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Exact solutions for interacting boson systems under rotation

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Abstract

We study a class of interacting, harmonically trapped boson systems at angular momentum L. The Hamiltonian leaves an L-dimensional subspace invariant, and this permits an explicit solution of several eigenstates and energies for a wide class of two-body interactions.

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The study of vortices in Bose–Einstein condensates of trapped atomic vapours is of considerable experimental [1, 2] and theoretical [3–13] interest. Present experiments and much theoretical work [3–7] focus on the Thomas–Fermi regime of short coherence length [14]. The opposite limit of perturbatively weak contact interactions has not yet been realized experimentally but is of theoretical interest as well [8–13]. The case of attractive interactions was studied by Wilkin *et al* [9]. Mottelson developed a theory for repulsive interactions [10], and Kavoulakis *et al* [12] and Jackson *et al* [13] compared mean field and exact numerical results. Exact diagonalization techniques by us showed that the ground state energy depends linearly on angular momentum and led to an analytical expression for the ground state wavefunction [11]. Recently, the form of this wavefunction and its eigenenergy were confirmed in analytic calculations [15–17]. However, we still lack insight about why certain eigenstates turn out to be simple analytical functions.

We show in this paper that there is a subspace structure that explains these findings. Interestingly, this structure is not limited to the two-body contact interaction considered in previous investigations but present in a rather large class of Hamiltonians. Let us consider a system of N harmonically trapped bosons in two spatial dimensions[1] at angular momentum L. The trap Hamiltonian is

$$\hat{H} = \tfrac{1}{2}\sum_{j=1}^{N}(-\nabla_j^2 + r_j^2 - 2) \tag{1}$$

[1] The three-dimensional problem is essentially two dimensional; see e.g. [13].

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where $r_j^2 = x_j^2 + y_j^2$. Let us use complex single-particle coordinates $z_j = x_j + \mathrm{i}y_j$, $j = 1, \ldots, N$, and fix the angular momentum to be L. The eigenfunctions of the trap Hamiltonian are of the form

$$\psi(z_1, \ldots, z_N) = \phi(z_1, \ldots, z_N) \exp\left(-\tfrac{1}{2} \sum_{j=1}^{N} |z_j|^2 \right) \tag{2}$$

and have degenerate energy $E = L$. In equation (2) $\phi(z_1, \ldots, z_N)$ denotes a homogeneous polynomial of degree L that is totally symmetric under permutation of particle indices. Suitable basis functions for such polynomials are products

$$\phi(z_1, \ldots, z_N) = e_{\lambda_1} e_{\lambda_2} \ldots e_{\lambda_k}$$

where e_λ denotes the elementary symmetric polynomial [18]

$$e_\lambda(z_1, \ldots, z_N) = \sum_{1 \leqslant p_1 < p_2 < \cdots < p_\lambda \leqslant N} z_{p_1} z_{p_2} \ldots z_{p_\lambda}$$

and $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ is a partition of L into at most N integers, e.g. $\sum_{j=1}^{k} \lambda_j = L$ with $N \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 0$. We set $e_0 = 1$. From here on we omit the notation of the ubiquitous exponentials in the wavefunction and restrict ourselves to the regime $0 \leqslant L \leqslant N$. In this regime, N is simply a parameter of the problem, and the dimension of Hilbert space is equal to the number of partitions of L into integers.

The operator for the total angular momentum is

$$\hat{L} = \sum_{j=1}^{N} z_j \partial_j \tag{3}$$

where ∂_j denotes the derivative with respect to z_j and acts only on the polynomial part of the wavefunction. Another important observable is the angular momentum of the centre of mass. Its operator is given by

$$\hat{L}_\mathrm{c} \equiv z_\mathrm{c} D_\mathrm{c} \tag{4}$$

where $z_\mathrm{c} = e_1/N$ denotes the centre of mass and

$$D_\mathrm{c} = \sum_{j=1}^{N} \partial_j. \tag{5}$$

Again, it is understood that derivatives act on the polynomial part of the wavefunction only. L_c has eigenvalues $0, 1, 2, \ldots, L-2, L$ and commutes with the harmonic trap Hamiltonian (1) and the angular momentum (3). We further have the following commutator relations:

$$[\hat{L}, D_\mathrm{c}] = -D_\mathrm{c} \qquad [\hat{L}, z_\mathrm{c}] = z_\mathrm{c} \qquad [\hat{L}_\mathrm{c}, D_\mathrm{c}] = -D_\mathrm{c} \qquad [\hat{L}_\mathrm{c}, z_\mathrm{c}] = z_\mathrm{c}. \tag{6}$$

The introduction of perturbatively weak interactions lifts the degeneracy of the trap Hamiltonian. In what follows we are interested in two-body interactions of the form

$$\begin{aligned} \hat{V} &= g \sum_{m=0}^{L} c_m \hat{A}_m \\ \hat{A}_m &= \sum_{1 \leqslant i < j \leqslant N} (z_i - z_j)^m \, (\partial_i - \partial_j)^m \end{aligned} \tag{7}$$

where the derivatives act only on the polynomial part of the wavefunction. We choose $gN \ll 1$ to be in the perturbative regime. Note that a large class of two-body interactions is of the

form (7). Consider for instance the two-body potential $V(r) = r^{2n}$. Comparison of matrix elements shows that

$$\left\langle \sum_{i<j} (r_i - r_j)^{2n} \right\rangle = \left\langle \sum_{i<j} [(z_i^* - z_j^*)(z_i - z_j)]^n \right\rangle$$
$$= \left\langle \sum_{i<j} (\partial_i - \partial_j)^n (z_i - z_j)^n \right\rangle$$
$$= \left\langle \sum_{m=0}^{n} \binom{n}{m} \frac{n!}{m!} 2^{n-m} \hat{A}_m \right\rangle.$$

Thus, any potential that depends analytically on the squared inter-particle distance r^2 can be written in the form of equation (7). Further examples include zero-range potentials of the form

$$V(r) = 2\pi g[a_0 \delta(r) + a_1 \nabla^2 \delta(r) + a_2 \nabla^4 \delta(r)].$$

For $c_m = (-1/2)^m/m!$ the interaction (7) corresponds to the two-body contact interaction $V = 2\pi g \delta(\vec{r})$. This can be seen by computing and comparing matrix elements of these two different representations of the interaction. It is also instructive to analyse the action of the operator (7) on the wavefunction. Inserting the appropriate coefficients for the contact interaction into the Hamiltonian (7) we obtain a Taylor series. Thus,

$$\sum_{m=0}^{L} \frac{1}{m!} \left(\frac{z_j - z_i}{2} \right)^m (\partial_i - \partial_j)^m \phi(z_1, \ldots, z_i, \ldots, z_j, \ldots, z_N)$$
$$= \sum_{m,n=0}^{L} \frac{(-1)^n}{m!n!} \left(\frac{z_j - z_i}{2} \right)^{m+n} \partial_i^m \partial_j^n \, \phi(z_1, \ldots, z_i, \ldots, z_j, \ldots, z_N)$$
$$= \phi\left(z_1, \ldots, z_i - \frac{z_i - z_j}{2}, \ldots, z_j - \frac{z_j - z_i}{2}, \ldots, z_N \right)$$
$$= \phi\left(z_1, \ldots, \frac{z_i + z_j}{2}, \ldots, \frac{z_j + z_i}{2}, \ldots, z_N \right).$$

In [17] we presented an alternative differential operator with this effect on wavefunctions.

The operators $\hat{A}_m$ commute with $\hat{L}$, $\hat{L}_c$, z_c and D_c. Equation (6) thus implies that the application of z_c to eigenstates with quantum numbers L, L_c and energy gE yields eigenstates with quantum numbers $L+1$, L_c+1, gE. Similarly, the application of D_c to such eigenstates yields eigenstates with quantum numbers $L-1$, L_c-1, gE. These properties are well known for the case of a two-body contact interaction [19, 20] and generalize to the Hamiltonian (7). Note that the operators A_0 and A_1 are simply given by

$$\begin{aligned} \hat{A}_0 &= \tfrac{1}{2} N(N-1) \\ \hat{A}_1 &= N(\hat{L} - \hat{L}_c). \end{aligned} \tag{8}$$

Applying the operators $\hat{A}_m$ to the elementary symmetric polynomials yields

$$\hat{A}_m \, e_1 = 0 \qquad \text{for} \quad m \geqslant 1. \tag{9}$$

Thus, the one-dimensional space $\mathcal{V}_0 = \text{span}\{e_1^L\}$ is an invariant subspace of the Hamiltonian and therefore an eigenstate. Further,

$$\hat{A}_m \, e_\lambda = 0 \qquad \text{for} \quad m \geqslant 3. \tag{10}$$

This immediately shows that the $(L-1)$-dimensional vector space

$$\mathcal{V}_1 = \text{span}\{e_{L-\lambda} \, e_1^\lambda : \lambda = 0, 1, 2, \ldots, L-2\} \tag{11}$$

is annihilated by A_m for $m \geqslant 3$. The space $\mathcal{W}_1 = \mathcal{V}_0 \cup \mathcal{V}_1$ is an invariant subspace of the Hamiltonian (7) since

$$\hat{A}_2\, e_\lambda = 2\lambda N\, e_\lambda - 2(N-\lambda+1)\, e_1\, e_{\lambda-1} \tag{12}$$

and

$$\hat{A}_1\, e_\lambda = N\lambda\, e_\lambda - (N-\lambda+1)\, e_1\, e_{\lambda-1}.$$

Eigenfunctions of the Hamiltonian (7) in the subspace $\mathcal{W}_1$ are obtained by constructing states with good L_c quantum numbers. For this purpose we project e_L onto the space with $L_\mathrm{c} = 0$ and obtain

$$\begin{aligned} e_L(z_1 - z_\mathrm{c}, \ldots, z_N - z_\mathrm{c}) &= \sum_{m=0}^{L} \frac{1}{m!}\, (-z_\mathrm{c})^m\, D_\mathrm{c}^m\; e_L(z_1, \ldots, z_N) \\ &= \frac{(-1)^L}{L!} (\hat{L}_\mathrm{c} - 1)\, (\hat{L}_\mathrm{c} - 2) \ldots (\hat{L}_\mathrm{c} - L)\, e_L(z_1, \ldots, z_N). \end{aligned} \tag{13}$$

The rhs of the first line is the Taylor expansion, yielding a shift in the single-particle coordinates by $-z_\mathrm{c}$. The second line is obtained by combining the operators D_c and z_c to the operator for the angular momentum of the centre of mass (4). This yields an operator that projects onto the subspace with $L_\mathrm{c} = 0$. The expansion [17]

$$e_L(z_1 - z_\mathrm{c}, \ldots, z_N - z_\mathrm{c}) = \sum_{m=0}^{L} \binom{N-m}{L-m} (-z_\mathrm{c})^{L-m} e_m(z_1, \ldots, z_N) \tag{14}$$

shows explicitly that only states of $\mathcal{W}_1$ are involved in the construction of a state with $L_\mathrm{c} = 0$. The states

$$z_\mathrm{c}^\lambda\, e_{L-\lambda}(z_1 - z_\mathrm{c}, \ldots, z_N - z_\mathrm{c}) \qquad \lambda = 0, 1, 2, \ldots, L-2, L \tag{15}$$

thus constitute an orthogonal basis of $\mathcal{W}_1$. Since they have different quantum numbers $L_\mathrm{c} = \lambda$ they are also eigenstates of the Hamiltonian (7). Note that these wavefunctions do not depend on the coefficients c_m in the Hamiltonian. The corresponding energies are obtained by applying the Hamiltonian (7) to the states (15). Using equations (8), (12) and (14) yields

$$E_\lambda = g[\tfrac{1}{2} N(N-1)c_0 + N(L-\lambda)(c_1 + 2c_2)] \qquad \lambda = 0, 1, 2, \ldots, L-2, L. \tag{16}$$

The states (15) thus form a ladder. Let us consider the important case of a two-body contact interaction. Analytical arguments [9] showed that the ground state is obtained for $\lambda = L$ in the case of attractive interactions $g < 0$. For repulsive interactions, a combination of analytical [15–17] and numerical calculations [11] showed that the ground state is obtained for $\lambda = 0$. It is interesting to note that the corresponding fermionic problem, i.e. the quantum Hall effect with zero-range interaction potentials, also has an exact ground state solution—the Laughlin wavefunction [21].

Note that the annihilation properties described in equations (9) and (10) may be generalized to further subspaces. Let

$$\mathcal{V}_k = \mathrm{span}\left\{ e_{\lambda_1}\, e_{\lambda_2} \ldots e_{\lambda_k}\, e_1^m : m = 0, 1, \ldots, L-2k; \sum_{j=1}^{k} \lambda_j = L - m; \lambda_j > 1 \right\} \tag{17}$$

denote the space of wavefunctions that contain products of k elementary polynomials different from e_1 and e_0. Using equations (9) and (10) one obtains

$$\hat{A}_m\, \mathcal{V}_k = 0 \qquad \text{for} \quad m \geqslant 2k+1. \tag{18}$$

However, these spaces are not invariant subspaces of the Hamiltonian, since e.g. $e_2^3(z_1,\ldots,z_6) \in \hat{V} e_3^2(z_1,\ldots,z_6)$. Let us consider the space that is generated when acting with the Hamiltonian (7) on $\mathcal{V}_k$ for $k > 1$. A direct calculation shows that

$$(\partial_i - \partial_j)^\mu (z_m - z_n)^\nu (\partial_m - \partial_n)^\nu \mathcal{V}_k = 0 \qquad \text{for} \quad \mu \geqslant \nu + 2k + 1.$$

Therefore,

$$\hat{A}_m \hat{V} \mathcal{V}_k = 0 \qquad \text{for} \quad m \geqslant 4k + 1.$$

Introducing the subspaces

$$\mathcal{W}_k = \cup_{j=0}^k \mathcal{V}_j$$

yields

$$\hat{V} \mathcal{W}_k \subset \mathcal{W}_{2k} \qquad \text{for} \quad k > 1.$$

The Hamiltonian matrix thus can be cast into a block-tridiagonal form. The invariant subspaces $\mathcal{W}_0$ and $\mathcal{W}_1$ yield a block-diagonal structure, and the remaining matrix is block tridiagonal.

The knowledge of L eigenstates (15) permits us to determine two additional eigenstates that are independent of the specific coefficients in the Hamiltonian (7). For $L = 4$ and 5, there are two basis states with $L_c = 0$. One of them has already been determined above, i.e. $e_L(z_1 - z_c, \ldots, z_N - z_c)$. The other two wavefunctions in question are

$$\phi_4 = N \sum_{j=1}^N (z_j - z_c)^4 + 3\left(\sum_{j=1}^N (z_j - z_c)^2 \right)^2$$

$$\phi_5 = N \sum_{j=1}^N (z_j - z_c)^5 + 2 \sum_{j=1}^N (z_j - z_c)^2 \sum_{i=1}^N (z_i - z_c)^3.$$

These wavefunctions are polynomials of degrees four and five, respectively. Therefore, one finds $\phi_4, \phi_5 \in \mathcal{W}_2$.

The results derived so far were obtained for $0 \leqslant L \leqslant N$. For $L > N$ the Hilbert space depends on both L and N. The definition of the spaces $\mathcal{V}_k$ has to be modified correspondingly, e.g. $\mathcal{V}_1 = \text{span}\{e_{L-\lambda}\, e_1^\lambda : \lambda = \max(0, L-N), \ldots, L-2\}$ to be compared with equation (11). However, the annihilation property described in equation (18) remains valid and the Hamiltonian matrix continues to be of block-tridiagonal form. Note that the wavefunctions (15) are restricted to parameter values $\lambda = \max(0, L-N), \ldots, L-3, L-2, L$ for $L > N$.

Quantum mechanical systems with invariant subspaces have received considerable attention in recent years, since (quasi-) exactly solvable systems fall into this class. For a general discussion see, for example, [22–25]. In contrast to this paper, the interest there is in *infinite*-dimensional Hilbert spaces with a nested sequence of finite-dimensional subspaces, and most research has been limited to systems with a few degrees of freedom. Note finally that partially solvable many-body systems [25] or systems with partial dynamical symmetry [26–28] also permit the analytical computation of selected eigenstates.

In summary, we have studied a large class of harmonically trapped, interacting boson systems at angular momentum L. The Hamiltonian leaves a L-dimensional subspace invariant, and the quantization of the angular momentum for the centre of mass yields several eigenstates that are independent of the details of the two-body interaction. Important examples of suitable interactions are two-body potentials of zero range and interactions that are analytical in the squared inter-particle distance.

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