Transmission coefficients in compound-nucleus reaction theory

Y. Alhassid,[1,*] G. F. Bertsch,[2,†] P. Fanto,[1,‡] and T. Kawano[3,§]

[1]*Center for Theoretical Physics, Sloane Physics Laboratory, Yale University, New Haven, Connecticut 06520, USA*

[2]*Department of Physics and Institute for Nuclear Theory, Box 351560, University of Washington, Seattle, Washington 98195, USA*

[3]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

(Received 16 January 2019; published 27 February 2019)

A recent article [D. A. Brown, G. P. A. Nobre, and M. W. Herman, Phys. Rev. C **98**, 024616 (2018)] proposed a modification of the cross section formula used in practical calculations of compound nucleus reactions. We discuss the main concepts and approximations of statistical reaction theory and conclude that the standard practical implementations of the cross section formula remain the preferred choice.

DOI: 10.1103/PhysRevC.99.024621

I. INTRODUCTION

Although statistical reaction theory has played a prominent role in nuclear physics since its beginnings in the 1930s, it has only come to maturity in the 1980s with the development and analytic solution [1,2] of the statistical model based on the Gaussian orthogonal ensemble (GOE) description of the underlying compound nucleus Hamiltonian. We shall refer to this model as the GOE theory of compound nucleus reactions. The analytic solution for the average compound-nucleus reaction cross section in the GOE theory requires the evaluation of a complicated three-dimensional integral that depends on the parameters of the model, and in practice simplified formulas are often implemented. The most common such simplified treatment is the Hauser-Feshbach plus width-fluctuation-correction formula, discussed in detail below.

II. GOE THEORY AND CALCULATION

Reference [3] proposed a modification to this formula but did not test its performance with respect to the underlying GOE theory. In this work, we benchmark both the standard and modified formulas against the GOE theory. We find that the standard formula is in good agreement with the GOE theory, while the modified formula deviates from the GOE theory at larger values of the coupling parameters.

The average compound-nucleus reaction cross section $\bar{\sigma}_{c,c'}$ to scatter from channel c to channel c' can be expressed in terms of the fluctuating part of the S matrix,

$$\bar{\sigma}_{c,c'} = \sigma_0 \overline{|\delta_{c,c'} - S_{c,c'}|^2}, \tag{1}$$

where $\sigma_0 = \pi g_c / k_c^2$ is the maximum inelastic cross section for incoming relative momentum k_c in a two-body entrance channel c, and g_c is a statistical spin factor in this channel. We assume N internal states described by a Hermitian Hamiltonian H coupled to Λ open channels. The coupling constants $W_{\mu,c}$ between the internal states μ ($1 \leqslant \mu \leqslant N$) and channels c ($1 \leqslant c \leqslant \Lambda$) form an $N \times \Lambda$ real matrix W. The unitary S matrix is expressed in terms of a Hermitian K matrix

$$S = \frac{1 - iK}{1 + iK}, \tag{2}$$

where the K matrix is given by

$$K = \pi W^T \frac{1}{E - H} W. \tag{3}$$

In the GOE theory, H is chosen to be a random matrix that belongs to the GOE and the average over energy is replaced by an average over the ensemble. Expressed in the basis of eigenstates μ of H, the coupling amplitudes $W_{\mu c}$ are characterized by channel-dependent Gaussian distributions in the limit of large N. We define the partial width amplitude of an eigenstate μ to decay into a channel c by

$$\gamma_{\mu c} = \sqrt{2\pi} W_{\mu c}. \tag{4}$$

A dimensionless coupling parameter x_c in channel c is defined by

$$x_c = \frac{\pi}{2D} \langle \gamma_c^2 \rangle, \tag{5}$$

where D is the mean level spacing in the middle of the GOE spectrum, and $\langle \gamma_c^2 \rangle$ is the mean square partial width amplitude in channel c:

$$\langle \gamma_c^2 \rangle = \frac{1}{N} \sum_{\mu} \gamma_{\mu c}^2. \tag{6}$$

The ensemble-average S-matrix elements can be expressed in terms of x_c (see Eq. (65) in Sec. V of Ref. [2]):

$$\langle S_{c,c'} \rangle = \delta_{c,c'} \frac{1 - x_c}{1 + x_c}. \tag{7}$$

*yoram.alhassid@yale.edu
†bertsch@uw.edu
‡paul.fanto@yale.edu
§kawano@lanl.gov

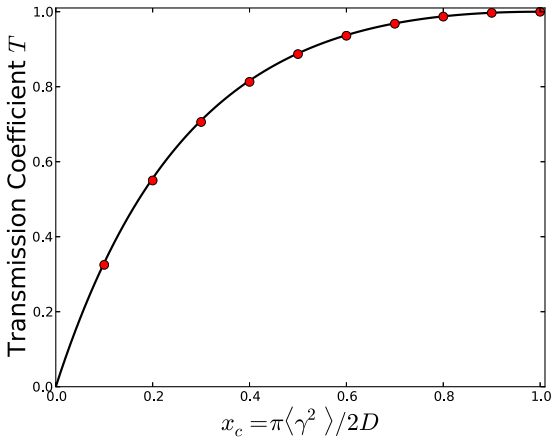

FIG. 1. Transmission coefficient T_c for an incoming channel c in the GOE model with six equivalent reaction channels as a function of x_c. Solid line: Eq. (9). Solid circles: results obtained by sampling H from the GOE.

The physical parameters are the transmission coefficients defined by

$$T_c = 1 - |\langle S_{c,c}\rangle|^2 . \tag{8}$$

We note that in the presence of direct reactions, the average S matrix is not diagonal and the transmission coefficient in channel c is usually approximated by $T_c = 1 - \sum_{c'} |\langle S_{c,c'}\rangle|^2$. A rigorous method to treat the effect of direct reactions is to use the Engelbrecht Weidenmüller transform [4].

Using Eq. (7), we have

$$T_c = \frac{4x_c}{(1+x_c)^2} . \tag{9}$$

Relations (7) and (9) hold for any values of the coupling parameters x_c. Thus the GOE theory is highly constrained; the complete structure of the average S-matrix elements at a fixed energy is determined by the the transmission coefficients T_c. Given the set of T_c, the average compound-nucleus reaction cross sections can be expressed in closed form by the three-dimensional integral derived in Ref. [1]. Alternatively, these cross sections can be calculated numerically by sampling the Hamiltonian matrix H from the GOE and using Eqs. (3), (2), and (1) as was done in Refs. [5,6]. This latter approach is demonstrated in Fig. 1, which shows the incoming channel transmission coefficient T_c for a reaction with six equivalent channels as a function of x_c. The numerical results obtained by sampling H from the GOE (solid circles) are in excellent agreement with the analytic expression (9) (solid line).

However, the main practical codes implementing statistical reaction theory follow the historical development of the theory and compute compound-nucleus reaction cross sections in two steps [7]. The first step is to evaluate the Hauser-Feshbach (HF) formula for the average cross sections or branching ratios, assuming that the relative decay probability in each channel is proportional to T_c,

$$\bar{\sigma}_{ab}^{\rm HF} = \sigma_0 \frac{T_a T_b}{\sum_c T_c} . \tag{10}$$

The average partial decay widths $\langle\Gamma_c\rangle$ may also be used as the input parameters (instead of T_c), but as discussed below, this is a more indirect approach. In the second step in constructing the practical formula, the HF cross section is corrected by a factor $\mathcal{W}_{ab}$, known as the width-fluctuation correction (WFC),

$$\bar{\sigma}_{ab} = \bar{\sigma}_{ab}^{\rm HF} \mathcal{W}_{ab} . \tag{11}$$

Since the complete GOE theory depends only on the parameters x_c (or alternatively T_c), the WFC factor $\mathcal{W}_{ab}$ can be calculated in terms of these parameters (e.g., using the triple integral formula of Ref. [1]). A particularly accurate approximation to $\mathcal{W}_{ab}$ is Moldauer's formula [8] with a particular parametrization of the number of degrees of freedom ν_c in channel c in terms of the transmission coefficients (see, e.g., Ref. [5]).

In Ref. [3], the average partial widths of the resonances $\langle\Gamma_c\rangle$ were taken as the parameters of the statistical theory. In principle, these partial widths can be calculated from the pole expansion of the S matrix, and the total width can be determined by the imaginary part of the S-matrix poles E_μ in the complex energy plane, $\Gamma_\mu = -{\rm Im}\, E_\mu/2$. In the weak-coupling limit where x_c are small in all channels, the relation between $\langle\Gamma_c\rangle$ and x_c is

$$\frac{\langle\Gamma_c\rangle}{D} \approx \frac{2}{\pi} x_c . \tag{12}$$

As discussed in Ref. [3], average partial widths can be measured in the unresolved resonance region where Eq. (12) no longer holds. It would be useful to relate the average partial widths to the transmission coefficients in order to test optical model parametrizations of the latter. This will be addressed in the framework of the GOE theory in future work. Here, we use the parameters x_c defined by Eqs. (4) and (6), or alternatively T_c, as the independent parameters of the statistical theory.

Reference [3] proposed a modification of the HF+WFC formula (11) by replacing T_c in Moldauer's formula by a certain function f_c of T_c. The proposed formula has the structure

$$\bar{\sigma}_{ab}^{\rm mod} = \sigma_0 \frac{T_a f_b}{\sum_c f_c} \mathcal{W}_{ab}(\vec{f}) . \tag{13}$$

Here, $\vec{f} = (f_1, \ldots, f_\Lambda)$ and

$$\mathcal{W}_{ab}(\vec{f}) = \left(1 + \delta_{ab}\frac{2}{\nu_a}\right) \times \int_0^\infty dz \prod_c \left(1 + \frac{2 f_c z}{\nu_c \sum_{c'} f_{c'}}\right)^{-\delta_{ac}-\delta_{bc}-\nu_c/2} , \tag{14}$$

where ν_c is the number of degrees of freedom in channel c. In particular, the authors of Ref. [3] find that the choice

$$f_c = \frac{T_c}{\sqrt{1-T_c}} \tag{15}$$

permits a better agreement with experiment in one of the reactions they studied.

To determine whether this modified formula provides better fits to cross sections of the statistical GOE theory when the coupling parameters x_c vary between the weak coupling limit

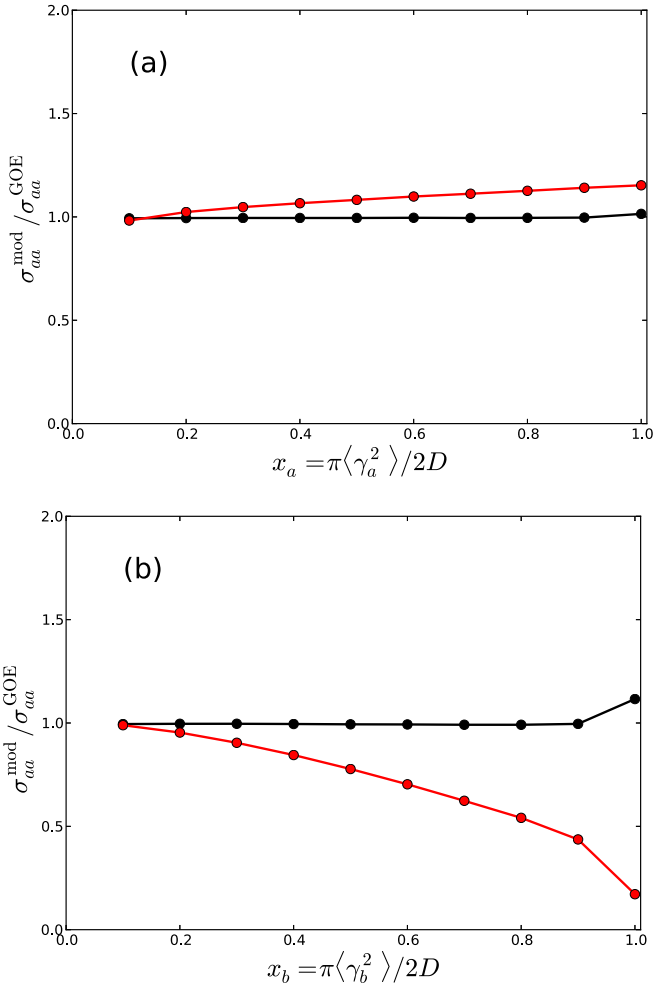

FIG. 2. Ratio of entrance channel cross section calculated from Eqs. (13) and (14) to the GOE statistical theory cross section for the two-channel example described in the text. Black circles: $f_c = T_c$; Red circles: $f_c = T_c/\sqrt{1-T_c}$. (a) and (b) show the ratios as a function of x_a and x_b, respectively.

$x_c \ll 1$ and $x_c = 1$, we consider a two-channel compound nucleus reaction with similar parameters to Example C of Ref. [5]. In our calculation we use the same definition of the WFC as in Refs. [3,5].

We first vary the transmission coefficient T_a in one channel from zero to one, setting $T_b = 0.26$ in the second channel. The results are shown in Fig. 2(a), where the ratio of the cross section computed using f_c in Eq. (15) to the cross section calculated in the GOE theory is compared with a similar ratio but using the standard formula for the WFC in which $f_c = T_c$. We observe that this ratio is close to 1 for the standard choice of variables $f_c = T_c$ (solid black circles). This is not surprising since the particular parametrization of the WFC was obtained by fitting such ratios. If we use f_c as in Eq. (15), we obtain the higher cross section ratios shown in solid red circles, indicating deviations from the GOE theory as x_a increases from its value at $x_a = 0.1$.

Next, we make a similar comparison by varying T_b while setting $T_a = 0.39$. The results are shown in Fig. 2(b). Again, the standard choice $f_c = T_c$ gives very good agreement with the GOE theory, while the choice in Eq. (15) leads a reduction in the ratio at higher values of x_b. Thus choosing a different parametrization for f_c does not improve the fits without making a compensating change in the parametrization of the WFC factor $\mathcal{W}$.

III. CONCLUDING REMARKS

We close with some remarks on the super-radiance phenomenon [9]. In statistical reaction theory, super-radiance refers to the development of broad resonances with simple structure for $x_c > 1$. However, these resonances do not affect the average cross sections of the statistical theory [2]. From Eq. (9), we see that the transmission coefficients T_c (9) are invariant, the transformation $x_c \to 1/x_c$. Since the analytic solution of the GOE theory depends only on T_c, changing from $x_c < 1$ to $1/x_c$ at which super-radiance occurs does not change the GOE theory cross section.

In conclusion, we advocate using the statistical GOE theory as far as possible in analyzing compound-nucleus reaction cross sections.

ACKNOWLEDGMENTS

We thank D. A. Brown for his useful comments on the manuscript. The work of Y.A. and P.F. was supported in part by the US DOE Grant No. DE-FG02-91ER40608 and by the US DOE NNSA Stewardship Science Graduate Fellowship under cooperative agreement No. NA-0003864. T.K. carried out this work under the auspices of the National Nuclear Security Administration of the US Department of Energy at Los Alamos National Laboratory under Contract No. 89233218CNA000001.

[1] J. Verbaarschot, H. A. Weidenmüller, and M. Zirnbauer, Phys. Rep. **129**, 367 (1985).
[2] G. E. Mitchell, A. Richter, and H. A. Weidenmüller, Rev. Mod. Phys. **82**, 2845 (2010).
[3] D. A. Brown, G. P. A. Nobre, and M. W. Herman, Phys. Rev. C **98**, 024616 (2018).
[4] C. A. Engelbrecht and H. A. Weidenmüller, Phys. Rev. C **8**, 859 (1973).
[5] T. Kawano and P. Talou, Nucl. Data Sheets **118**, 183 (2014).
[6] T. Kawano, P. Talou, and H. A. Weidenmüller, Phys. Rev. C **92**, 044617 (2015).
[7] P. A. Moldauer, Phys. Rev. C **11**, 426 (1975).
[8] P. A. Moldauer, Nucl. Phys. A **344**, 185 (1980).
[9] N. Auerbach and V. Zelevinsky, Rep. Prog. Phys. **74**, 106301 (2011).