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NUCLEAR HYDRODYNAMICS

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Abstract: Continuum equations are derived for nuclear motion in the framework of time-dependent Hartree-Fock theory. The equations describe an elastic medium at low frequencies and independent particle motion at high frequencies. The giant quadrupole and monopole states are derived from the theory. The quadrupole energy in the Fermi gas model is given by $E = (\hbar^2/m)(6k_F^2/5\langle r^2\rangle)^{\frac{1}{2}}$. The monopole energy depends on the details of the underlying Hartree-Fock theory. For higher multipoles, the theory only gives a mean energy of the strength function. We derive the Wigner density matrix of the theory. From this it is argued that all the modes except the giant quadrupole and monopole are subject to Landau damping.

1. Introduction

To describe such complex phenomena as nuclear fission or heavy ion collisions, it is essential that one deal with fewer variables than are contained in the many-body problem. In classical physics, it is a simple matter to reduce the many-body problem to hydrodynamics via the Boltzmann equation. For describing the dynamics of nuclei the situation is not so simple because the mean free path of the particles is not small compared to the size of the nucleus.

Nevertheless, much theoretical work has been done using hydrodynamic equations as a model for the dynamics. In many cases the parameters such as compressibility, viscosity are treated as adjustable parameters. It is important therefore to derive as far as possible the continuum mechanics of finite Fermi systems, both to obtain the form of the equations as well as the theoretical values of the constants that appear. We shall first discuss a fundamental constraint on what can be accomplished by a hydrodynamic description. From the hydrodynamic point of view, this constraint is the equation of continuity. On the microscopic quantum mechanical level, the constraint appears in the energy-weighted sum rules. We then derive the hydrodynamics of the time-dependent Hartree-Fock equations. The result may appear disappointing, for it does not give low energy vibrations or moments of inertia. However, in view of the constraint we cannot expect to do better with the most basic hydrodynamic

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variables. Following this, we discuss the dissipative mechanisms in the Fermi system. The Landau damping is well understood in the framework of the quantum mechanical Boltzmann equation, but the subject of collisional damping is still open.

2. The equation of continuity

The type of equations we wish to derive will certainly involve as field quantities the mass density ρ and the momentum density $\boldsymbol{p}$. For a more complete description, other variables might be required in addition. Examples are the strain tensor, deformation of the Fermi surface, or the pairing field. In any case, we would expect the theory to have ρ and $\boldsymbol{p}$ as fundamental field quantities, and that the equation of continuity would be satisfied. This is

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \frac{\boldsymbol{p}}{m}. \tag{1}$$

Classically, the equation of motion is derived by combining this with an equation for the force on the medium,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F}(\rho, \ldots). \tag{2}$$

In the simplest case, $\boldsymbol{F}$ is the gradient of a scalar,

$$\boldsymbol{F} = \nabla P, \tag{3}$$

and the equation for ordinary sound may be easily derived [33]).

Turning to the quantum mechanical problem, we may derive the same equations with the usual definitions of ρ and $\boldsymbol{p}$:

$$\rho(r, t) = \sum_i \rho_i(r, t), \tag{4}$$

where

$$\rho_1(r, t) = \int \mathrm{d}r_2 \ldots \mathrm{d}r_N \psi^* \psi,$$

$$\boldsymbol{p}(r, t) = \sum_i \boldsymbol{p}_i(r, t), \tag{5}$$

where

$$\boldsymbol{p}_1(r, t) = \int \mathrm{d}r_2 \ldots \mathrm{d}r_N \frac{1}{2i} [\psi^* \nabla_1 \psi - (\nabla_1 \psi^*)\psi].$$

Here $\psi(r_1 \ldots r_N, t)$ is the many-body wave function satisfying the Schrödinger equation,

$$\left[-\sum_i \frac{\nabla_i^2}{2m} + \sum_{ij} V(r_{ij})\right] \psi = i \frac{\mathrm{d}}{\mathrm{d}t} \psi. \tag{6}$$

We prove the equation of continuity by using the Schrödinger equation to evaluate

$\partial\rho/\partial t$. This simple exercise, which should be familiar to most readers [1]), goes as follows:

$$\frac{\partial\rho_1}{\partial t} = \frac{1}{i}\int \mathrm{d}r_2 \ldots \mathrm{d}r_N[\psi^* H\psi - (H\psi)^*\psi]$$

$$= -\frac{1}{2mi}\int \mathrm{d}r_2 \ldots \mathrm{d}r_N[\psi^*\nabla_1^2\psi - (\nabla_1^2\psi)^*\psi]$$

$$= -\nabla\cdot\frac{1}{m}\int \mathrm{d}r_2 \ldots \mathrm{d}r_N \frac{[\psi^*\nabla_1\psi - (\nabla_1\psi)^*\psi]}{2i} = -\nabla\cdot\boldsymbol{p}_1/m. \tag{7}$$

In the same way, we can evaluate the time derivative of the momentum density to obtain an equation of type eq. (2). The left hand side is rather complicated; it is not in general the gradient of a scalar as is required for ideal fluid behavior. To make further progress, we must impose a specific model of ψ. This will be done in sect. 3.

Before proceeding, it is useful to derive generalized energy-weighted sum rules [2,3]) from the equation of continuity [4]). The sum rules are relations involving matrix elements of position operators between eigenstates of the Hamiltonian:

$$\sum_\alpha (E_\alpha - E_0)\langle\psi_0|\hat{\rho}(x)|\psi_\alpha\rangle\langle\psi_\alpha|M(r)|\psi_0\rangle = \frac{-1}{2m}\nabla_x\cdot\langle\psi_0|\nabla M\hat{\rho}(x)|\psi_0\rangle. \tag{8}$$

Here $\hat{\rho}(x)$ is the density operator, with matrix elements defined in eq. (4), and the sum is over all excited states of the system. Also, $M(r)$ is an arbitrary function of position, such as $r^2Y^2(r)$, $j_0(kr)$, etc. To derive the relation, we start with the system in the ground state and excite it with a weak impulsive potential of the form

$$V(r,t) = M(r)\delta(t-t_0). \tag{9}$$

In terms of the eigenstates of the Hamiltonian, the wave function after t_0 is

$$\psi = |\psi_0\rangle + \sum_\alpha \frac{\langle\psi_0|M|\psi_\alpha\rangle}{i}\,\mathrm{e}^{-i(E_\alpha - E_0)t}|\psi_\alpha\rangle. \tag{10}$$

We now calculate the rate of change of density just after t_0:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -2\sum_\alpha (E_\alpha - E_0)\langle\psi_0|M|\psi_\alpha\rangle\langle\psi_\alpha|\hat{\rho}(x)|\psi_0\rangle. \tag{11}$$

The divergence of the momentum field is similarly expressed,

$$-\frac{\nabla\cdot\boldsymbol{p}}{m} = \frac{-1}{im}\nabla\cdot\sum_\alpha\langle\psi_0|M|\psi_\alpha\rangle\sum_i\frac{\int\prod_{j\neq i}\mathrm{d}^3r_j[(\nabla_i\psi_0^*)\psi_\alpha - \psi_0^*\nabla_i\psi_\alpha]}{2i}$$

$$= \frac{\nabla}{m}\cdot\langle\psi_0|\hat{\rho}(x)[\nabla M]|\psi_0\rangle. \tag{12}$$

We have used completeness in the last step. The equation of continuity then requires that the left-hand sides of eqs. (11) and (12) be equal, which is just the sum rule eq. (8). The standard sum rule [5]) is obtained by integrating both sides of eq. (8) with $\int dx M(x)$,

$$\sum_{\alpha} (E_\alpha - E_0)\langle\psi_0|M|\psi_\alpha\rangle^2 = \frac{1}{2m}\langle\psi_0|(\nabla M)^2|\psi_0\rangle. \tag{13}$$

Of course, this could have been derived directly from the Schrödinger equation without using the equation of continuity. The purpose of the demonstration was to show that a hydrodynamic theory, with the usual interpretations of ρ and $\boldsymbol{p}$, must produce excitations satisfying the sum rules. Empirically, the low-lying collective excitations in nuclei typically have only a fraction of the sum rule strength. The lowest quadrupole excitations, for example, typically exhaust 5 % of the sum rule [6]). The low-lying collective octupole has a similar fraction of the octupole sum. Therefore, as Lane [5]) observes, these states cannot be described by a simple hydrodynamic theory. Only by going to a multicomponent fluid description could one hope to have a comprehensive description. It is an important unsolved problem to elucidate the continuum theory of these lower states, because they play an important role in the dynamics of fission.

3. Dynamics of Hartree-Fock theory

A simple quantum mechanical model which displays interesting dynamics is time-dependent Hartree-Fock theory. Another study of this theory somewhat related to ours is given in ref. [40]). The many-body wave function is approximated by a determinant of single-particle wave functions ψ_i. The ψ_i are solutions of the Hamiltonian,

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi_i = \left(-\frac{\nabla^2}{2m} + V[\rho] - \varepsilon_i\right)\psi_i. \tag{14}$$

The ε_i are determined by the time-independent solutions. These solutions, which we refer to with a superscript zero, give the ground state wave function. We also make use of the ground state density,

$$\rho_0 = \sum_i |\psi_i^{(0)}|^2. \tag{15}$$

To impart motion to a system initially in its ground state, we multiply each single-particle wave function by the factor $\mathrm{e}^{im\xi}$, where ξ is an arbitrary real function of coordinate and time. The wave function will then acquire a momentum density

$$\boldsymbol{p} = m\rho_0\nabla\xi, \tag{16}$$

with

$$\psi_i(r) = \mathrm{e}^{im\xi}\psi_i^{(0)}.$$

Thus ξ may be interpreted as a velocity potential. With the phase factor $\mathrm{e}^{im\xi}$ the system has been given a coherent velocity field, but its density is still the same as in the

ground state. To describe density changes, we shall have to modify the wave function in a different way. In the theory of the Bose quantum liquid, it is sufficient to multiply the single-particle wave function by a real function, $R(x)$. The density variable is R^2, and the hydrodynamic theory follows easily [7]). This method is inadequate for the Fermi system, because the nodes in the wave functions are unchanged by multiplication. A more useful method is to consider the coordinate transformation on the wave functions,

$$\psi_i(r) = \psi_i^{(0)}(r+\nabla\Phi). \tag{17}$$

However, the transformation (17) is insufficient because it does not conserve the normalization of the wave function. This may easily be seen for the case where Φ is a stretching field. Then ψ covers more volume but since its magnitude is the same, the integrated probability must be larger. The normalization can be conserved for small Φ by using the following transformation instead:

$$\psi_i(r) = \frac{1}{\sqrt{1-\nabla^2\Phi}}\psi_i^{(0)}(r+\nabla\Phi). \tag{18}$$

We shall now assume that the changes in the wave function are small. We may then expand the single-particle wave functions†

$$e^{im\xi}\frac{1}{\sqrt{1-\nabla^2\Phi}}\psi_i^{(0)}(r+\nabla\Phi) = \psi_i^{(0)}+im\xi\psi_i^{(0)}+(\nabla\Phi)\cdot\nabla\psi_i^{(0)}+\tfrac{1}{2}(\nabla^2\Phi)\psi_i^{(0)}. \tag{20}$$

The density is expanded to first order in Φ as

$$\sum_i \psi_i^*\psi_i = \rho_0+\delta\rho, \tag{21}$$

where

$$\delta\rho = (\nabla\Phi)\cdot\nabla\rho_0+(\nabla^2\Phi)\rho_0.$$

The continuity equation requires that

$$\frac{d\rho}{dt} = \frac{d\delta\rho}{dt} = -\frac{1}{m}\nabla\cdot p = -\nabla\cdot\rho\nabla\xi. \tag{22}$$

This equation can be satisfied with the density of eq. (21) if only

$$\xi = -d\Phi/dt. \tag{23}$$

We now derive the equation of motion by applying the Hartree Hamiltonian, eq. (14), to the wave function (20). The imaginary part of the resulting equation is equiv-

† The reader may be concerned about violations of the Pauli principle in these wave functions. In fact there are none, because all the wave functions are changed the same way. Algebraic details showing the cancellation of Pauli-violating terms are given in ref. [11]).

alent to eq. (23). The real part of the Hamiltonian equation yields

$$m\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2}\psi_i^{(0)} = \left(-\frac{\nabla_i^2}{2m} + V[\rho_0] - \varepsilon_i\right)(\nabla\Phi\cdot\nabla\psi_i^{(0)} + \tfrac{1}{2}\nabla^2\Phi\psi_i^{(0)}) + \frac{\delta V}{\delta\rho}\delta\rho\psi_i^{(0)}. \tag{24}$$

The last term arose from the time-dependent part of the Hartree-Fock potential, $V[\rho]$. Obviously, these equations cannot all be solved simultaneously for one function Φ. The best we can do is to solve them in some average sense, by adding together the equations for the different orbits, suitably weighted.

We shall also assume that Φ has a harmonic time dependence,

$$\Phi(r, t) = \Phi(r)\sin\omega t.$$

This assumption, together with the small amplitude assumption, reduces the general time-dependent Hartree-Fock theory to RPA [8]). With the further assumption that all the particles are governed by the same velocity potential, the theory reduces further to its hydrodynamic limit. This encompasses only the spin-independent, isoscalar excitations of the nucleus. To include spin or isospin degrees of freedom, we would need to consider independent velocity fields for particles of different spin and isospin projection.

First let us simplify the appearance of eq. (24) somewhat with the notation

$$\boldsymbol{u} = \nabla\Phi. \tag{25}$$

Since $\dot{\Phi}$ is a velocity field, $\boldsymbol{u}$ may be interpreted as a field of displacement vectors. Making use of the ground state condition,

$$(-\nabla^2/2m + V[\rho_0] - \varepsilon_i)\psi_i^{(0)} = 0, \tag{26}$$

the first term on the right-hand side of eq. (24) becomes

$$\left(-\frac{\nabla^2}{2m} + V[\rho_0] - \varepsilon_i\right)(\boldsymbol{u}\cdot\nabla\psi_i^{(0)} + \tfrac{1}{2}(\nabla\cdot\boldsymbol{u})\psi_i^{(0)}) = -\frac{\nabla^2\boldsymbol{u}}{m}\cdot\nabla\psi_i^{(0)}$$

$$-\sum_{\alpha\beta}\frac{(\nabla_\alpha u_\beta)\nabla_\alpha\nabla_\beta\psi_i^{(0)}}{m} - \frac{\nabla^2(\nabla\cdot\boldsymbol{u})}{4m}\psi_i^{(0)} - \boldsymbol{u}\cdot(\nabla V)\psi_i^{(0)}. \tag{27}$$

To reduce eqs. (24) to a single equation, we multiply (24) by

$$(\boldsymbol{u}\cdot\nabla\psi_i^{(0)} + \tfrac{1}{2}(\nabla\cdot\boldsymbol{u})\psi_i^{(0)})^* \tag{28}$$

and sum over occupied orbits †. We then integrate over the space coordinate. The

† The choice (28) for a weighting factor may seem quite arbitrary. We use it because the result comes out simply, and because the result is equivalent to using Thouless' RPA variational principle[10]).

result for the left-hand side of the equation is

$$-m\omega^2 \int \sum_i (\boldsymbol{u}\cdot\nabla\psi_i^{(0)} + \tfrac{1}{2}(\nabla\cdot\boldsymbol{u})\psi_i^{(0)})^* \Phi\psi_i^{(0)} \mathrm{d}^3 r.$$

This is integrated by parts to obtain

$$\tfrac{1}{2}\omega^2 \int m\rho_0 \boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}^3 r. \tag{29}$$

Interpreting $\omega\boldsymbol{u}$ as the velocity vector, this is no more than the kinetic energy, as an integral over the kinetic energy density. The right-hand side of our equation is

$$I[\boldsymbol{u}] = -\frac{1}{2m}\int \mathrm{d}^3 r(\boldsymbol{u}\cdot\nabla\psi_i^{(0)} + \tfrac{1}{2}(\nabla\cdot\boldsymbol{u})\psi_i^{(0)})^*(2(\nabla^2\boldsymbol{u})\cdot\nabla\psi_i^{(0)} + 2\sum_\beta^\alpha (\nabla_\alpha u_\beta)\nabla_\alpha\nabla_\beta\psi_i^{(0)}$$
$$+\tfrac{1}{2}(\nabla^2\nabla\cdot\boldsymbol{u})\psi_i^{(0)} + 2m(\nabla V)\cdot\boldsymbol{u}\psi_i^{(0)} - 2m\frac{\partial V}{\partial\rho}\delta\rho\psi_i^{(0)}), \tag{30}$$

where the derivative operators act only on the function immediately following them.

Before reducing this further, we shall make some general observations. The equation may be solved for ω^2,

$$\omega^2 = \frac{I[\boldsymbol{u}]}{\frac{1}{2}m\int \boldsymbol{u}\cdot\boldsymbol{u}\rho_0\,\mathrm{d}^3 r}. \tag{31}$$

This equation is just Rayleigh's variational principle [9]) for the vibrational energy, if we interpret $I[\boldsymbol{u}]$ as the potential energy associated with a trial displacement $\boldsymbol{u}$. In fact, it can be shown from Thouless' variational principle [10]) that eq. (31) is a quantum mechanical upper bound on the energy of the lowest RPA mode [11]). This may be considered the justification for our reduction procedure. Treating (31) as a variational principle, we can recover a differential equation for $\boldsymbol{u}$ by standard techniques [12]). However, it is more transparent to keep the dynamic equation in integral form.

To simplify the expression for the potential energy $I[\boldsymbol{u}]$, we shall replace the square of the wave functions by the density,

$$\sum_i |\psi_i^{(0)}|^2 = \rho_0,$$
$$\sum_i (\nabla_\alpha\psi_i^{(0)})^*\psi_i^{(0)} = \tfrac{1}{2}\nabla_\alpha\rho. \tag{32}$$

Furthermore, we shall assume an isotropic momentum tensor, so that second and third derivatives on the wave function are evaluated as†

$$\sum_i (\nabla_\alpha\psi_i^{(0)})^*\nabla_\beta\psi_i^{(0)} = \delta_{\alpha\beta}\kappa, \tag{33a}$$

$$\sum_i (\nabla_\alpha\nabla_\beta\psi_i^{(0)*})\nabla_\gamma\psi_i^{(0)} = \tfrac{1}{2}\delta_{\beta\gamma}\nabla_\alpha\kappa + \tfrac{1}{2}\delta_{\alpha\gamma}\nabla_\beta\kappa - \tfrac{1}{2}\nabla_\gamma\kappa\delta_{\alpha\beta}, \tag{33b}$$

$$\sum_i \psi_i^{(0)*}\nabla_\alpha\nabla_\beta\psi_i^{(0)} = \tfrac{1}{2}\nabla_\alpha\nabla_\beta\rho - \delta_{\alpha\beta}\kappa. \tag{33c}$$

† The derivation in ref. [11]) erred by a factor of 2, because only one term was considered in (33b).

Here κ is a function of position; in the Fermi gas model

$$\kappa = \tfrac{1}{5}k_F^2\rho. \tag{34}$$

In the expression for I, we integrate by parts whenever a gradient of the density or the momentum tensor occurs, to make the final expression less dependent on surface quantities. The resulting expression for I contains terms of three types, depending on κ, ρ and the potential, respectively,

$$I = I_\kappa + I_\rho + I_V. \tag{35}$$

The algebraic expressions for these terms are:

$$I_\kappa = \int \mathrm{d}^3r\, \frac{\kappa}{m}\left[\sum_\beta^\alpha \left(\frac{\partial u_\alpha}{\partial x_\beta}\right)^2 - \tfrac{1}{2}\boldsymbol{u}\cdot\nabla^2\boldsymbol{u}\right], \tag{36}$$

$$I_\rho = \int \mathrm{d}^3r\, \frac{\rho}{2m}\{-\tfrac{1}{4}(\nabla\cdot\boldsymbol{u})\nabla^2\nabla\cdot\boldsymbol{u} + \tfrac{1}{2}\sum_{\alpha\beta}\nabla_\alpha\nabla_\beta((\nabla\cdot\boldsymbol{u})\nabla_\alpha u_\beta)$$
$$+\tfrac{1}{4}\sum_{\alpha\beta}\nabla_\alpha(u_\alpha\nabla^2\nabla\cdot\boldsymbol{u}) + \tfrac{1}{2}\sum_\alpha\nabla_\alpha((\nabla^2 u_\alpha)\nabla\cdot\boldsymbol{u})\}, \tag{37}$$

$$I_V = \int \mathrm{d}^3r\left\{\frac{\delta V}{\delta\rho}\,\frac{(\delta\rho)^2}{2} - (\nabla V)\cdot\boldsymbol{u}\,\frac{\delta\rho}{2}\right\}. \tag{38}$$

Let us examine the physical significance of these terms in more detail. The second derivative term may be compared to the potential energy function of the theory of elasticity, which is given by [13])

$$V = \int \mathrm{d}^3r\left[\tfrac{1}{2}\lambda(\nabla\cdot\boldsymbol{u})^2 + \mu\sum_{\alpha\beta}\left(\frac{\partial u_\alpha}{\partial x_\beta}\right)^2\right]. \tag{39}$$

Comparing this with eq. (34), we see that the elasticity parameters due to the single-particle kinetic energy in Hartree theory are

$$\mu_\kappa = \frac{\kappa}{m} \approx \frac{k_F^2\rho}{5m}, \qquad \lambda_\kappa = 0. \tag{40}$$

The quantum expression has an additional term proportional to $u\nabla^2 u$, which has no classical analog.

We can understand the necessity of the fourth derivative term, I_ρ, if we note that the theory is asymptotically correct in the high momentum limit, where

$$\boldsymbol{u} = \nabla\sin\boldsymbol{q}\cdot\boldsymbol{r}, \qquad q \gg k_F. \tag{41}$$

Then the excitation energy must equal the single-particle kinetic energy:

$$\omega^2 = (q^2/2m)^2. \tag{42}$$

It may be easily verified that eq. (31) with I_ρ evaluated from eq. (37) gives just this limit.

Finally, we turn to the potential energy term. This can be simplified in the limit of short range interactions, whence

$$\nabla V = \frac{\delta V}{\delta \rho} \nabla \rho, \tag{43}$$

and we find

$$I_V \approx \int \mathrm{d}^3 r \left(\tfrac{1}{2} \rho^2 (\nabla \cdot \boldsymbol{u})^2 \frac{\delta V}{\delta \rho} + \tfrac{1}{2} \rho (\nabla \cdot \boldsymbol{u}) \boldsymbol{u} \cdot \nabla \rho \frac{\delta V}{\delta \rho} \right). \tag{44}$$

Unfortunately, the surface term cannot be eliminated because of the dependence of $\delta V/\delta \rho$ on position. The first term has the form of an elastic potential, with moduli

$$\lambda_V = \rho^2 \frac{\delta V}{\delta \rho}, \qquad \mu_V = 0. \tag{45}$$

Before applying the theory to a realistic nuclear equation of state, it is instructive to consider the simple example of a single particle in an external oscillator well. The fields $\boldsymbol{u}_0 = \frac{1}{2} \nabla r^2$ and $\boldsymbol{u}_2 = \nabla (r^2 Y^2)$ create eigenstates of the system – namely the $2\ \hbar\omega$ excitations of the ground state.

The ground state wave function and Hamiltonian may be chosen as

$$\psi^{(0)} = N \mathrm{e}^{-\frac{1}{2} \nu r^2}, \tag{46}$$

$$H = \left(- \frac{\nabla^2}{2m} + \frac{\nu^2}{2m} r^2 \right).$$

The oscillator energy is given by

$$\omega_0 = \nu / m. \tag{47}$$

Let us now substitute the monopole field in eq. (31). We find for the kinetic energy

$$\tfrac{1}{2} m \int \rho \, u \cdot u \, \mathrm{d}^3 r = 3m/4\nu. \tag{48}$$

The potential energy is calculated as

$$\kappa = \tfrac{1}{3} \nu^2 r^2 \rho, \qquad I_\kappa = \frac{3}{2} \frac{\nu}{m}, \qquad I_\rho = 0, \tag{49}$$

$$I_V = - \frac{1}{2} \int \frac{\nu^2}{m} \boldsymbol{r} \cdot \boldsymbol{r} \nabla \cdot (\boldsymbol{r} \rho) \mathrm{d}^3 r = \frac{3\nu}{2m}.$$

Substituting in eq. (31),

$$\omega^2 = \frac{(\nu/m)(\frac{3}{2} + \frac{3}{2})}{\frac{3}{4} m/\nu} = (2\omega_0)^2, \tag{50}$$

as it must. Half the collective potential energy was actually kinetic energy of the

single-particle wave function, and half came from the external potential. The quadrupole field $\nabla r^2 Y^2$ gives the same result.

We now see why the elastic modulus λ must not depend on the single-particle kinetic energy. The coefficient of the modulus λ vanishes for the quadrupole field but not for the monopole. If $\lambda_\kappa \neq 0$, the $L = 2$ and $L = 0$ oscillator states would have different energies.

We also gain an understanding of the simple result of the self-consistent oscillator model,

$$\omega = \sqrt{2}\omega_0 . \tag{51}$$

This was derived by Hamamoto [14]) for the quadrupole and by Suzuki [15]) for both the quadrupole and monopole. The calculation includes the single-particle kinetic energies, I_κ, but the self-consistent approximation used in effect assumes that the two terms in eq. (38) for I_V cancel completely. It is then obvious that ω^2 will be half of ω_0^2 for both $L = 0$ and $L = 2$. The result (51) is reasonable for $L = 2$, but not for $L = 0$, since λ_V is non-vanishing. Zamick has treated the residual interaction more carefully in the oscillator model [16]). He shows explicitly that the quadrupole energy is independent of the interaction if it is short range, and that the monopole depends very much on the interaction.

Let us now apply the hydrodynamic theory to the lowest multipole excitations. The classical theory of oscillations of a sphere gives the lowest monopole frequency as [18])

$$\omega^2 = \frac{\lambda+2\mu}{\rho}\,\frac{6.57}{R^2} \qquad (\lambda = \mu),$$

where R is the radius of the sphere. The displacement vector $\boldsymbol{u}$ satisfies

$$\boldsymbol{u} = \nabla j_0(qr),$$

with

$$q^2 = \omega^2 \frac{\lambda+2\mu}{\rho} .$$

Various authors have constructed hydrodynamic [19–22]) and other collective models [37–39]) of the nuclear monopole vibration. Many early treatments postulated that the potential energy of the collective mode is determined by the bulk modulus of compressibility of nuclear matter. For example, Zamick [22]) writes for the monopole

$$\omega^2 = \frac{\hbar^2}{m}\,\frac{K}{\langle r^2\rangle}, \tag{52}$$

where K is the bulk modulus. The bulk modulus K is defined†

$$K = 9V^2 \frac{\partial E/A}{\partial V^2} \equiv 9\frac{k}{\rho} .$$

† Realistic theories [17,24,25]) and the Thomas-Fermi analysis of the liquid drop formula [34]) give K in the range 200–300 MeV.

The relationship with Lamé's elastic constants is

$$k = \lambda + \tfrac{2}{3}\mu.$$

Let us examine this relationship in more detail. In a Hartree-Fock theory the binding energy per particle is

$$E/A = \frac{3}{5}\frac{\hbar^2}{2m}k_F^2 + \mathscr{V}[\rho],$$

where $\mathscr{V}$ is the potential energy per particle. The bulk modulus in such a theory is

$$\frac{k}{\rho} = \frac{\partial}{\partial\rho}\rho^2\frac{\partial E/A}{\partial\rho} = \frac{2}{3}\frac{\hbar^2}{2m}k_F^2 + 2\rho\frac{\partial\mathscr{V}}{\partial\rho} + \rho^2\frac{\partial^2\mathscr{V}}{\partial\rho^2}. \tag{53}$$

To relate this to the present theory, we write the Hartree-Fock potential in terms of the potential energy,

$$V = \partial\mathscr{V}\rho/\partial\rho.$$

Then the "bulk modulus" as given by the elasticity parameters is

$$k_{HF} = \lambda_V + \tfrac{2}{3}\mu_\kappa = \rho\left(2\rho\frac{\partial\mathscr{V}}{\partial\rho} + \rho^2\frac{\partial^2\mathscr{V}}{\partial\rho^2} + \frac{2}{15}\frac{\hbar^2}{m}k_F^2\right).$$

This is nearly identical to the classically derived bulk modulus, eq. (53).

Let us now make an estimate of the monopole frequency, using the trial velocity field $\boldsymbol{u} = \frac{1}{2}\nabla r^2$. With this choice I_ρ vanishes. We find

$$\begin{aligned} T &= \tfrac{1}{2}m\int\rho_0 r^2\,\mathrm{d}^3r = \tfrac{1}{2}m\langle r^2\rangle A, \\ I_\kappa &= \frac{3}{m}\int\kappa\,\mathrm{d}^3r \approx 3\frac{k_F^2}{5m}A, \\ I_V &= \frac{1}{2}\int\left(9\rho^2\frac{\delta V}{\delta\rho} + 3\rho\boldsymbol{r}\cdot\nabla\rho\frac{\delta V}{\delta\rho}\right)\mathrm{d}^3r. \end{aligned} \tag{54}$$

Short range interactions that saturate nuclear matter seem to have a change in sign in $\delta V/\delta\rho$ in the surface region. We will therefore neglect the second term in I_V. Then

$$I_V \approx \tfrac{1}{2}9A\left(\rho\frac{\delta V}{\delta\rho}\right),$$

and the monopole frequency is

$$\omega^2 = \frac{1}{m\langle r^2\rangle}\left(\frac{6}{5m}k_F^2 + 9\rho\frac{\delta V}{\delta\rho}\right) = \frac{9}{m\langle r^2\rangle}\frac{k_{HF}}{\rho}. \tag{55}$$

This is identical to Zamick's formula, eq. (52). Let us now make a numerical estimate.

According to the interactions I and II of the Hartree-Fock theory of Vautherin and Brink [17]), the parameter λ is

$$\frac{\lambda}{\rho} = \rho \frac{\delta V}{\delta \rho} = \begin{cases} 17.0 \text{ MeV}, & \text{Skyrme I} \\ 14.6 \text{ MeV}, & \text{Skyrme II.} \end{cases}$$

We estimate the mean square radius of the nucleus as $r^2 = \frac{3}{5}(1.2)^2 A^{\frac{2}{3}}$. This is substituted in eq. (55), and the monopole frequency is

$$\hbar\omega_{\mathrm{I}} = \frac{107}{A^{\frac{1}{3}}} \text{ MeV}, \qquad \hbar\omega_{\mathrm{II}} = \frac{102}{A^{\frac{1}{3}}} \text{ MeV}. \tag{56}$$

The actual solution of the RPA equations [23]) for these interactions gives an isolated monopole state. In Pb, the frequency is about 10 % higher than given by eq. (56).

We now estimate the giant quadrupole energy. The classical theory is quite complicated, since the shearing field is coupled to the compressive field by the boundary condition at the surface. Again, we shall be content with a numerical estimate based on the trial velocity field,

$$\boldsymbol{u} = \nabla(z^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2). \tag{57}$$

Both I_ρ and I_V vanish, and we are left with the simple formula

$$\omega^2 = \frac{6k_{\mathrm{F}}^2}{5m^2\langle r^2\rangle}. \tag{58}$$

Numerically, this becomes

$$\hbar\omega_{L=2} = 66/A^{\frac{1}{3}},$$

which is in reasonable agreement with the oscillator model, eq. (51). The empirical energy of the giant quadrupole is closer to $60/A^{\frac{1}{3}}$ MeV.

4. Dissipation

In the preceding section we derived an equation for the displacement vector $\boldsymbol{u}$ of nuclear motion, assuming that all the particles move coherently. In fact the particles lose coherence, either from differences in single-particle energies or from collisions, so that eq. (31) can only describe some average frequency associated with the displacement field $\boldsymbol{u}$.

The dissipation of energy into the single-particle degrees of freedom, known as Landau damping, can be treated within the framework of time-dependent Hartree-Fock theory. The damping due to collisions is more difficult and we defer consideration until later.

A formalism general enough to include both the hydrodynamic RPA and damping effects is given by the quantum Boltzmann theory, based on Wigner's representation

of the density† matrix [26])

$$f(r, p, t) = \int \mathrm{d}x' \, \mathrm{e}^{ip \cdot x'} \langle \psi^\dagger(r+\tfrac{1}{2}x')\psi(r-\tfrac{1}{2}x') \rangle, \tag{59}$$

where the ψ are nucleon field operators, and the expectation is in the time-dependent state. This function satisfies a Boltzmann equation [27]),

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \nabla_x f = \text{collision term.} \tag{60}$$

If there is no interaction and the left-hand side vanishes, then $f(x, p, t) = f(x-(p/m)t, p)$ and the particles propagate ballistically. The time-dependent Hartree theory can be obtained by considering an independent-particle limit of the collision terms. The equation then reduces to the Vlasov equation [25]),

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \nabla_x f - \nabla_p f \cdot \nabla_x \int V \, \mathrm{d}^3 p \, f(R, p) = 0. \tag{61}$$

We shall now display hydrodynamic RPA in this formalism. First consider the density matrix in the interior of a nucleus in its ground state. The function $f(r, p, t)$ is roughly a constant within a sphere in p-space, and zero outside,

$$f(r, p, t) \approx \theta(|p| - p_\mathrm{F}) f(r). \tag{62}$$

We now apply the familiar transformation to the wave function

$$\psi'(r) = \mathrm{e}^{-im\dot{\Phi}} \frac{\psi(r+\nabla\Phi)}{\sqrt{1-\nabla^2\Phi}}. \tag{63}$$

The new expectation of the field operators is

$$\langle \psi^+(r+\tfrac{1}{2}x)\psi(r-\tfrac{1}{2}x) \rangle' = \mathrm{e}^{-im\dot{\Phi}(r-\frac{1}{2}x)+im\dot{\Phi}(r+\frac{1}{2}x)} \frac{\langle \psi^+(r+\frac{1}{2}x+\nabla\Phi(r+\frac{1}{2}x))\psi(r-\frac{1}{2}x+\nabla\Phi(r-\frac{1}{2}x)) \rangle}{1-\nabla^2\Phi}. \tag{64}$$

We expand $\dot{\Phi}$ and $\nabla\Phi$ to first order in x, and change the integration variable from x to $x' = x+\nabla\Phi$. The distribution function then becomes

$$f'(r, p, t) = \int \mathrm{d}x' \, \mathrm{e}^{\mathrm{i}(p-p\cdot\nabla\nabla\Phi - m\nabla\dot{\Phi})\cdot x'} \langle \psi^+(r+\tfrac{1}{2}x'+\nabla\Phi(r))\psi(r-\tfrac{1}{2}x'+\nabla\Phi(r)) \rangle \approx \theta(|p-p\cdot\nabla\nabla\Phi - m\nabla\dot{\Phi}| - p_\mathrm{F}) f(r+\nabla\Phi(r)). \tag{65}$$

Next we assume that the change in the distribution function is small, so that we may expand it as follows

$$f' = f_0 + \delta(|p| - p_\mathrm{F})(p \cdot \nabla(\hat{p} \cdot \nabla\Phi) + m\hat{p} \cdot \nabla\dot{\Phi}).$$

† I am indebted to P. Carruthers [28]) and J. Negele for pointing out to me the usefulness of this object.

Inserting this in the Vlasov equation, and keeping only the first order term, we find

$$\delta(|p|-p_F)(\boldsymbol{p}\cdot\nabla(\hat{\boldsymbol{p}}\cdot\nabla\dot{\Phi})+m\hat{\boldsymbol{p}}\cdot\nabla\ddot{\Phi})+\left(\frac{\boldsymbol{p}}{m}\cdot\nabla\right)(-\boldsymbol{p}\cdot\nabla(\hat{\boldsymbol{p}}\cdot\nabla\Phi)+m\hat{\boldsymbol{p}}\cdot\nabla\dot{\Phi})$$

$$=\delta(|p|-p_F)\hat{\boldsymbol{p}}\cdot\nabla\int d^3p\,V\delta(|p|-p_F)(\boldsymbol{p}\cdot\nabla(\hat{\boldsymbol{p}}\cdot\nabla\Phi)+m\hat{\boldsymbol{p}}\cdot\nabla\dot{\Phi}). \tag{66}$$

The terms with $\dot{\Phi}$ cancel out on the left-hand side, and vanish when integrated on the right-hand side. We could now derive the hydrodynamic theory by integrating out the dependence on the direction of the momentum vectors, $\hat{\boldsymbol{p}}$. Instead, let us examine under which conditions the dependence on $\hat{\boldsymbol{p}}$ can be neglected. This will tell us when the mode retains coherence. The form of the equation is

$$m\hat{\boldsymbol{p}}\cdot\nabla\ddot{\Phi}+\frac{\boldsymbol{p}}{m}\cdot\nabla(-\boldsymbol{p}\cdot\nabla)(\hat{\boldsymbol{p}}\cdot\nabla)\Phi=\hat{\boldsymbol{p}}\cdot\nabla U[\Phi]. \tag{67}$$

This can only be satisfied for all $\hat{\boldsymbol{p}}$ if the second term on the left can be neglected. The collective modes of the Landau theory emerge when U is repulsive. Clearly, for large repulsive U, the frequency is high and thus the $\hat{p}^3$ term can be neglected. However, more relevant for nuclear motion, the $\hat{p}^3$ will vanish if Φ is a quadratic function of r. Thus the giant quadrupole and giant monopole should be undamped in this theory. Empirically, the quadrupole is observed as a fairly well defined mode having 50 % of the sum rule in medium to heavy nuclei [35, 36]). Physically, the reason for the stability of the mode is that the distortion of the Fermi sphere is uniform throughout the nucleus. Then the ballistic propagation of particles does not lead to a loss of coherence.

Note that except for these special cases, damping will be severe. The ballistic propagation of the particles damp out any spatially non-uniform distortion of the Fermi sphere, on a time scale of the order $t \sim mL/p_F$, where L is a length characterizing the spatial scale of non-uniformity. This is at variance with the usual notion of viscosity, which implies a damping time of the order of

$$t = \rho L^2/\nu,$$

where ν is the coefficient of viscosity.

In the Hartree-Fock treatment the off-diagonal interaction between the particles has largely been ignored. The incoherent treatment of this gives another damping mechanism. The theory is well developed for infinite systems at finite temperature [29-31]). Using the Boltzmann equation, one finds the expected result that the rate of energy dissipation is proportional to the energy in the collective mode. The dependence on the wavelength and frequency of the mode is given by

$$\frac{dE}{dt}=cE\left(\frac{v_F q}{\omega_q}\right)^2,$$

where v_F is the Fermi velocity. At zero temperature, the damping can be calculated

by the Golden Rule with a microscopic treatment of the states and the interaction,

$$\Gamma = 2\pi\langle \text{collective}\,|V|\text{2p-2h}\rangle^2 \rho_{\text{2p2h}}.$$

Qualitatively it is found that the decay rate of collective oscillations in uncharged systems is [33])

$$\Gamma \approx \omega_q q^2.$$

For the charged Fermi gas, Dubois [32]) has calculated the damping of the plasmon. Here the dependence of q and ω_q is

$$\Gamma \approx \omega_q^2 q^2.$$

Presumably the collective excitations can be treated independently in a state with many quanta excited, so that the rate of energy dissipation is proportional to the number of quanta and therefore the total energy in the collective mode. Unfortunately, if we apply the Boltzmann theory to systems at zero temperature, this simple result is not obtained. Because of the density of states at zero temperature, the damping is proportional to the fourth power of the deformation. Thus even the linearity of the hydrodynamic theory is brought into question. Clearly more work remains to be done in this area.

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