# The collective Hamiltonian in a schematic realization of the GCM

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Here I apply the GCM/GOA theory worked out in [1] to a model having degenerate particle-hole excitations. The Hamiltonian and matrix element:

$$H_0|0\rangle = 0;\;\; H_0|ph\rangle = E_{ph}|ph\rangle;\;\; \langle ph|Q|0\rangle = Q_{ph}$$

I using a different collective variable $q$, one defined to satisfy

$$\langle q|Q|q\rangle = q$$

The collective wave function:

$$|q\rangle = \frac{1}{(1+q^2/(4Q_{ph}^2)^{1/2}}|0> - \frac{q}{2Q_{ph}}|ph>$$

The collective parameter $q$ has been defined to satisfy First task, find $\alpha$ for the GOA. The overlap is

$$\langle q|q'\rangle = 1 - \frac{(q-q')^2}{8Q_{ph}^2}$$

and the Gaussian approximation $\langle q|q'\rangle = \exp(-\alpha(q-q')^2)$ requires

$$\alpha = \frac{1}{8Q_{ph}^2}$$

Now find $h_2$ and $\epsilon$.

$$\frac{\langle q|H_0|q'\rangle}{\langle q|q'\rangle} \approx E_{ph}\frac{qq'}{4Q_{ph}^2} = -\frac{(q-q')^2}{16Q_{ph}^2}E_{ph} + \frac{(q+q')^2}{16Q_{ph}^2}E_{ph}$$

$$h_2 = \frac{E_{ph}}{16Q_{ph}^2};\;\; \epsilon = \frac{E_{ph}}{4Q_{ph}^2}$$

We then use Eq. (4) of [1] to get $\omega$:

$$\omega = \sqrt{\frac{h_2\epsilon}{\alpha^2}} = E_{ph}$$

as expected. The energy of the ground state is obtained from eq. (3) of [1], it is:

$$E_0 = 0$$

as expected.

This can be easily generalized to the RPA Hamiltonian, Eq. (5). The parameter $h_2$ is the same as before but $\epsilon$ changes to

$$\epsilon = \frac{E_{ph} + 2v}{4Q_{ph}}.$$

The spectrum is harmonic with a frequency given by the same formula

$$\omega = \sqrt{\frac{\epsilon h_2}{\alpha^2}} = \sqrt{(E_{ph} + 2v)E_{ph}}$$

and ground state energy

$$E_0 = \frac{1}{2}\omega - \frac{1}{2}(E_{ph} + v).$$

We now ask, is there a collective Hamiltonian having this harmonic spectrum with the given ground state energy? The answer is obviously yes, we just have to choose the parameters appropriately. The Hamiltonian has the form

$$-\frac{1}{2M}\frac{\partial^2}{\partial q^2} + \frac{1}{2}\kappa q^2 - V(0).$$

However, the energetics alone are not enough to determine a length scale for the variable $q$. We therefore demand in addition

$$\langle 1|Q|0\rangle_{RPA} = \langle 1|q|0\rangle_{col}.$$

The two sides are given by

$$\langle 1|Q|0\rangle^2_{RPA} = \frac{Q_{ph}E_{ph}}{\omega},$$

$$\langle 1|q|0\rangle^2_{col} = \frac{1}{2m\omega}.$$

The deduced parameters are:

$$\frac{1}{M} = 2Q^2_{ph}E_{ph}$$

$$\kappa = 2\frac{\partial^2}{\partial q^2}\langle q|H|q\rangle = \frac{E_{ph}}{2Q^2_{ph}}$$

$$V(0) = \frac{1}{4}E_{ph} + \frac{1}{4}(E_{ph} + 2v).$$

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[1] http://http:www.phys.washington.edu/users/bertsch/rpa3.pdf