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Capillary waves in a quantum liquid

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Capillary waves in a superfluid are shown to propagate with the classical form of the dispersion, $\omega^2 \sim k^3$, but the coefficient in the model is not given by the surface tension.

The Bose quantum liquid behaves like a classical irrotational fluid in many respects. It is of interest to see whether the surface modes of the fluid are the same as the classical ones, since it is now possible to study surface waves of rather short wavelength.[1] I will derive here the dispersion of capillary waves in a microscopic model. Using a variational principle, we find that the dispersion has the same functional form as in the classical liquid,[2]

$$\omega^2 = Ak^3 . \tag{1}$$

We do not get the same coefficient A as appears in the classical formula,

$$\omega^2 = (\sigma/\rho)k^3 , \tag{2}$$

where σ is the surface tension of the fluid and ρ is the density. However, since we only have a variational energy, we cannot assert that the capillary-wave propagation in a superfluid is nonclassical.

This derivation of quantum-mechanical capillary waves follows the method Pitaevskii used[3] in his calculation of the dispersion of waves on a vortex line. However, his dynamical equations must be generalized somewhat in order for the system to have a free surface. We consider an energy functional of the system of the form

$$E = \int dv \left(-\psi \frac{\hbar^2}{m} \nabla^2 \psi + V[\rho] \right) , \quad \text{with } \rho = \psi^* \psi . \tag{3}$$

In Eq. (3) V is an operator on the density function; for example, $V[\rho] = \int d^3r_1\, \rho(r_1)\rho(r)\, V(r_{12})$. The usual microscopic theory assumes an interaction energy density

$$V[\rho] = \tfrac{1}{2} V_0\, \rho^2(r) ,$$

with V_0 repulsive. Recently, progress has been made in theoretical nuclear calculations[4] using a more general form of V, such as $V[\rho] \sim \frac{1}{2} V_0 \rho^2 + \frac{1}{3} V_1 \rho^3$. A free surface is possible with suitably chosen V of this form. Minimization of the energy yields a Schrödinger-like equation for the condensate wave function of the ground state, $\psi_0(r)$,

$$-\frac{\hbar^2}{2m} \nabla^2 \psi_0 + \frac{\delta V[\rho]}{\delta\rho} \psi_0 = \lambda \psi_0 . \tag{4}$$

We assume that the excited state is given by a condensate wave function of the form

$$\psi_{\text{ex}} = \psi_0 + e^{i\omega t} f(\vec{\mathbf{x}}) + e^{-i\omega t} \epsilon f^*(\vec{\mathbf{x}}) . \tag{5}$$

Here the function f and the constant $\epsilon < 1$ are as yet undetermined. It is convenient to make a change of variables as follows:

$$\eta = f + \epsilon f^* , \quad \zeta = f - \epsilon f^* . \tag{6}$$

The variable η is proportional to the density fluctuation in the excitation, and the variable ζ is proportional to the fluctuation in the velocity potential. The random-phase-approximation (RPA) equations that Pitaevskii obtains may then be expressed as follows, setting $\hbar^2/m = 1$:

$$\Delta\zeta \equiv -\tfrac{1}{2}\nabla^2\zeta + \frac{\delta V}{\delta\rho}\zeta - \lambda\zeta = \omega\eta , \tag{7a}$$

$$\Sigma\eta \equiv -\tfrac{1}{2}\nabla^2\eta + \frac{\delta V}{\delta\rho}\eta + 2\frac{\delta V^2}{\delta\rho^2}\rho\eta - \lambda\eta = \omega\zeta . \tag{7b}$$

These equations may also be obtained by linearizing the hydrodynamic equation of Gross.[5] He derives the pair of equations

$$\frac{\partial}{\partial t} R = -\nabla R \cdot \nabla S - \tfrac{1}{2} R \nabla^2 S \qquad \text{(continuity)} , \tag{8a}$$

$$-\frac{\partial}{\partial t} S = (\nabla S)^2 + \frac{\delta V}{\delta\rho}(R^2) - \frac{\nabla^2 R}{R} \quad \text{(Bernoulli)} . \tag{8b}$$

To make the connection between Eq. (7) and Eq. (8), we identify R with $\psi_0 + \eta e^{i\omega t}$, S with $\zeta e^{i\omega t}/\psi_0$, and linearize.

An upper bound on the energy of the excitation may be found from the Thouless variational principle,[6] which uses a trial η_t and a trial ζ_t. If linear combinations of η_t and ζ_t are orthogonal to all vectors associated with zero energy, such as uniform translation or rotation, then the variational ω will be given by the upper bound

$$\frac{\langle \eta_t \Sigma \eta_t \rangle + \langle \zeta_t \Delta \zeta_t \rangle}{|\langle \eta_t \zeta_t \rangle|} \geqslant \omega . \tag{9}$$

To use this principle let us assume that the surface moves infinitesimally in the z direction, with an amplitude proportional to e^{ikx}. This is given by a trial η of the form

$$\eta_t = \frac{\partial \psi_0}{\partial z} e^{ikx}. \tag{10}$$

We also assume that the velocity potential of the flow is the same as in the classical fluid, i.e., $\phi = e^{ikx} e^{-kz}$, if the surface is in the x-y direction. Then the trial velocity fluctuation is

$$\zeta_t = \alpha \psi_0 e^{ikx} e^{-kz}. \tag{11}$$

We have included in (11) a parameter α which will be the variational parameter.

Considerable simplification of (9) is possible by using the relations

$$\Delta \psi_0 = 0, \tag{12a}$$

$$\Sigma \frac{d\psi_0}{dz} = 0. \tag{12b}$$

Equation (12a) is nothing more than the Hartree-Fock equation, and Eq. (12b) is easily proved by differentiating (12a). It asserts that the state made by an infinitesimal translation of the system has zero excitation energy. In the limit of small k, we find the following expressions for the integrals in (9):

$$\langle \eta_T \Sigma \eta_T \rangle = k^2 \left[\int \frac{d^3r}{2} \left(\frac{d\psi_0}{dz} \right)^2 - \tfrac{1}{4} \int d^3r_1 d^3r_2\, x_{12}^2 \frac{\delta^2 V}{\delta \rho^2} \frac{d\rho(r_1)}{dz} \frac{d\rho(r_2)}{dz} \right],$$

$$\langle \zeta_T \Delta_T \rangle = \alpha^2 k \int d^3r \frac{d\psi_0}{dz} \psi_0 = \alpha^2 k \rho_\infty, \tag{13}$$

$$\langle \zeta_T \eta_T \rangle = \alpha \int d^3r \frac{d\psi_0}{dz} \psi_0 = \alpha \rho_\infty,$$

where ρ_∞ is the density deep in the interior of the liquid.

Minimizing ω with respect to variations in α then yields the bound

$$\omega^2 \leqslant \frac{k^3}{\rho_\infty} \left[\int \frac{d^3r}{2} \left(\frac{d\psi_0}{dz} \right)^2 - \tfrac{1}{4} \int d^3r_1 d^3r_2\, x_{12}^2 \frac{\delta^2 V}{\delta \rho^2} \frac{d\rho(r_1)}{dz} \frac{d\rho(r_2)}{dz} \right]. \tag{14}$$

The term in brackets, which is the classical expression would be the surface energy, has some similarity to a quantum formula derived by Brout and Nauenberg,[7]

$$\sigma = \int \frac{d^3r}{2} \left(\frac{d\psi_0}{dz} \right)^2 - \tfrac{1}{2} \int d^3r_1 d^3r_2\, \rho^{(2)}(r_1, r_2) \times \left(z_{12} \frac{\partial}{\partial z_{12}} - \frac{x_{12}}{2} \frac{\partial}{\partial x_{12}} - \frac{y_{12}}{2} \frac{\partial}{\partial y_{12}} \right) \frac{\delta V}{\delta \rho^{(2)}}. \tag{15}$$

Feibelman[8] has also derived a formula quite similar to (14) in the context of a Fermi liquid, and shown that it is equivalent to (15) in the limit of a short-range interaction. The two expressions are not equivalent in general, however.

Examining our trial wave function, we find that it carries an oscillation in surface thickness, unlike the classical wave. This is indicated in Fig. 1, where equivalent equidensity points are sketched for a surface wave of our type and of the classical type. The energy of this thickness oscillation is independent of k, and it only requires an admixture of this mode of order k^2 in the potential energy to modify the dispersion of the capillary wave. From the figure it can be seen that the thickness oscillation is out of phase with the surface oscillation. It is therefore impossible to obtain the classical motion in an RPA calculation, since the RPA excitation can always be represented by a real amplitude multiplied by a sinusoidal function of time. While it is most likely that the failure to reproduce the classical motion is just due to the inadequacy of the Hartree-Fock theory, the possibility of a fundamental difference between classical and quantum surface motion is not necessarily excluded. As an example, the Fermi liquid has a completely different dispersion from either the classical or the quantum Bose liquid.

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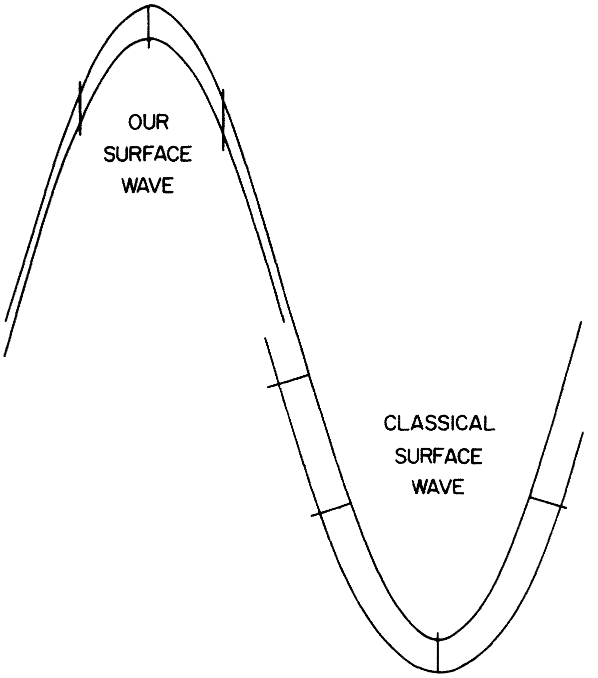

FIG. 1. Exaggerated equidensity profile of a classical surface wave and an RPA quantum surface wave.

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[1] G. B. Benedek, in *In Honor of Philip N. Morse,* edited by H. Feshbach and K. U. Ingard (M.I.T., Cambridge, Mass., 1969).

[2] H. Lamb, *Hydrodynamics* (University Press, Cambridge, Mass., 1930).

[3] L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. 40, 646 (1961) [Sov. Phys.—JETP 13, 451 (1961)].

[4] See, for example, J. W. Negele and D. Vautherin, Phys. Rev. C 5, 1472 (1972).

[5] E. P. Gross, J. Math. Phys. 4, 195 (1964).

[6] D. J. Thouless, Nucl. Phys. 22, 78 (1961).

[7] R. Brout, and M. Nauenberg, Phys. Rev. 112, 1451 (1958).

[8] P. Feibelman, Ann. Phys. (N.Y.) 48, 369 (1968).