Nishida-Son from a BCS perspective

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These notes are to make a link between traditional many-body theory and the recent work [1] of Nishida and Son (NS) on the two-component Fermi system in the unitary limit. Using a field theoretic formulation, they treat the problem by considering the fermion propagator in $d = 4 - \epsilon$ dimensions, expanding about $d = 4$. Their lowest-order result for the fermion chemical potential μ is

$$\mu = \epsilon^{3/2} e_F/2 \tag{1}$$

where $e_F = k_F/2m$ is the Fermi energy of a noninteracting gas of the same density. The same result can be obtained in a straightforward way from the BCS theory, as will be shown below. Note that this does not establish the validity of BCS as the leading order theory. Nevertheless there is a good motivation to use BCS. For the 3-dimensional problem, Leggett showed that the BCS approximation gave the correct leading-order theory when the interaction is strong enough to make a compact bound state in the two-particle system. And in 4 dimensions Nussinov and Nussinov [3] showed that the theory at the unitary limit is trivial: fermions condense into noninteracting bosonic pairs.

There is only one independent variable in BCS theory, the chemical potential μ. Making the BCS ansatz for the wave function, one is led to the consistency condition determining the gap parameter Δ,

$$1 = v_0 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{\Delta^2 + (k^2/2m - \mu)^2}} \tag{2}$$

where v_0 is an interaction strength. The integral is divergent and must be cut off in some way. I do not have to specify the details of the cutoff, however. In the unitary limit the two-particle system has a zero-energy bound state, and this condition can be used to transform the consistency condition to a form that does not depend on the cutoff. The regularized consistency condition is

$$0 = \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{\sqrt{\Delta^2 + (k^2/2m - \mu)^2}} - \frac{1}{k^2/2m} \right). \tag{3}$$

I can now go to continuous dimensions by replacing the integration variable $d^3k/(2\pi)^3$ by $\Omega_d k^{d-1} dk/(2\pi)^d$ where Ω_d is the solid angle in d dimensions. It can be divided out and disappears from the equation. It is also convenient to transform the integral to dimensionless variables $z = k^2/2m\mu$ and $x = \Delta/\mu$. In terms of these, the equation is

$$0 = \int z^{d/2-1} dz \left(\frac{1}{\sqrt{x^2 + (z-1)^2}} - \frac{1}{z} \right), \tag{4}$$

This equation is to be solved for x after the integral has been carried out. The fast way to treat the integral is to use dimensional regularization, which gives an exact analytic expression. But it is instructive to treat it in a more pedestrian way, approximating the integrand by a power series. The square root term has different series expansion depending on whether $z >> x$ or vice versa. This requires that the integration range be separated into low-z and high-z regions. I separate the two at $z = x$,

$$\int_0^\infty = \int_0^x + \int_x^\infty \tag{5}$$

For the high-z integral, the expansion of the integrand is

$$\frac{1}{\sqrt{x^2 + (z-1)^2}} - \frac{1}{z} \approx \frac{1}{z^2} - \frac{x^2}{2z^3} + ... \tag{6}$$

and the integral is

$$\frac{x^{d/2-2}}{2 - d/2} - \frac{x^{d/2-1}}{6-d}. \tag{7}$$

Notice that the integral is only convergent for $d < 4$. Near $d = 4$ the first term becomes large due to the vanishing denominator.

In the low-z integral the integrand is approximated taking

$$\frac{1}{\sqrt{x^2 + (z-1)^2}} \approx \frac{1}{x}. \tag{8}$$

Both terms in the integrand terms gives integrals of order $x^{d/2-1}$, the same as the second term in eq. 7. Combining all the terms, the result is $x \sim 1/\epsilon$ or

$$\Delta \sim \epsilon^{-1}\mu. \tag{9}$$

I write the result in this way because the approximations made are not accurate enough to get the correct prefactor in the equation ($x = 2/\epsilon$ from NS).

The density of the system n is determined from another formula of BCS theory,

$$n = \int \frac{d^d k}{(2\pi)^d} \left(1 - \frac{k^2/2m - \mu}{\sqrt{\Delta^2 + (k^2/2m - \mu)^2}} \right). \tag{10}$$

Converting to the variables x and z as before, the equation for d dimensions can be expressed

$$\frac{4}{d}\left(\frac{e_F}{\mu}\right)^{d/2} = \int z^{d/2-1} dz \left(1 - \frac{z-1}{\sqrt{x^2+(z-1)^2}}\right). \tag{11}$$

The integral is estimated in the same way as was done for the self-consistency condition. Again, there is a term in the power series expansion that diverges at $d = 4$; it gives the leading term in the integral, $x^{\epsilon/2}/2\epsilon$ Combining this with eq. (9) one finds $\mu \sim \epsilon^{3/2} e_F$. Note that the chemical potential goes to zero at $d = 4$, as required by the Nussinov analysis. However, the power law is 3/2 rather than 1 as they conjectured.

As mentioned earlier, the above estimates are not accurate enough to extract the prefactors of the ϵ dependence. The exact integrals can be derived by dimensional regularization and are obtained from the generic formula [2]

$$\int_0^\infty dz \frac{z^\alpha}{[x^2+(z-1)^2]^2} = -\frac{\pi}{\sin \pi\alpha}(1+x^2)^{\alpha/2} P_\alpha(-1/\sqrt{1+x^2}). \tag{12}$$

Here P_α is the Legendre function of order α. The self-consistency condition becomes

$$0 = P_{1-\epsilon/2}(-1/\sqrt{1+x^2}) \tag{13}$$

Expanding this in powers of ϵ and x^{-1} one finds

$$x = 2/\epsilon \tag{14}$$

in leading order, in agreement with NS. The corresponding density integral in eq. 11 is [5]

$$-\frac{\pi}{\sin \pi d/2}(1+x^2)^{d/4} P_{d/2}(-1/\sqrt{1+x^2}) \tag{15}$$

This is evaluated for small ϵ and x^{-1} to be x^2/ϵ. Combining this with eqs. 11 and 14 yields

$$\mu = \epsilon^{3/2} e_F/2 \tag{16}$$

This is the lowest-order result obtained by Nishida and Son.

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[1] Y.Nishida and D. Son, "An epsilon expansion for Fermi gas at infinite scattering length", cond-mat/0605600 (2006).

[2] I.S. Gradshteyn and I.M. Ryzhik, "Table of Integrals, Series and Products," (Academic Press, New York, 1965), eq. 3.252.11.

[3] Z. Nussinov and S. Nussinov, cond-mat/0410597 (2004)

[4] A.J. Leggett, in "Modern Trends in the Theory of Condensed Matter (Springer, Berlin, 1980).

[5] I omit a term proportional to $P_{1/2}$ that vanishes in the unitary limit.