Eur. Phys. J. A (2019) **55**: 248
DOI 10.1140/epja/i2019-12764-8

THE EUROPEAN PHYSICAL JOURNAL A

Regular Article – Theoretical Physics

Monopole moments and the β-vibration in deformed nuclei*

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Received: 15 April 2019
Published online: 4 December 2019
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Communicated by N. Alamanos

Abstract. I revisit the theory of the β-vibration in deformed nuclei focusing on the transition matrix elements to the ground band. The relation between monopole and quadrupole matrix elements is derived from an incompressibility assumption and is validated by self-consistent mean-field theory with several energy density functionals. With estimates of the deformation-dependent mean-field potential energy function, it is found that the excited 0^+ band in ^{156}Gd comes fairly close to meeting the criteria to be classified as a β-vibration.

1 Introduction

P.F. Bortignon had a great interest in the interplay between collective and single-particle aspects of nuclear structure and their relation to the nuclear Hamiltonian. Among the many topics was the giant monopole vibration and its dependence on nuclear compressibility [1–3]. In this note, I want to revisit the topic focusing on the low-lying monopole transitions in the structure of deformed nuclei. These transitions are prominent in the search for low-frequency β-vibrations, reviewed two decades ago by Garrett [4].

The existence of deformed nuclei has been known almost since the beginnings of nuclear physics [5], but the theoretical understanding of nuclear shapes and sizes has greatly evolved over time. A common assumption is the idea that the nucleus is incompressible when undergoing shape changes [6,7]. A good way to implement this idea is to treat quadrupole shape changes as taking place by an irrotational and divergence-free flow. For axial deformations, a density distribution that starts out as spherically symmetric

$$\rho(\vec{r}) = \rho(|r|) \tag{1}$$

is transformed to an axially deformed shape $\rho(\vec{r}\,')$ by the scaling transformation where

$$\vec{r}\,' = (x', y', z') = \left(xe^{\varepsilon/2}, ye^{\varepsilon/2}, ze^{-\varepsilon}\right). \tag{2}$$

Here ε is the deformation parameter. The mean square radius

$$\langle r^2\rangle_\varepsilon = \frac{1}{A}\int \mathrm{d}^3r\, r^2\rho(\vec{r}\,') \tag{3}$$

of the deformed nucleus ^{A}X is easily expressed in terms of mean square radius $R_0^2 = \langle r^2\rangle_0$ of the spherical nucleus. Namely,

$$\langle r^2\rangle_\varepsilon = \left(e^{2\varepsilon} + 2e^{-\varepsilon}\right) R_0^2/3. \tag{4}$$

The quadrupole moment

$$Q_2 = \int \mathrm{d}^3r\, \rho(\vec{r}\,')(z^2 - (x^2+y^2)/2) \tag{5}$$

can be expressed similarly as

$$Q_2 = \left(e^{2\varepsilon} - e^{-\varepsilon}\right) AR_0^2/3. \tag{6}$$

Combining eqs. (4) and (6), there is a parameter-free analytic relationship F between $q = Q_2/AR_0^2$ and $\langle r^2\rangle_\varepsilon/R_0^2$,

$$\langle r^2\rangle_\varepsilon/R_0^2 = F(q). \tag{7}$$

The formula for F is derived in the appendix.

2 $\langle r^2\rangle$ from self-consistent mean field theory

Our first task is to see how well eq. (7) is satisfied in microscopic theory of nuclear structure. The method of choice for properties of heavy nuclei is self-consistent mean field theory based on nuclear energy density functionals. Of these, I have tested three different functionals: the Gogny D1S [8], the BCPM functional [9], and the Skyrme SLy4 [10]. The functionals are similar in that they all require a density-dependent contact interaction to achieve

* Contribution to the Topical Issue "Giant, Pygmy, Pairing Resonances and Related Topics" edited by Nicolas Alamanos, Ricardo A. Broglia, Enrico Vigezzi.

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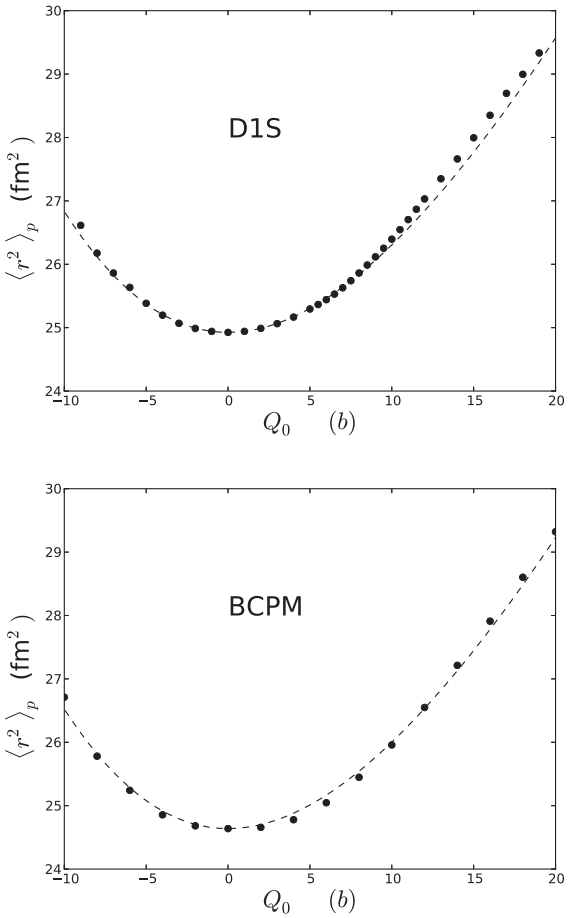

Fig. 1. Mean square charge radius in ^{156}Gd as a function of mass quadrupole moment. The black circles show the radii calculated by self-consistent mean field theory. The dashed line shows the relationship according to the incompressibility formula eq. (A.3).

nuclear saturation. However, the details of the contact interaction are rather different. In the Gogny D1S and the Skyrme SLy4 the form of the density dependence is a power law ρ^α with $\alpha = 1/3$ for D1S and 1/6 for SLy4. In the BCPM the density dependence is fitted to the equation of state of nuclear matter. Fluctuations in shape are accessed by the Generator Coordinate Method (GCM), which involves minimizing the energy of the system while constraining some shape variable(s).

The calculations were carried out for the nucleus ^{156}Gd, the subject of a recent experiment to measure transition properties of a possible β vibration band [11]. The energy minimizations were performed in the Hartree-Fock-Bogoliubov approximation constraining the mass quadrupole moment, eq. (5). The code `HFBaxial` [12] was used for the D1S and the BCPM functionals, while the SLy4 functional was treated by the code `ev8` [13].

Comparison of the mean square charge radii with the incompressibility formula are shown in fig. 1. For all three energy functionals the mean square moments are close to that predicted by incompressible flow. I conclude that incompressible flow is a justified approximation for treating fluctuations in quadrupolar deformations.

3 The β-vibration and its matrix elements

A simple way to define the shape variable β is by scaled matrix elements of the quadrupole operator. A common definition for the scaling which I follow is

$$Q_2 = A\frac{3R^2}{2(5\pi)^{1/2}}\beta. \tag{8}$$

Here $R = r_0 A^{1/3}$ with $r_0 = 1.2$ fm. In the collective model, the coordinate becomes an operator, but its matrix elements are calculated through the quadrupole operator $\hat{Q}_2 = \hat{z}^2 - (\hat{x}^2 + \hat{y}^2)/2$.

The incompressibility assumption allows one to calculate the monopole matrix element between the ground band and a β-vibrational band if the deformation of the bands and the quadrupole transition matrix element are known. The fluctuations about the equilibrium shape turn out to be small, so one can expand the operators to first order as

$$\begin{aligned} \hat{r}^2 &= r^2(0) + \delta\hat{r}^2, \\ \hat{\beta} &= \beta_0 + \delta\hat{\beta}, \end{aligned} \tag{9}$$

where $r^2(0)$ and β_0 are the expectation values in the ground state band, assumed to be the same in the excited band.

By orthogonality, the matrix elements of the operators between the bands depend only on their fluctuating parts. Then the matrix elements between the ground state band $|g\rangle$ and the β-vibration band $|\beta\rangle$ can be expressed

$$\langle g|\hat{r}^2|\beta\rangle \approx \frac{1}{A}\frac{\mathrm{d}Q_2}{\mathrm{d}\beta}F'(Q_2(0)/AR^2)\langle g|\hat{\beta}|\beta\rangle, \tag{10}$$

where F' is the derivative of the incompressibility function. Its value is $F' \approx 0.22$ for deformations around $\beta = 0.34$. I will assume that the charge moment Q_2^p is given by the same β as the mass moment according to the formula

$$Q_2^p = Z\frac{3R^2}{2(5\pi)^{1/2}}\beta. \tag{11}$$

With these definitions the charge monopole matrix element can be estimated as

$$\langle g|\hat{r}_p^2|\beta\rangle \approx \frac{3}{4(5\pi)^{1/2}}ZR^2\beta\langle g|\hat{\beta}|\beta\rangle. \tag{12}$$

Without any further theory, the Ansatz of a β-vibration implies a relation between the quadrupole and monopole transition matrix element between the bands [4, 14]. It is

$$\langle g|\hat{r}_p^2|\beta\rangle = F'(Q(0)/AR^2)\langle g|\hat{Q}_2^p|\beta\rangle. \tag{13}$$

Both matrix elements have been measured experimentally. Reference [11] established that many $E(2)$ transitions between the 0_2^+ band and the ground band are consistent with a single band-to-band matrix element of $\langle g|\hat{Q}_2^p|\beta\rangle = 30\,\mathrm{fm}^2$. On the other side, the tabulation [15] gives a range $\langle g|\hat{r}_p^2|\beta\rangle = 6$–$10\,\mathrm{fm}^2$. Plugging this into eq. (13) with $F' = 0.22$ we find $\langle g|\hat{r}_p^2|\beta\rangle = 6.6\,\mathrm{fm}^2$. Thus, the measurements satisfy the consistency check.

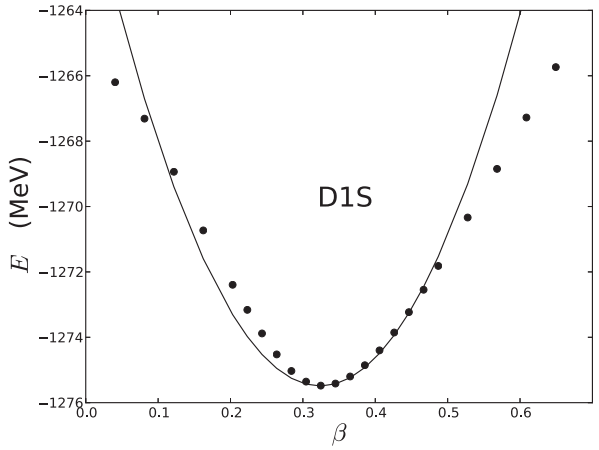

Fig. 2. ^{156}Gd energy as a function of β in self-consistent mean field theory, shown as filled circles. The line is a fit to a quadratic dependence on β, eq. (14). The k parameter of the fit is $k = 298$ MeV, 225 MeV, and 337 MeV for the D1S, BCPM, and SLy4 functionals, respectively.

4 Where is the β vibration?

More theoretical input is needed to calculate the actual values of the two matrix elements. The ingredients are theoretical potential energy function $V(\beta)$ together with a corresponding collective inertial function. Both can be calculated in the self-consistent mean-field approximation by the GCM. In practice the GCM is quite reliable for the potential energy function, reproducing for example the deformations of lanthanide and actinide nuclides and fission barriers of heavy nuclei. On the other hand, the inertial function is less well understood. It depends strongly on the pairing field, which is fairly weak and fluctuating.

So for this contribution I will take $V(\beta)$ from GCM theory, but leave it to Nature to tell us the inertial parameter. If the experimental excitation energy of the vibration is known, one can deduce an average inertial parameter and determine the corresponding wave function. The matrix elements of the relevant operators can then be easily extracted.

Figure 2 shows the GCM energies for the three functionals. These curves are to be considered as slices through a collective potential energy surface (PES). The accuracy of the theory hardly warrants treating the PES in full detail. To keep the presentation on an analytic footing as far as possible, I shall fit the PES around the minimum by a harmonic potential. This is parameterized as

$$V(\beta) = \frac{1}{2}k(\beta - \beta_0)^2 + V(0). \tag{14}$$

The fitted k parameter is given in the figure caption.

As mentioned above, the inertial function $I(\beta)$ in the collective model is the most problematic aspect of the theory; I do not attempt to calculate it here. To make a simple estimate, assume it to be independent of deformation. Then the excitation energy $\hbar\omega$ of the β-vibration is given by the harmonic oscillator formula

$$\hbar\omega = \sqrt{\frac{k}{I}}\,. \tag{15}$$

The corresponding matrix element of the β operator between the ground state and the vibrational state is given by

$$\langle g|\hat{\beta}|\beta\rangle = \sqrt{\frac{\hbar\omega}{2k}}\,. \tag{16}$$

This formula tells us what quadrupole matrix element we should expect, given the excitation energy of the vibration.

We can now see how well the first excited 0^+ state in ^{156}Gd satisfies the conditions for a β-vibration. Its excitation energy is 1.049 MeV. With the harmonic oscillator potential energy parameter in the range $k = 225$–337 MeV, eq. (16) gives $\langle g|\hat{\beta}|\beta\rangle \approx 0.039$–$0.047$. Converting that to the quadrupole matrix element with the operator form of eq. (11) gives $\langle g|\hat{Q}_2|\beta\rangle = Q_2 = 39$–$47\,\mathrm{fm}^2$, to be compared with $30\,\mathrm{fm}^2$ extracted from the measured interband transitions [11]. Thus, the predicted β-vibration matrix element is only slightly higher than experiment. A similar agreement would be obtained for the monopole matrix elements, since we have confirmed that the two are consistent with each other. The predicted range from eq. (12) is $\langle g|\hat{r}_p^2|\beta\rangle = 10$–$12\,\mathrm{fm}^2$.

5 Conclusion

While the transitions between bands seem consistent with the interpretation of the 0_2^+ band as a β-vibration, a more complete theory is still needed to draw any firm conclusions. In particular, the frequency of the vibration should be predicted by theory rather than taken from experiment. This can be achieved in several ways. A comprehensive study in the framework of GCM was carried out by mapping the GCM Hamiltonian onto the five-parameter collective space of the Bohr Hamiltonian [16]. That work obtained an excitation energy of the ^{156}Gd β-vibration of 1.27 MeV, not far from the observed energy. Another GCM approach that might be implemented is the Hill-Wheeler method applied directly to a small set of intrinsic-frame wave functions.

Another possibility that is likely to be the case in many deformed nuclei, is that the first excited 0^+ band is a two-quasiparticle excitation. In this sense, its collective analog might bear some resemblance to a pairing vibration [17]. Multi-quasiparticle excitations of pairing condensates appear naturally in the framework of ref. [18], and it would be interesting to see where such states appear in ^{156}Gd.

It should be mentioned that ^{156}Gd is an unusual example. The systematics of the monopole matrix elements between ground and the 0_2^+ bands are difficult to understand theoretically. Typically, the calculated matrix elements are much larger than experiment (see [16], p. 20).

The author thanks L. Robledo for providing the code HFBaxial used for the calculations with the D1S and BCPM energy func-

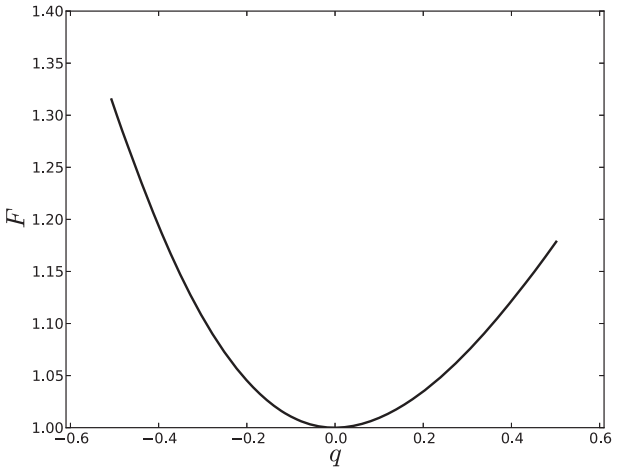

Fig. 3. The incompressibility function $F(q)$ in the range $-1/2 < q < 1/2$.

tionals. The author also thanks A. Aprahamian for discussions motivating the present study.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Author's comment: All data generated during this study are contained in the article.]

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Appendix A.

The incompressible monopole formula can be easily derived from eqs. (6) and (4). First, change the deformation variable from ε to $w = e^{\varepsilon}$. Then eq. (6) is a cubic equation for w in terms of $q = Q_2/AR^2$. The physical solution is

$$w = \frac{q}{D} + D, \tag{A.1}$$

where

$$D = \left(\frac{1 + \sqrt{1 - 4q^3}}{2}\right)^{1/3}. \tag{A.2}$$

Substituting eq. (A.2) in eq. (4) yields

$$F(q) = \frac{1}{3}\left(\frac{q}{D} + D\right)^2 + \frac{2}{3}\left(\frac{q}{D} + D\right)^{-1}. \tag{A.3}$$

The function $F(q)$ is shown in fig. 3.

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