tau_-|0\rangle-\langle 0|\sigma_z\tau_-|i\rangle\langle i|\sigma_z\tau_+|0\rangle)\\&\quad=\sum_i\langle i|\sigma_z\tau_-|0\rangle^2-\sum_i\langle i|\sigma_z\tau_+|0\rangle^2=2\langle 0|\tau_z|0\rangle=2(N-Z).\end{aligned} \tag{35}$$

The operators on the second line are the Gamow-Teller beta decay operators, and the sum rule relates the total strengths for β^- and β^+ transitions to the neutron excess. This sum rule provides a useful lower bound on the β^- strength in nuclei with neutron excesses, because the β^+ strength is insignificant in comparison.

§ 3. RPA

There are many ways to formulate RPA and to derive the equations of motion. The RPA is the small amplitude limit of the time-dependent Hartree-Fock theory, and I will derive the equations by this method.[3a] This formulation is particularly useful for deriving simple formulas for the giant vibrational frequencies. I will also make the equivalence of this formulation with the particle-hole representation in configuration space, and with the Landau kinetic equation for an infinite medium. Finally, I will derive the response function for RPA. This last formulation is the most efficient for treating simplified interactions.

We begin with the time-dependent Hartree-Fock equations,

$$i\frac{\partial}{\partial t}\phi_i(t)=\left[-\frac{\nabla^2}{2m}+V[\rho]\right]\phi_i(t). \tag{36}$$

Here the $\phi_i(t)$ are single-particle wave functions which depend on $\boldsymbol{r}$ and t. I write the potential as a functional of ρ to suggest that V can be nonlocal, depending on the full density matrix. We start with an equilibrium solution

*) This identity is a nonrelativistic limit of the current algebra relation $[A_\mu^i, A_\nu^j]=if_{ijk}V_4^k$ of high energy physics.[10]

$\phi_i^{(0)}(t)$ having time dependence

$$i\frac{\partial}{\partial t}\phi_i^{(0)}(t)=\varepsilon_i\phi_i^{(0)}(t). \tag{37}$$

The RPA is the small amplitude limit of vibrations about this equilibrium. Let us perturb the system with an external potential $\lambda V_{\text{ext}}(\boldsymbol{r})e^{-i\omega t}$, where λ is a small parameter. The perturbed wave function is expressed to first order in λ as

$$\phi_i(t)=e^{-i\varepsilon_i t}(\phi_i^{(0)}+\lambda\phi_i'(t)). \tag{38}$$

Given these wave functions with $\phi_i^{(0)}$ real, we may compute the density and current to first order in λ as

$$\langle\psi|\hat{\rho}|\psi\rangle=\rho_0+\lambda\delta\rho+O(\lambda^2)$$

with

$$\delta\rho=2\sum_i^A\phi_i^{(0)}\,\mathrm{Re}\,\phi_i'(t), \tag{39}$$

$$\langle\psi|\boldsymbol{j}|\psi\rangle=\lambda\delta\boldsymbol{j}+O(\lambda^2)$$

with

$$\delta\boldsymbol{j}=\frac{1}{2}\sum_i(\vec{\nabla}\phi_i^{(0)}\,\mathrm{Im}\phi_i'-(\vec{\nabla}\,\mathrm{Im}\,\phi_i')\phi_i^{(0)}). \tag{40}$$

The time varying potential in principle depends on the full density matrix, but we assume that we need only consider the dependence on $\rho^{(2)}(R+(r/2), R-(r/2))$ up to the quadratic terms in r. (This is exact for Skyrme Hamiltonians). Then the potential may be expanded as

$$V[\rho]=V(\rho_0)+\lambda\left[\frac{\delta V}{\delta\rho}\delta\rho+V_\tau\delta\tau+V_\tau\delta\rho\hat{\tau}+\frac{1}{2i}(V_j\delta\boldsymbol{j}\cdot\vec{\nabla}+\vec{\nabla}\cdot V_j\delta\boldsymbol{j})+V_{\text{ext}}\right]$$
$$+O(\lambda^2)\vec{\nabla}V_\tau\vec{\nabla}, \tag{41}$$

where $\hat{\tau}$ is the kinetic energy operator and $\delta\tau$ is its transition density. We now insert (38) and (41) into (36) and extract the coefficient of λ,

$$i\frac{\partial}{\partial t}\phi_i'(t)+\varepsilon_i\phi_i'(t)=\left[+\vec{\nabla}\left(-\frac{1}{2m}+V_\tau\right)\vec{\nabla}+V(\rho_0)\right]\phi_i'(t)$$
$$+\left[\frac{\delta V}{\delta\rho}\delta\rho+\frac{1}{2i}(V_j\delta\boldsymbol{j}\cdot\vec{\nabla}+\vec{\nabla}\cdot V_j\delta\boldsymbol{j})+V_\tau\delta\tau+V_\tau\delta\rho\hat{\tau}+V_{\text{ext}}\right]\phi_i^{(0)}. \tag{42}$$

Next we separate ϕ_i' into its real and imaginary parts. Equation (42) then gives two equations,

$$-\frac{\partial}{\partial t}\mathrm{Im}\,\phi_i'(t)=[H_0-\varepsilon_i]\mathrm{Re}\,\phi_i'(t)$$
$$+\left[\frac{\delta V}{\delta\rho}\delta\rho+V_\tau\delta\rho\vec{\nabla}^2+V_\tau\delta\tau+V_{\mathrm{ext}}\right]\phi_i^{(0)}\,, \tag{43}$$

$$\frac{\partial}{\partial t}\mathrm{Re}\,\phi_i'(t)=[H_0-\varepsilon_i]\mathrm{Im}\,\phi_i'(t)+V_j\delta\boldsymbol{j}\cdot\vec{\nabla}\phi_i^{(0)}\,, \tag{44}$$

where

$$H_0=\vec{\nabla}\cdot\left(-\frac{1}{2m}+V_\tau\right)\vec{\nabla}+V(\rho_0)\,.$$

If (44) is multiplied by $\phi_i^{(0)}$, the left-hand side is just the time derivative of the density of particle i. Let us sum over i, and Eq. (44) gives

$$\delta\dot{\rho}=\sum_i\{\phi_i^{(0)}[H_0-\varepsilon_i]\ \mathrm{Im}\,\phi_i'(t)+\phi_i^{(0)}V_j(\delta\boldsymbol{j})\cdot\vec{\nabla}\phi_i^{(0)}\}\,. \tag{45}$$

If V is purely local, the V_j and V_τ terms are absent and the right-hand side can be simplified

$$\begin{aligned}\delta\dot{\rho}&=\sum_i(\phi_i^{(0)}H_0\mathrm{Im}\,\phi_i'-\mathrm{Im}\,\phi_i'H_0\phi_i^{(0)})\\&=\frac{1}{2m}\sum_i(\phi_i^{(0)}\vec{\nabla}^2\mathrm{Im}\,\phi_i'-\mathrm{Im}\,\phi_i'\vec{\nabla}^2\phi_i^{(0)})\\&=\frac{1}{2m}\sum_i\vec{\nabla}\cdot[\phi_i^{(0)}\vec{\nabla}\ \mathrm{Im}\,\phi_i'-\mathrm{Im}\,\phi_i'\vec{\nabla}\phi_i^{(0)}]\\&=-\frac{1}{m}\vec{\nabla}\cdot\delta\boldsymbol{j}\,.\end{aligned} \tag{46}$$

Thus the equation of continuity follows directly from the TDHF equations. The RPA conserves current and will obey the energy weighted sum rules discussed in the first lecture. This key fact makes RPA the most useful theory for vibrations.

With nonlocal interactions, the baryon current is still conserved. The additional terms in Eq. (44) have the relationship $V_\tau=V_j$, and will cancel so that (46) remains true. Of course, for spin- and isospin-currents, there need not be any such relation and these currents are not conserved in general.

3.1. *Impulsive excitation and the spurious state*

I now want to make the physics of RPA more concrete and describe the situation where the vibration has been excited by a potential field $V(\boldsymbol{r})\delta(t)$. Then at $t=0_+$, there is a velocity field but no change in density,

$$\mathrm{Re}\,\phi_i'(0_+)=0,\quad \mathrm{Im}\,\phi_i'(0_+)=V(\boldsymbol{r})\phi_i^{(0)}\,. \tag{47}$$

If the vibration has a frequency ω, these functions will vary in time as

$$\text{Im}\,\phi_i'(t) = \cos\omega t\,\text{Im}\,\phi_i', \qquad \text{Re}\,\phi_i'(t) = \sin\omega t\,\text{Re}\,\phi_i', \tag{48}$$

where the symbol ϕ_i' without a time label denotes a time-independent function of position. We substitute (48) into (43) and (44), and for simplicity neglect the nonlocality in V. Then the RPA equations become

$$\omega\,\text{Im}\,\phi_i' = [H_0 - \varepsilon_i]\text{Re}\,\phi_i' + \frac{\delta V}{\delta\rho}\delta\rho\phi_i^{(0)}, \tag{49}$$

$$\omega\,\text{Re}\,\phi_i' = [H_0 - \varepsilon_i]\text{Im}\,\phi_i'. \tag{50}$$

The fact that there must be a spurious state at zero frequency is seen in this representation by considering the excitation to be a uniform force field, which results in a pure translation,

$$\text{Im}\,\phi_i' = 0 \qquad \text{Re}\,\phi_i' = \boldsymbol{a}\cdot\vec{\nabla}\phi_i^{(0)}. \tag{51}$$

Equation (50) is automatically satisfied for $\omega = 0$, while (49) is satisfied if

$$[H_0 - \varepsilon_i]\boldsymbol{a}\cdot\vec{\nabla}\phi_i^{(0)} + \frac{\delta V}{\delta\rho}\delta\rho\phi_i^{(0)} = 0. \tag{52}$$

The first term is manipulated to obtain

$$[H_0 - \varepsilon_i]\boldsymbol{a}\cdot\vec{\nabla}\phi_i^{(0)} = -\boldsymbol{a}\cdot\vec{\nabla}V(\rho_0)\phi_i^{(0)} = -\frac{\delta V}{\delta\rho}(\boldsymbol{a}\cdot\vec{\nabla}\rho_0)\phi_i^{(0)} \tag{53}$$

which cancels the second term in (52) because

$$\delta\rho = 2\sum_i \phi_i^{(0)}\,\text{Re}\,\phi_i' = 2\sum_i \phi_i^{(0)}\boldsymbol{a}\cdot\vec{\nabla}\phi_i^{(0)} = \boldsymbol{a}\cdot\vec{\nabla}\rho_0.$$

Later on, we shall invoke this consistency requirement between $\delta V/\delta\rho$, $\vec{\nabla}V$ and $\vec{\nabla}\rho_0$, to define simplified interactions.

3.2. *Configuration space representations*

Starting with (49) and (50), it is a simple matter to write down the RPA equations in the familiar matrix representation. We add and subtract the two equations, and integrate the resulting equations with an arbitrary Hartree-Fock wave function $\phi_i^{(0)}$. The amplitudes are expressed as

$$\frac{\lambda}{2}\int\phi_j^{(0)}(\text{Re}\,\phi_i' + \text{Im}\,\phi_i')\,d^3r = X_{ij},$$

$$\frac{\lambda}{2}\int\phi_j^{(0)}(\text{Re}\,\phi_i' - \text{Im}\,\phi_i')\,d^3r = Y_{ij}. \tag{54}$$

In terms of these amplitudes Eqs. (49) and (50) become

$$\omega X_{ij} = (\varepsilon_j - \varepsilon_i) X_{ij} + \sum_{kl} (X_{kl} + Y_{kl}) v_{ik,jl} ,$$
$$-\omega Y_{ij} = (\varepsilon_j - \varepsilon_i) Y_{ij} + \sum_{kl} (X_{kl} + Y_{kl}) v_{ik,jl} , \qquad (55)$$

where $v_{ij,i'j'}$ is the matrix element of $\delta V/\delta\rho$. For nonlocal potentials this is given by

$$v_{ik,jl} = \int d^3r_1 d^3r_2 d^3r_3 d^3r_4 \phi_i^{(0)}(\boldsymbol{r}_1) \phi_k^{(0)}(\boldsymbol{r}_2) \phi_j^{(0)}(\boldsymbol{r}_3) \phi_l^{(0)}(\boldsymbol{r}_4) \frac{\delta V(\boldsymbol{r}_1, \boldsymbol{r}_3)}{\delta\rho(\boldsymbol{r}_2, \boldsymbol{r}_4)} . \qquad (56)$$

The normalization of the X and Y amplitudes can be determined by requiring that the energy associated with the vibration equals $\hbar\omega$. This gives the usual condition

$$\sum_{ij} (X_{ij}^2 - Y_{ij}^2) = 1 . \qquad (57)$$

As expressed in (55), the RPA is solved by diagonalizing a matrix. The matrix is not symmetric, so the usual Householder matrix diagonalization method cannot be applied directly. However, the matrix can be formulated as a product of two symmetric matrices, for which standard techniques can be applied.[11] The matrix form, Eq. (55), gives the only practical method of solution if the full nonlocality of the potential, as expressed in (56), is important for the physics. The continuum must be treated by a discrete set of states, and then the number of configurations required to give an adequate approximation to the true solution is of the order of 20 for excitations in ^{16}O to 1000 for excitations in ^{208}Pb.

3.3. *Landau theory*

The RPA assumes a simple form in infinite systems, when only long wavelength excitations are considered. Then the orbits are labelled by momentum $\boldsymbol{k}$, and the excitation can be characterized by a momentum $\boldsymbol{q}$. We now go back to (36), and instead of considering real and imaginary parts of $\phi_i'(t)$, we divide it into positive and negative frequency components,

$$\phi_k' = (x_k e^{i\boldsymbol{q}\cdot\boldsymbol{r} - i\omega t} + y_k^* e^{-i\boldsymbol{q}\cdot\boldsymbol{r} + i\omega t}) \phi_k^{(0)} . \qquad (58)$$

The equation of motion for the coefficients x and y are

$$\omega x_k = \left(\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{m^*} + \frac{q^2}{2m^*}\right) x_k + \sum_{k'} \frac{\delta V_k}{\delta\rho_{k'}} (x_{k'} + y_{k'}) n_k^{(0)} , \qquad (59)$$

$$-\omega y_k = \left(-\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{m^*} + \frac{q^2}{2m^*}\right) y_k + \sum_{k'} \frac{\delta V_k}{\delta\rho_{k'}} (x_{k'} + y_{k'}) n_k^{(0)} , \qquad (60)$$

where $n_k^{(0)} = \theta(k_F - k)$, and we have included a possible momentum dependence in the central potential with an effective mass m^* in the single-particle energy.

We now use a trick to write the equations in a form that only involves the amplitudes in the vicinity of the Fermi surface. In (60), change the variable from $\boldsymbol{k}$ to $\boldsymbol{k}+\boldsymbol{q}$, and subtract the equation from (59),

$$\omega(x_k+y_{k+q})=\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{2m^*}(x_k+y_{k+q})+\sum_{k'}(x_{k'}+y_{k'})\frac{\delta V}{\delta\rho_{k'}}(n_k^{(0)}-n_{k+q}^{(0)}). \tag{61}$$

Thus (x_k+y_{k+q}) is only nonvanishing in the region about the Fermi surface where $n_k^{(0)}-n_{k+q}^{(0)}\neq 0$. A differential equation equivalent to (61) can be made by the replacements

$$\begin{aligned}&(x_k+y_{k+q})\to\delta n,\quad \omega\to\frac{\partial}{i\partial t},\quad \frac{\boldsymbol{k}\cdot\boldsymbol{q}}{m^*}\to\frac{\boldsymbol{v}\cdot\vec{\nabla}}{i},\\&(n_k^{(0)}-n_{k+q}^{(0)})\to\boldsymbol{q}\cdot\vec{\nabla}_{\boldsymbol{k}}n^{(0)},\\&\sum_{k'}\frac{\delta V}{\delta\rho_{k'}}(x_{k'}+y_{k'})=\delta V\to\frac{\vec{\nabla}U}{i\boldsymbol{q}}.\end{aligned} \tag{62}$$

Then (61) becomes the linearized Vlasov equation

$$\frac{\partial}{\partial t}\delta n+\boldsymbol{v}\cdot\vec{\nabla}\delta n-\vec{\nabla}U\cdot\vec{\nabla}_{\boldsymbol{k}}n^{(0)}=0\,. \tag{63}$$

For the Landau Fermi liquid equation, we write in (62)

$$n_k^{(0)}-n_{k+q}^{(0)}\approx q\cos\theta_{kq}\delta(k-k_F),\quad (x_k+y_{k+q})\approx\delta n_k(n_k^{(0)}-n_{k+q}^{(0)}). \tag{64}$$

Then the sharply peaked function $(n_k^{(0)}-n_{k+q}^{(0)})$ factors out of (61) and we are left with

$$\begin{aligned}\omega\delta n_k-v_Fq\cos\theta\delta n_k&=g_s\int\frac{d^3k'}{(2\pi)^3}\frac{\delta V_k}{\delta\rho_{k'}}\delta n_{k'}q\cos\theta_{kq}\delta(k-k_F)\\&=\frac{k_F{}^2qg_s}{(2\pi)^3}\int d\Omega\frac{\delta V_k}{\delta\rho_{k'}}\cos\theta_{kq}\delta n_{k'}\,,\end{aligned} \tag{65}$$

where g_s ($=4$ in nuclear matter) is the spin-isospin-degeneracy of the system. Landau next defines a multipole expansion of the interaction,

$$\frac{\delta V_k}{\delta\rho_{k'}}=\frac{(2\pi)^3v_F}{4\pi k_F{}^2g_s}\sum_L\mathcal{F}_LP_L(\cos\theta_k)\cdot P_L(\cos\theta_{k'}) \tag{66}$$

to get the final equation for δn_k

$$\delta n_k=\frac{1}{\omega/v_Fq\cos\theta}\sum_L\mathcal{F}_LP_L(\cos\theta)\int d\cos\theta'\cos\theta'P_L(\cos\theta')\delta n_{k'}\,. \tag{67}$$

We see that we only need to know the interaction in the vicinity of the Fermi surface, and then only the diagonal interaction as a function of relative

momentum. This is valid only if the momentum q is much smaller than any other length scale in the application.

Numerical values for the Landau parameters in nuclear matter have been calculated by Sjöberg, et al.,[12),13)] based on the Brueckner theory and a realistic nucleon-nucleon interaction. These are given in Table I.

Table I. Nuclear matter Landau parameters, from Refs. 12) and 13). The spin-dependent parameters are defined by the expansion, $\mathcal{F}_L = F_L + G_L \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + F_L'\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 + G_L'\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$. The numbers in the table were calculated from Eq. (66) assuming $v_F = \frac{k_F}{m}$, i.e., $\frac{(2\pi)^3 v_F}{4\pi k_F^2 g_s} = 153$ MeV-fm^3.

L	F_L	G_L	F_L'	G_L'
0	−0.48	.45	.77	1.29
1	−0.90	.76	.59	.07
2	−0.33			

3.4. *Green's functions and the density response*

To find the vibration induced by an external field in RPA, we must solve coupled inhomogeneous differential equations based on the homogeneous equations (49) and (50). The standard technique for dealing with such situations utilizes Green's functions. To solve the one-particle problem

$$(H_0 - \varepsilon_i - \omega)\phi_i' = -V_{\text{ext}}\phi_i^{(0)},$$

we define the inverse operator

$$g(\varepsilon_i + \omega) = (H_0 - \varepsilon_i - \omega)^{-1}. \tag{68}$$

The brute force way to construct this operator is to find the eigenstates $\phi_j^{(0)}$ and eigenvalues ε_j of H_0, and then the Green's function is

$$g(\boldsymbol{r}, \boldsymbol{r}', \varepsilon_i + \omega) = \sum_j \frac{\phi_j^{(0)}(\boldsymbol{r})\phi_j^{(0)}(\boldsymbol{r}')}{\varepsilon_j - \varepsilon_i - \omega}. \tag{69}$$

The cleverer way is to use the solutions of $(H_0 - \varepsilon_i - \omega)\phi = 0$ that satisfy each boundary condition,

$$g(r, r', \varepsilon_i + \omega) = \frac{\phi^-(r_<)\phi^+(r_>)}{\left(\phi^+\frac{d}{2mdr}\phi^- - \phi^-\frac{d}{2mdr}\phi^+\right)_R}. \tag{70}$$

To solve RPA with the Green's function, we substitute (49) into (50) and solve for $\mathrm{Re}\,\phi_i'$,

$$\mathrm{Re}\,\phi_i' = [\omega^2 - (H_0 - \varepsilon_i)^2]^{-1}[H_0 - \varepsilon_i]\left(\frac{\delta V}{\delta\rho}\delta\rho\phi_i^{(0)} + V_{\text{ext}}\phi_i^{(0)}\right). \tag{71}$$

The operator is simplified using

$$(A^2-B^2)^{-1}B=\frac{1}{2}[(A-B)^{-1}-(A+B)^{-1}]. \tag{72}$$

The equation for $\mathrm{Re}\,\phi_i'$ is then

$$\mathrm{Re}\,\phi_i'=-\frac{1}{2}(g(\varepsilon_i-\omega)+g(\varepsilon_i+\omega))\left(\frac{\delta V}{\delta\rho}\delta\rho+V_{\mathrm{ext}}\right)\phi_i^{(0)}. \tag{73}$$

Since the equation involves $\delta\rho$ on the right-hand side, let us make an equation with $\delta\rho$ on the left by multiplying (73) by $\phi_i^{(0)}$ and summing over i. The result can be expressed compactly in terms of the free density response function,

$$G^0(\boldsymbol{r},\boldsymbol{r}',\omega)=-\sum_i^A\phi_i^{(0)}(\boldsymbol{r})\,(g(\boldsymbol{r},\boldsymbol{r}',\varepsilon_i-\omega)+g(\boldsymbol{r},\boldsymbol{r}',\varepsilon_i+\omega))\,\phi_i^{(0)}(\boldsymbol{r}'). \tag{74}$$

Then (73) becomes

$$\delta\rho=-G^0\left(\frac{\delta V}{\delta\rho}\delta\rho+V_{\mathrm{ext}}\right). \tag{75}$$

With another operator inversion we finally arrive at

$$\delta\rho=\left[1+G^0\frac{\delta V}{\delta\rho}\right]^{-1}G^0V_{\mathrm{ext}}\equiv G^{\mathrm{RPA}}V_{\mathrm{ext}}. \tag{76}$$

We only need the free density response G^0 in (76) because I assumed that $\delta V/\delta\rho$ depends only on the local density $\hat{\rho}(\boldsymbol{r})$. The theory can be generalized to dependence on $\boldsymbol{j}(\boldsymbol{r})$, $\boldsymbol{\sigma}(\boldsymbol{r})$, etc. which requires then additional Green's functions. It does not seem feasible to generalize to an arbitrary dependence on the full density matrix $\hat{\rho}(\boldsymbol{r},\boldsymbol{r}')$, for then G^0 would be a function of four coordinate variables.

There are several ways G^{RPA} can be used. Its poles identify the eigenmodes, according to the representation

$$G^{\mathrm{RPA}}(\boldsymbol{r},\boldsymbol{r}',\omega)=\sum_f\frac{\langle i|\hat{\rho}(\boldsymbol{r})|f\rangle\langle f|\hat{\rho}(\boldsymbol{r}')|i\rangle}{\pm\omega-(\omega_f-\omega_i)}. \tag{77}$$

The probability of exciting the system by an external field is given by

$$S(\omega)=\sum_i\langle 0|V_{\mathrm{ext}}|i\rangle^2\delta(E_i-\omega)=\frac{1}{\pi}\int d^3r\,d^3r'\,V_{\mathrm{ext}}(\boldsymbol{r})\,\mathrm{Im}\,G(\boldsymbol{r},\boldsymbol{r}')\,V_{\mathrm{ext}}(\boldsymbol{r}'). \tag{78}$$

Here G is given an imaginary part either by adding a small $i\eta$ to the energy, or, in the case of continuum states, using outgoing wave boundary conditions in the definition of ϕ^+ in (70).

To actually solve (76), we first make a spherical harmonic decomposition of the dependence on $\hat{r}$. Then G^0 is expanded

$$G^0(\boldsymbol{r}, \boldsymbol{r}', \omega) = \sum_L G_L{}^0(r, r', \omega) Y^L(\hat{r}) \cdot Y^L(\hat{r}'), \tag{79}$$

where

$$G_L{}^0(r, r', \omega) = -\sum_j \sum_i^A \frac{\varepsilon_j - \varepsilon_i}{\omega^2 - (\varepsilon_j - \varepsilon_i)^2} \rho_{ji}^L(r) \rho_{ji}^L(r').$$

Expanding $\delta V(\boldsymbol{r})/\delta\rho(\boldsymbol{r}')$ in the same way, the angle-decomposed Green's function is

$$G^{\mathrm{RPA}}(r, r') = [1 + G_L{}^0 v_L]^{-1} G_L{}^0. \tag{80}$$

This still looks a formal equation, but it can be easily implemented representing the operators by matrices in some suitable vector space of functions. For large enough systems, (80) is found to be superior to the configuration representation. This is because the dimensionality depends on how finely the interaction must be described by the matrix representations, and not on the number of configurations, which grows very quickly with the size of the system. My collaborators and I have used a representation by a discrete mesh in coordinate space.[14),15)] The unit operator is of course the unit matrix, and $G_L{}^0(r, r')$ is computed at each pair of mesh points. This works quite well with rather coarse meshes. As the practitioners of time-dependent Hartree-Fock theory have discovered, it is possible to get good accuracy with mesh sizes up to 1 fm spacing.

Another representation, used by Knüpfer and Huber,[16)] is a discrete mesh in momentum space. In principle it should not make any real difference which one of these is used. However, it is easier to visualize the physics in position space and the self-consistency of the potential is easier to control in position space.

3.5. *Separable interactions and self-consistency*

The solution of Eq. (80) becomes trivial if the interaction $\delta V/\delta\rho$ can be approximated by a separable function

$$v_L(r, r') = \kappa f(r) f(r'). \tag{81}$$

This is equivalent to making (80) a one-dimensional matrix problem. To obtain the solution, we make use of the following formula for the inverse of a dyadic operator,

$$[1 - |\eta\rangle\langle\xi|]^{-1} = 1 + |\eta\rangle \frac{1}{1 - \langle\eta|\xi\rangle} \langle\xi|. \tag{82}$$

This may be verified by a power series expansion. Substituting (81) in (80)

we find

$$\delta\rho_L(r) = \int G_L^0(r,r')V_{\text{ext}}(r')dr' - \frac{\kappa\int G_L^0(r,r')f(r')dr'\int f(r'')G_L^0(r'',r')V_{\text{ext}}(r')dr'dr''}{1+\kappa\int dr'dr''f(r')G_L^0(r',r'')f(r'')}. \quad (83)$$

It is then only necessary to solve the algebraic equation

$$1+\kappa\int fG_L^0 f=0 \quad (84)$$

to find the poles of G^{RPA}. The physics problem is now in deciding an optimum form for $f(r)$, and the best value for κ in the approximate interaction, Eq. (81). It is here that self-consistency provides a powerful constraint, and leads to Bohr and Mottelson's treatment of vibrations.[15] We first note that if the transition density $\delta\rho$ has the Tassie or macroscopic form,

$$\delta\rho\cong d\frac{d\rho_0}{dr}, \quad (85)$$

then consistency demands that δV be related to the static potential in the same way,

$$\delta V\cong d\frac{dV_0}{dr}. \quad (86)$$

This can be achieved with the separable interaction (81) if we choose $f=dV_0/dr$ and κ so that

$$v(r,r')=\frac{\left.\frac{dV_0}{dr}\right|_r\left.\frac{dV_0}{dr}\right|_{r'}}{\int r^2dr\frac{dV_0}{dr}\frac{d\rho_0}{dr}}. \quad (87)$$

The theory with the interaction (87) used in (84) is mathematically equivalent to the collective coordinate theory of Bohr and Mottelson.[18]

3.6. *Some results of microscopic calculations*

I will now show a couple examples of experimental transition densities, and the comparison with theory. One of the strongest transitions observed in spherical nuclei is the $0^+\rightarrow 3^-$ octupole transition in ^{208}Pb, at excitation energy 2.74 MeV. The transition density measured in electron scattering is shown in Fig. 1, from Ref. 17). Notice first of all the strong surface peaking of the experimental transition density. The theory curves are the RPA calculations of Gogny,[19] Bertsch and Tsai,[14] Ring and Speth,[20] and Hamamoto.[21]

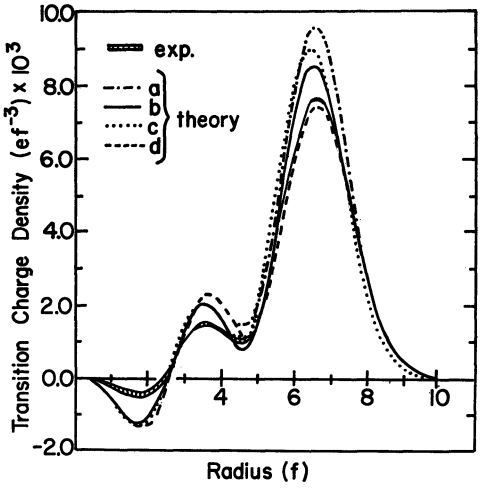

Fig. 1. Transition charge density for ^{208}Pb(g.s.→3$^-$), as measured by electron scattering, Ref. 17). The thin lines are predictions of various RPA models.

The calculations of Refs. 14) and 19) are fully self-consistent in that the same Hamiltonian is used to describe the ground state as the excited state. Hamamoto's calculation[21] is based on the separable form of the interaction described in § 3. 5. Despite the seeming coarseness of this approximation, the results are surprisingly accurate.

Figure 2 shows the transition density of the 2$^+$ state in ^{64}Zn. Again there is strong surface peaking. The theory curve is a modified RPA which allows partial occupancies of the different single-particle orbits.[22] In both examples, we see that RPA gives the correct surface transition density. However, there is an interesting systematic discrepancy in the interior: RPA usually

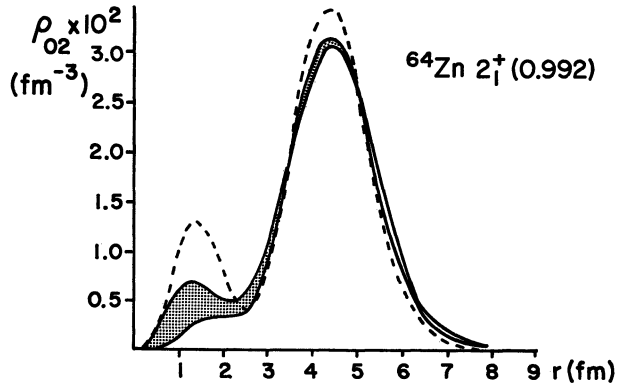

Fig. 2. Transition charge density for ^{64}Zn(g.s.→2$^+$), measured by electron scattering. The line is the prediction of an RPA calculation.[22]

predicts more structure than is observed experimentally. The oscillations associated with the single-particle levels near the Fermi surface are suppressed in the experimental transition density. This is also true of the static densities, and remains a problem to be solved.

3.7. *Coherent state limit*

If we assume that the motion initiated by some potential field V remains coherent, i.e., is an eigenstate of the system, then the RPA can be reduced to simple integral formulas. This assumption was made in Eqs. (47)$\sim$(50). The imaginary part of the time-varying wave function is given in Eq. (47), and the real part may be found from Eq. (50),

$$\mathrm{Re}\,\phi_i' = \left(\boldsymbol{u}\cdot\vec{\nabla} + \frac{1}{2}(\vec{\nabla}\cdot\boldsymbol{u})\right)\phi_i^{(0)}\,, \tag{88}$$

where

$$\boldsymbol{u} = \frac{\vec{\nabla}V}{m\omega}\,. \tag{89}$$

If we go back to the representation with a time-varying wave function $\phi_t(\boldsymbol{r}, t)$, we find that (88) represents a displacement of the wave function by a distance $\boldsymbol{u}$. The $(\vec{\nabla}\cdot\boldsymbol{u})$ term is the change in amplitude due to the stretching or squeezing of the wave function under a (nonuniform) displacement. Substituting (88) into (49), we find an equation to be satisfied by the field V,

$$\omega V\phi_i^{(0)} = \left[H_0,\ \boldsymbol{u}\cdot\vec{\nabla} + \frac{1}{2}(\vec{\nabla}\cdot\boldsymbol{u})\right]\phi_i^{(0)} + \frac{\delta V}{\delta\rho}\delta\rho\phi_i^{(0)}\,. \tag{90}$$

If we multiply this by $\mathrm{Re}\,\phi_i$, sum over i and integrate over $\boldsymbol{r}$, the equation assumes a very nice form,[23]

$$\omega_u^{\,2} = \frac{I[\boldsymbol{u}, \vec{\nabla}, \rho]}{\frac{m}{2}\int \boldsymbol{u}\cdot\boldsymbol{u}\rho_0 d^3r}\,, \tag{91}$$

where the numerator is an integral over the ground state depending on $\boldsymbol{u}$. This has the form of Rayleigh's variational principle, with the numerator representing a potential energy functional of the displacement field*) $\boldsymbol{u}$, and the denominator the usual inertia. Equation (91) together with (89) is in fact a variational principle, giving an upper bound on the frequency of the lowest excitation. A differential equation for $\boldsymbol{u}$ can be derived in the usual way from such a variational principle. We will not discuss this any further except

*) In the theory of Holzwarth and collaborators,[24] $\boldsymbol{u}$ is treated as a variational field instead of V, which allows more kinds of vibration to be considered. However, it appears that this assumption is not justified from the microscopic RPA.[25]

to mention that the differential equation is that of vibrations in an elastic medium rather than in a fluid. In the case that V does not excite a single eigenstate, Eq. (91) still has meaning as the average frequency associated with V, obtained by dividing the ω^3-weighted strength by the ω-weighted strength.[26),27)]

$$\omega_u^2=\frac{\langle i|[[H,V],H],[H,V]|i\rangle}{\langle i|[[H,V],V]|i\rangle}=\frac{\sum(E_f-E_0)^3\langle f|V|0\rangle^2}{\sum(E_f-E_0)\langle f|V|0\rangle^2}\,. \tag{92}$$

The giant quadrupole vibration is described by a field $\boldsymbol{u}=\vec{\nabla}r^2Y^2$. Applying (88) with such a field, we find that a short-range potential in the Hamiltonian gives no contribution to I, for the same reason that (52) is satisfied. Thus the restoring force in the giant quadrupole is the single-particle kinetic energy. The actual formula for the quadrupole frequency is found to be

$$\omega_Q^2=\frac{2\langle T\rangle}{m\langle r^2\rangle}\,, \tag{93}$$

where $\langle T\rangle$ is the average single-particle kinetic energy. In the harmonic oscillator model, this gives the famous result[28)]

$$\omega_Q^2=2\omega_{\rm osc}^2\,. \tag{94}$$

The expected coefficient of 4, for noninteracting particles in an oscillator well, is cut in half by throwing away the potential energy associated with the quadrupole distortion. The formula (93) is however in no way based on the oscillator model; substitution of Fermi gas parameters in (93) gives an equally good account of the empirical giant quadrupole state.

If the potential is momentum-dependent, its effects do not cancel out completely. Because of current conservation, the denominator of (93) remains the same. The numerator, however, is modified with $\langle T-\vec{\nabla}V_v\vec{\nabla}\rangle$ replacing $\langle T\rangle$.

The nonlocality of Brueckner theory gives an increase in (94) by $\sim 30\%$, which makes the theory disagree with experiment. A possible resolution of this was pointed out by Kohno and Andō,[29)] who considered contributions to nonlocality of higher order in the expansion of the density matrix $\rho(R+(r/2), R-(r/2))$ in powers of r.

§ 4. Isospin- and spin-excitations

4.1. *The giant dipole vibration*

In the Goldhaber-Teller model, the giant dipole state is the coherent state excited by the potential field $V(\boldsymbol{r})=z\tau_z$. The displaced field $\boldsymbol{u}$ is then a uniform displacement of protons against neutrons. The energy of the state, as given by the integral formula (91), is[30)]

$$\omega_{GT}^2=\frac{16\pi}{3mA}\frac{\int r^2dr\left(\frac{d\rho_n}{dr}\right)\frac{\delta V_p}{\delta\rho_n}}{\langle(1+2m(V_\kappa-V_j^\tau))^{-1}\rangle}. \tag{95}$$

The momentum-dependent factor can be couched in Landau language, identifying $2mV_\kappa$ as $F_1/3$ and $2m\rho V_j^\tau$ as $F_1'/3$,

$$\omega_{GT}^2=\frac{(1+F_1'/3)}{(1+F_1/3)}\omega^2, \tag{96}$$

where ω is the frequency in the absence of momentum dependence. From the signs of these parameters in Table I, it may be seen that both terms increase the energy of the state. The increase associated with F_1 is due to the larger single-particle energy splittings in the momentum-dependent potential, while the increase associated with F_1' is a residual interaction effect. From (95) it may easily be seen that ω_{GT}^2 varies with A as $A^{-1/3}$. This is not found to be the case empirically for heavy nuclei; the empirical giant dipole in heavy nuclei has a displacement field which is somewhat reduced in the surface region. A corresponding formula to Eq. (96) for that case has been given by Andō.[31]

4.2. *The Gamow-Teller vibration*

Recently, there has been much progress in measuring the strength function for the Gamow-Teller operator $\sigma\tau_-$, using the (p, n) reaction at proton energies around 200 MeV.[32] Figure 3 shows some experimental data,[33] with the prominent peak being the giant Gamow-Teller vibration. We have calculated the Gamow-Teller response in RPA using as a residual interaction the

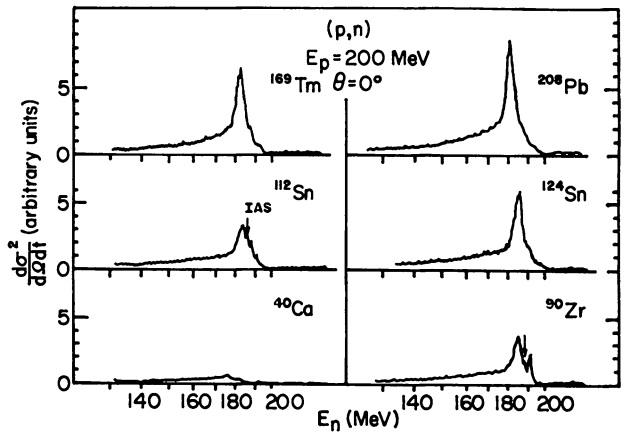

Fig. 3. The strength function for the (p, n) reaction, measured at 0° with incident proton energy of 200 MeV.[33] The prominent peak is the giant Gamow-Teller vibration. Note that it is absent from ^{40}Ca, an $N=Z$ nucleus.

direct part of a δ-function potential,[34]

$$\frac{\delta V}{\delta \rho} = v_{\sigma\tau} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) \ . \tag{97}$$

We found that the empirical data on the entire range of nuclei in Fig. 3 could be described with a value for the parameter $v_{\sigma\tau} = 220$ MeV-fm^3. This is certainly consistent with other sources of information about $v_{\sigma\tau}$,[35] such as the Landau parameter G_0'. From Table I,

$$v_{\sigma\tau} = \frac{(2\pi)^3}{16\pi k_F m} G_0' = (150 \text{ MeV-fm}^3)\ (1.37) \simeq 200 \text{ MeV-fm}^3 \ .$$

The Gamow-Teller state is not as strongly collective as the vibrations we discussed earlier, because most of the nucleons are spin-paired in the ground state. Consequently, much of the apparatus of RPA is unnecessary. In fact quite adequate estimates of the energy of the Gamow-Teller state can be obtained from the expectation value of H,

$$E = \frac{\langle 0 | O^\dagger [H, O] | 0 \rangle}{\langle 0 | O^\dagger O | 0 \rangle} \ . \tag{98}$$

The Hamiltonian is divided into an orbital energy part, a one-body spin-orbit part, and a residual interaction. Then these terms contribute separately to the energy,

$$E = E_{\text{orbital}} + E_{\text{spin-orbit}} + \overline{V} \ . \tag{99}$$

The Gamow-Teller operator has no spatial dependence and the orbital energy is therefore zero. The residual interaction (97) can be expressed

$$\overline{V} = \langle \text{GT} | \sum v_{\sigma\tau} \delta(\boldsymbol{r}_i - \boldsymbol{r}_j) \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j | \text{GT} \rangle \approx v_{\sigma\tau} \int (\delta \rho_{\sigma\tau}(\boldsymbol{r}))^2 d^3 r \ . \tag{100}$$

The evaluation of $\overline{V}$ can be simplified even further by relating to the matrix element of $\sigma\tau_-$,

$$\begin{aligned} &\int \delta \rho_{\sigma\tau}(\boldsymbol{r})\, d^3 r = \langle \text{GT} | \sigma\tau_- | 0 \rangle \ , \\ &\int (\delta \rho_{\sigma\tau}(\boldsymbol{r}))^2 d^3 r = \langle \delta \rho_{\sigma\tau} \rangle \langle \text{GT} | \sigma\tau_- | 0 \rangle \\ &\qquad \approx \frac{\langle \rho \rangle}{A} \langle \text{GT} | \sigma\tau_- | 0 \rangle^2 \ . \end{aligned} \tag{101}$$

Here $\langle \rho \rangle$ is the average density weighted by the radial distribution of the transition density. The $\langle \rho \rangle$ is roughly independent of nucleus and is of the order

$$\langle\rho\rangle\approx 0.12\ \mathrm{fm}^{-3}\,.$$

The transition strength $\langle \mathrm{GT}|\sigma\tau_-|0\rangle^2$ can be estimated from the sum rule limit Eq. (35)

$$\langle \mathrm{GT}|\sigma\tau_-|0\rangle^2\approx 2(N-Z)\,. \tag{103}$$

We conclude that the residual interaction contributes to the energy of the Gamow-Teller state according to

$$\overline{V}\approx\kappa_{\sigma\tau}\frac{2(N-Z)}{A}\,, \tag{104}$$

where $\kappa_{\sigma\tau}=v_{\sigma\tau}\langle\rho\rangle\approx(220\ \mathrm{MeV\text{-}fm}^3)\ (0.12\ \mathrm{fm}^{-3})\approx 25\ \mathrm{MeV}$. Equation (104) gives an estimate of 6 MeV in ^{90}Zr.

The contribution of the spin-orbit potential to the mean energy is also easy to compute. In the valence shells with the highest l, the nucleons travel with the Fermi momentum along the surface, and therefore have an A-independent spin-orbit splitting,

$$\varepsilon_{j_>}-\varepsilon_{j_<}\approx 6\ \mathrm{MeV}\,. \tag{105}$$

In a typical situation in heavy nuclei, the parent nucleus has the $j_>$ shell filled, and the daughter has both spin-orbit partners empty. The operator $\sigma\tau_-$ excites the configurations $(j_>{}^p j_>{}^{n-1})$ and $(j_<{}^p j_>{}^{n-1})$ with roughly equal strength, so the expectation value of the spin-orbit potential is

$$E_{\mathrm{S.O.}}\approx\frac{1}{2}(\varepsilon_{j_>}-\varepsilon_{j_<})\approx 3\,\mathrm{MeV}. \tag{106}$$

Thus the total excitation in ^{90}Zr is about 9 MeV, as is in fact measured.

The approximations (99) and (104) were applied by Suzuki[36] to the study of the energy splitting between the isobaric analog state (excited by the operator τ_-) and the Gamow-Teller state. The IAS energy has no spin orbit contribution, and its potential has a coefficient κ_τ which is slightly larger than $\kappa_{\sigma\tau}$. The result is

$$E_{\mathrm{GT}}-E_{\mathrm{IAS}}=(E_{\mathrm{S.O.}})_{\mathrm{GT}}+\frac{\kappa_{\sigma\tau}-\kappa_\tau}{A}\langle\sigma\tau\rangle^2\,. \tag{107}$$

For nuclei with small neutron excesses, the $(E_{\mathrm{S.O.}})_{\mathrm{GT}}$ is larger and the GT state lies above the IAS. In the heaviest nuclei, such as ^{208}Pb, the second term balances the first and the Gamow-Teller state is at the same energy as the IAS. This was of course hypothesized by Fujita, et al.,[37] in very early theoretical work on the GT.

A global study of the Gamow-Teller strength function was made by Klapdor, et al.,[38] who used a separable interaction of a form similar to

Eqs. (103) and (104),

$$\langle ij^{-1}|v|kl^{-1}\rangle = \kappa_{\sigma\tau}\langle ij^{-1}|\sigma\tau_-|0\rangle\langle kl^{-1}|\sigma\tau_-|0\rangle\,. \tag{108}$$

The shell structure, of course, plays some role in the detailed distribution of the strength function. It is possible to see some of these details with our analytic estimates. The major factor disturbing the Gamow-Teller vibration is the competition with the spin-orbit potential, which favors j-coupling, as opposed to the residual interaction of the Gamow-Teller vibration, which favors L-S coupling. When the Hamiltonian matrix for the configurations $(j_>^p j_>^{n-1})$, $(j_<^p j_>^{n-1})$ is diagonalized, the residual interaction by itself gives the upper state a predominant $L=0$ $S=1$ character, and the lower state a predominant $L=1$ $S=1$ character, with an energy splitting of $\sim$6 MeV. The spin-orbit potential mixes these states, with a matrix element of the order of

$$\frac{1}{2}(\varepsilon_{j_>} - \varepsilon_{j_<}) \sim 3 \text{ MeV}\,. \tag{109}$$

The mixing probability P of the L-S coupled states in perturbation theory is then given by

$$P \sim \left(\frac{3}{6}\right)^2 \sim \frac{1}{4}\,. \tag{110}$$

Thus the lower state should have about one-quarter of the Gamow-Teller strength. This is indeed observed, at least for the lighter nuclei.[34]

4.3. *Momentum-dependence of the τ- and $\sigma\tau$-interactions*

In the last section we saw that the Gamow-Teller energetics determined the strength of a momentum-independent interaction, equivalent to G_0' in the Landau language. We now ask whether anything can be learned about non-locality of the interaction, i.e., G_1'. We saw in the case of the τ operator, the giant dipole state was sensitive to momentum-dependent parts of the interaction, because the operator has a spatial variation. We could make the same kind of analysis of the v_τ interaction that we made for the $v_{\sigma\tau}$, fitting the energy of the isobaric analog state to determine v_τ. We would then expect not to reproduce the giant dipole with this interaction, because it lacks momentum dependence. I tried this out with a calculation of the τ excitation in ^{208}Pb. The v_τ was fit to the energy of the analog state, and gives results shown in Table II. Note that the required interaction strength depends on the single particle Hamiltonian; a Hamiltonian with $m^*/m<1$ requires less residual interaction than does a local Hamiltonian. These interactions and single-particle Hamiltonians are then used to calculate the giant dipole state in RPA, with results shown on the fourth line of Table II. In both cases the predicted energy is too low. This indicates the need for a repulsive momentum-dependent

Table II. Models for the τ excitations.

	Single-Particle Hamiltonian		Exp
	Skyrme I	Skyrme II	
$\frac{m^*}{m}$	1	0.76	
v_τ	305 MeV-fm^3	248 MeV-fm^3	
Analog State energy (τ_-)	15.2 MeV	15.2 MeV	15.2 MeV
Dipole State energy ($z\tau_z$)	11.6 MeV	12.7 MeV	13.5 MeV
$L=1$ τ_-	22 MeV	23.7 MeV	24.7 MeV (Ref. 39))

Table III. Models for the $\sigma\tau_-$ excitations.

	Single-Particle Hamiltonian		Exp
	Skyrme I	Skyrme II	
$\frac{m^*}{m}$	1	0.76	
$v_{\sigma\tau}$	220 MeV-fm^3	170 MeV-fm^3	
Gamow-Teller energy ($\sigma\tau_-$)	15.6 MeV	15.6 MeV	15.6 MeV (Ref. 40))
$L=1$ $\sigma\tau_-$	20~21 MeV	21 MeV	21.5 MeV (Ref. 40))

interaction in the τ-channel, as for example the F_1' of Table I. From Eq. (96) we see that the influence of the theoretical F_1' would be to raise the dipole energy by $\sim 10\%$. Parenthetically, we note that the repulsive momentum-dependence brings the RPA theory closer to TDA, since the ground state correlation matrix elements are reduced.

We now go back to the $\sigma\tau$ operator, and ask whether some momentum-dependence can be deduced from the comparison of $L=0$ and $L=1$ energies. As before, we fit the $v_{\sigma\tau}$ to the $L=0$ energy, and then compare the $L=1$ prediction with experiment. The results of this exercise are shown in Table III. We see that the empirical $v_{\sigma\tau}$ fits the $L=1$ strength quite well. Thus there is no empirical need for a G_1' interaction. This accords well with the theory of Table I, which has small G_1'.

§ 5. Conclusion

I have presented some ideas and techniques of response function theory that have proved very useful in the last decade. As a conclusion, it might be helpful to list what I consider as some of the main open problems.

A. *Beyond RPA* The effective mass m^*/m at the Fermi surface is one or larger, due to correlations. We need a simple theory for dealing with these correlations to go beyond the RPA. Some requirements to impose on the simple theory are that it reproduces empirical single-particle energies, and that it respects the conservation laws.

B. *Missing strength* There is disagreement between electron scattering results and hadron scattering results on the amount of strength in the giant quadrupole, with electron scattering claiming less than 50%. We need to go beyond RPA to predict how much strength is lost to the main peaks, and where that missing strength is located.

C. *Suppression of interior density fluctuations* We saw in Figs. 1 and 2 that the RPA predicts measurable transition densities deep in the nuclear interior, while experiment shows interior transition densities to be strongly suppressed. In fact, the same thing happens with the ground state density distribution: Theory predicts interior fluctuations due to specific shell fillings, but these are not seen experimentally. The correlations between particles would tend to smooth out these fluctuations. But no one to my knowledge has presented a plausible case that the correlations should be strong enough to cause the degree of suppression seen experimentally.

Acknowledgements

The author acknowledges support by the Japan Society for the Promotion of Science, and by the National Science Foundation under grant PHY-80-17605.

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