ASYMPTOTIC FORM OF MONOPOLE VIBRATIONS

F.E. SERR, G.F. BERTSCH and J. BORYSOWICZ
Cyclotron Laboratory and Physics Department, Michigan State University, East Lansing, MI 48824, USA

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The velocity field for monopole vibrations of large nuclear systems is shown to obey the classical equations for an elastic medium, rather than a fluid. In the surface the velocity field becomes an increasing exponential. For ^{208}Pb, these effects combine to give motion similar to the Tassie model.

Highly collective nuclear states, such as the giant monopole vibration, are often described in fluid dynamical terms [1,2]. It is also possible to cast microscopic calculations in forms which give meanings to many classical continuum variables [3–7]. The liquid drop description is commonly thought to be correct for such macroscopic variables, at least in the limit of large nuclei. However, the Tassie model [8–10] also seems to give good results for heavy nuclei. For the monopole, the Tassie model is a simple radial scaling of the wavefunctions. These two models yield quite different forms for the velocity field of the collective motion, especially near the surface. Detailed microscopic calculations of the monopole vibration in ^{208}Pb [11] give a velocity field with an intermediate form. (See fig. 1.) A clear determination of the motion in the large A limit, with no assumptions on the form of the exciting field, has not been made. This is important if monopole vibrations are used to specify properties of nuclear matter. There also remains the question of whether ^{208}Pb is large enough to exhibit this asymptotic behavior.

We show below that the velocity field for monopole vibrations of large spherical nuclear systems is neither liquid drop nor Tassie. The motion of the bulk of the particles is well described by the classical equations for vibrations of an elastic medium, rather than a fluid. The surface boundary condition for the velocity field is an increasing exponential. We express this in terms of binding energies of the ground and excited states.

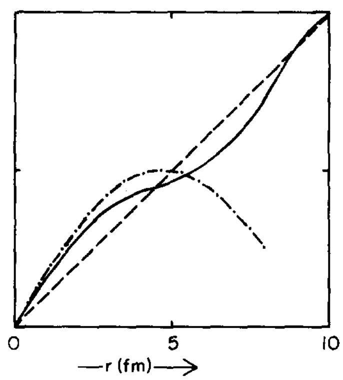

Fig. 1. Velocity fields for monopole vibration of ^{208}Pb: solid line, full RPA calculation (from ref. [11]); dashed line, Tassie model ($u_r \propto r$); dash-dot line, liquid drop model ($u_r \propto j_1(\pi r/R)$).

For systems as small as ^{208}Pb, this surface condition still masks the bulk behavior.

We have made variational calculations of the velocity field of the monopole vibration in both Bose and Fermi systems with density-dependent interactions. The RPA theory for a coherent collective state has previously been expressed in terms of the velocity field [3]. The derivation starts with the time-dependent Hartree–Fock formalism, then assumes small amplitude harmonic motion. The velocity field is then a small displacement vector, $\boldsymbol{u}(\boldsymbol{r}) \equiv \nabla\Phi(\boldsymbol{r})$, with a harmonic time dependence. Further assumption of a single velocity field for all the particles leads to a compact expression for the frequency of vibration

$$\omega^2 = I[u]/\tfrac{1}{2}m\int \rho_0 \boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}^3r\,, \tag{1}$$

with

$$I[u] \equiv m^2 \sum_i \int \mathrm{d}^3r\{([H_0,\Phi]\,\psi_i^0) \times ([H_0,[H_0,\Phi]] - (\delta V/\delta\rho)\,\delta\rho)\,\psi_i^0\} \tag{2}$$

$$= \int \mathrm{d}^3r \Big\{ \tfrac{1}{2}(\nabla\cdot\boldsymbol{u})^2 \rho_0^2 \frac{\delta V}{\delta\rho} - \frac{1}{m}\sum_i \Big[(\boldsymbol{u}\cdot\nabla\psi_i^0)(\nabla\psi_i^0)\cdot\nabla(\nabla\cdot\boldsymbol{u}) + (\boldsymbol{u}\cdot\nabla\psi_i^0 + \tfrac{1}{2}(\nabla\cdot\boldsymbol{u})\psi_i^0)\sum_{\alpha\beta}\nabla_\alpha u_\beta \nabla_\beta\nabla_\alpha\psi_i^0 \Big] + \tfrac{1}{2}\rho_0(\nabla\cdot\boldsymbol{u})\,\boldsymbol{u}\cdot(\nabla\rho_0)\frac{\delta V}{\delta\rho} + \frac{\rho_0}{8m}\boldsymbol{u}\cdot\nabla(\nabla^2(\nabla\cdot\boldsymbol{u})) - \frac{1}{4m}(\nabla\cdot\boldsymbol{u})(\nabla\rho_0)\cdot\nabla(\nabla\cdot\boldsymbol{u}) \Big\}. \tag{3}$$

(We follow the notation of ref. [3].) Note that the density, the hamiltonian and the variation of the potential are all evaluated in the self-consistent ground state. The approximation of a single velocity field is also made in a calculation of the monopole vibration by Eckart and Holzwarth [18].

Eq. (1) has the form of Rayleigh's variational principle [12] for the vibrational energy, leading us to interpret $I[u]$ as the potential energy associated with a displacement $\boldsymbol{u}$. It can be shown from Thouless' variational principle [13] that eq. (1) is a quantum mechanical upper bound on the energy of the lowest RPA model [6]. Insertion of the Tassie velocity potential, $\Phi(r) = r^2$, in eq. (1) gives the $\tilde{E}_3$ sum rule result of ref. [14]. However, the eigenstate need not have this form.

For monopole vibrations of a spherical system, symmetry requires that $\boldsymbol{u} = \hat{r}u_r(r)$. We solve eq. (1) for the variational minimum with a velocity field of the form

$$u_r = \sum_n a_n r^n\,. \tag{4}$$

The coefficients a_n are the variational parameters. Minimization is equivalent to solving for the lowest eigenstate of a matrix equation for the a_n. The variational principle gives the energy of the state and the functional form of the velocity field. It does not give the magnitude of the displacement, which for small amplitude harmonic motion does not affect the frequency.

The calculations must start with the self-consistent ground state wavefunctions, ψ_i^0. We look first at a model Bose system described by the hamiltonian

$$H_0 = -\nabla^2 + a\rho_0 + b\rho_0^2\,, \tag{5}$$

where a and b are constants. There is only one wavefunction. We show in fig. 2 the ground state density for different values of the system radius R. Results of the variational calculations for the velocity fields are also shown. Inclusion of 7 to 10 terms in eq. (4) gave ω to better than 0.1% and u_r to within 1%. Concentrating for the moment on the interior of the system, where the density (and the wavefunction) is nearly constant, we find that as R becomes large the velocity field approaches a spherical Bessel function:

$$u_r \propto \nabla_r j_0(qr) \propto j_1(qr)\,\hat{r}\,. \tag{6}$$

For R = 19.6, the wavenumber q satisfies the condition

$$qR = \pi \tag{7}$$

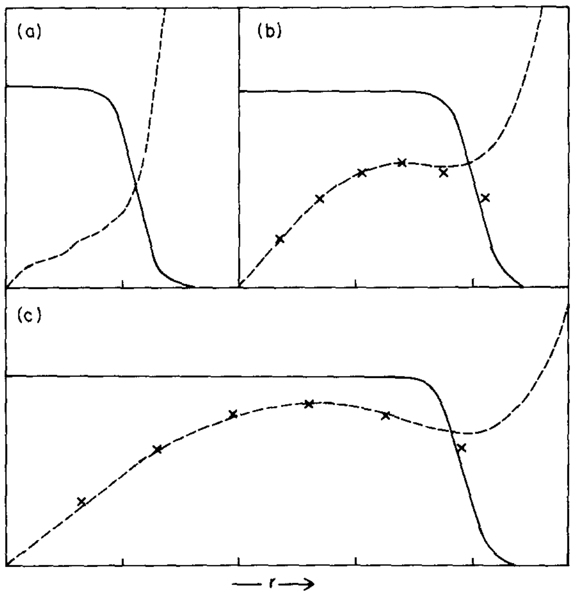

Fig. 2. (a) Ground state density (solid line) and velocity field (dashed line) for model Bose system with radius R = 5.5. Vertical scale is arbitrary. (b) Same as (a), with R = 10.1. Crosses (x) indicate values of j_1 (qr) with qR = 2.99. (c) Same as (b) with R = 20 and qR = 3.136.

to within 0.2%. Also, the frequency is within 4% of the value

$$\omega = 2\pi/R = (K/9m)^{1/2} q \ , \tag{8}$$

where $K = 9V^2 \partial^2(E/A)/\partial V^2$ is the bulk compressibility for a system described by the hamiltonian (5). These values are just the hydrodynamical results for vibrations of a sphere of radius R. This is as expected, since it is well known that a superfluid Bose system follows irrotational hydrodynamics [15].

We next examine large Fermi systems. The specific example is an A = 6100 nucleus using a Skyrme-type force with $t_0 = -1024$, $t_3 = 14604$, and $t_1 = t_2 = x = 0$. We set $N = Z$, neglect Coulomb forces and include no spin-orbit splitting. A self-consistent Hartree–Fock calculation of the ground state yields a density with radius R = 20.9 fm and surface thickness t = 1.6 fm, and a binding energy for the least bound nucleons of 17 MeV. The ground state density and the velocity field are shown in fig. 3. We again find that the motion of the bulk of the particle fits a spherical Bessel function, $j_1(qr)$. However, the wavenumber is shifted from the hydrodynamical value, eq. (7), by about 15%, so that $qR = 2.63$.

We can understand the results for both systems in terms of elastic vibrations. The classical expression for the potential energy of an elastic medium has the form [16]

$$W = \int d^3r \left[\tfrac{1}{2}\lambda(\nabla \cdot u)^2 + \mu \sum_{\alpha\beta} (\nabla_\alpha u_\beta)^2\right] , \tag{9}$$

with u the local displacement from equilibrium. Here λ and μ are Lamé's elastic constants. The bulk modulus is given by

$$K = 9(\lambda + \tfrac{2}{3}\mu)/\rho \ . \tag{10}$$

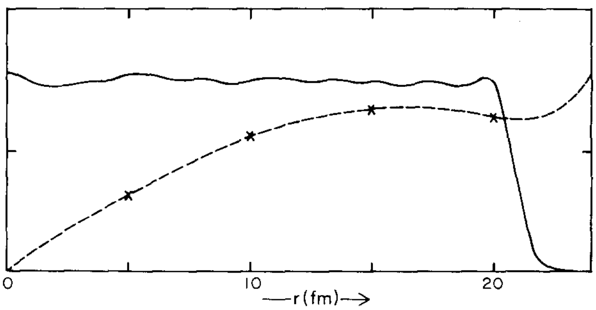

Fig. 3. (a) Ground state density (solid line) and velocity field (dashed line) for A = 6100 nucleus. Crosses (x) indicate values of j_1 (qr) with $qR = 2.63$. Vertical scale is arbitrary.

In a fluid, the ordinary sound velocity is $c^2 = K/9m$. However, when $\mu \neq 0$ we should consider both shear and compressional waves:

$$c_\ell^2 = (\lambda + 2\mu)/m\rho \qquad \text{(compressional)} \tag{11a}$$

$$c_t^2 = \mu/m\rho \qquad \text{(shear)} \tag{11b}$$

Classically, the monopole vibrations of a uniform elastic sphere are purely compressional modes. The frequency of the lowest mode is given by

$$\omega^2 = c_\ell^2 q^2 = q^2(\lambda + 2\mu)/m\rho \ . \tag{12}$$

The displacement vector has the form of eq. (6), with q determined by the condition that the stress vanish at the surface [17]:

$$\frac{\tan qR}{qR} = \left[1 - \left(\frac{qR}{2}\right)^2 \frac{\lambda + 2\mu}{\mu}\right]^{-1} . \tag{13}$$

If $\mu = 0$ (no shear), the boundary condition (13) reduces to eq. (7) and c_ℓ equals c, the usual hydrodynamical results.

This is precisely what occurs for the Bose system. Compare the quantum mechanical expression for the potential energy, eq. (3), with eq. (9). The terms with more than two derivatives on u will become small for low energy vibrations ($q \to 0$). We see that all the terms with cross derivatives of the displacement ($\nabla_\alpha u_\beta$) involve derivatives of the wavefunction. For the Bose system, $\nabla\psi^0$ is nonzero only in the surface. In the limit of large A (large R), the coefficient μ in the interior is negligible in comparison to the dominant term, $\lambda = \rho_0^2 \delta V/\delta\rho$. Thus we recover the hydrodynamical equations (6)–(8).

For the Fermi system, however, $\nabla\psi^0 \neq 0$ even where the density is constant. Thus $\mu \neq 0$, even in the interior. Both λ and μ now have contributions from the kinetic energy of the wavefunctions. This shifts the wavelength of the vibration. In the Fermi gas model, $\mu = \rho_0 k_f^2/5m$ [6]. If we take this value as an approximation to μ in the interior, for our Skyrme interaction we find $\lambda = 2.17\ \mu$. Eq. (13) then implies $qR = 2.76$, in fairly good agreement with the fitted value.

We can also compare the predictions of the various theories for the frequency of the state. For the Tassie model, where $u = \nabla r^2$ is a simple scaling field, the result is

$$\omega_T^2 = K/m\langle r^2\rangle \ . \tag{14}$$

Taking K and $\langle r^2 \rangle$ from our nuclear system, we find $\omega_T = 7.71$ MeV. The hydrodynamical formula

$$\omega_{HD}^2 = (\pi^2/15)\, K/m\langle r^2 \rangle \qquad (15)$$

gives $\omega_{HD} = 6.25$ MeV, while eq. (12) with $\lambda = 2.17\ \mu$ gives $\omega_{el} = 6.67$ MeV, in better agreement with the variational result for the energy of the state, 6.54 MeV. The hydrodynamical ω_{HD} is also a reasonable approximation to the variational result. We note that (15) is obtained from (12) by replacing q^2 with $\pi^2/(\frac{5}{3}\langle r^2 \rangle)$ and c_ℓ^2 with c^2. The errors from these two substitutions tend to cancel. Thus ω_{el} and ω_{HD} differ by only 6%, although q is shifted by 14% and $c_\ell = 1.21\ c$. This compensation is also noted in ref. [14].

We have thus far ignored any effects due to the finite surface thickness and exponential tail of the density. In fact, outside the constant interior the displacement u is a rapidly increasing function of r. This characteristic is evident in both Bose and Fermi systems (figs. 2 and 3), and in ^{208}Pb (fig. 1). Examining the model Bose system with $R = 10$ more closely, we find that u_r is actually an increasing exponential, $u_r \propto e^{\beta r}$, with $\beta = 0.4$. This behavior is necessary to match the exponentially decaying tails of the ground state ψ_0 and excited state ψ_1. Well beyond the classical turning points, we know that

$$\psi_i \sim \exp(-\alpha_i r)/r\ , \qquad (16)$$

where $\alpha_i = (2mE_i/\hbar^2)^{1/2}$. The excited state is related to the ground state wavefunction and the velocity field by [3]

$$\psi_1 \approx (1 + \boldsymbol{u} \cdot \nabla + \tfrac{1}{2}(\nabla \cdot \boldsymbol{u}))\, \psi_0\ . \qquad (17)$$

As r becomes large, the solution to eqs. (16) and (17) is $u_r \propto e^{\beta r}$, with $\beta = \alpha_0 - \alpha_1 = 0.45$, in excellent agreement with the variational result. A similar exponential growth is obtained in ref. [5]. Interestingly, this behavior can also be derived from the purely classical vibration of a diffuse surface whose mass density decreases exponentially.

We now see why the Tassie model seems to do well for "small" nuclei, such as ^{208}Pb. The surface condition shifts q in the interior to even smaller values than given by the elastic theory and prevents u_r from turning down at the surface. As shown in fig. 1, the resulting velocity field follows a straight line through most of the nucleus.

We have shown that, in the large A limit, the motion of the bulk of the nucleons follows the equations for elastic vibrations. However, the velocity field for large r is actually an increasing exponential. Are experimentally observable nuclei large enough to expect that their bulk motion is not significantly altered by the surface behavior? This would require that two quantities neglected in the analysis of the bulk behavior be small: the size of the surface, as reflected by the ratio of the surface thickness to the radius, and the energy of vibration, in comparison to the binding energy. Neither of these criteria is well satisfied for ^{208}Pb. In particular, the energy of the monopole vibration is greater than the single particle binding energies of a significant fraction of the particles. Thus, at least when considering monopole vibrations, ^{208}Pb is a "small" nucleus.

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References

[1] A. Bohr and B. Mottelson, Nuclear Structure Vol. II, (Benjamin, Reading, Mass., 1975) p. 654.
[2] J.D. Walecka, Phys. Rev. 126 (1962) 653.
[3] G. Bertsch, Nucl. Phys. A249 (1975) 253.
[4] G. Holzwarth and G. Eckart, Z. Physik A283 (1977) 219; Z. Physik A284 (1978) 291.
[5] G. Eckart and G. Holzwarth, Siegen preprint Si-79-8.
[6] G. Bertsch, Annals of Physics 86 (1974) 138.
[7] J.P. Blaizot, Phys. Lett. 78B (1978) 367.
[8] L.J. Tassie, Aust. J. Phys. 9 (1956) 407.
[9] J. Martorell, O. Bohigas, S. Fallieros and A.M. Lane, Phys. Lett. 60B (1976) 313.
[10] H. Sagawa and G. Holzwarth, Progr. Theor. Phys. 59 (1978) 1213.
[11] J.P. Blaizot, Nuclear Compressibilities, Physics Reports, to be published.
[12] J.W.S. Rayleigh, Theory of Sound (Dover, New York, 1945) p. 109.
[13] D.J. Thouless, Nucl. Phys. 22 (1961) 78.
[14] B.K. Jennings and A.D. Jackson, Sum Rules and the Breathing Mode, SUNY Stony Brook preprint.
[15] E. Gross, J. Math. Phys. 4 (1964) 195.
[16] I. Sokolnikov, Mathematical theory of elasticity (McGraw-Hill, New York, 1956).
[17] L.D. Landau and E.M. Lifschitz, Theory of elasticity (Pergamon Press, Oxford, 1970) p. 106.
[18] G. Eckart and G. Holzwarth, Phys. Lett. 87B (1979) 147.