Extracting particle freeze-out phase-space densities and entropies from sources imaged in heavy-ion reactions

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The space-averaged phase-space density and entropy per particle are both fundamental observables which can be extracted from the two-particle correlation functions measured in heavy-ion collisions. Two techniques have been proposed to extract the densities from correlation data: either by using the radius parameters from Gaussian fits to meson correlations or by using source imaging, which may be applied to any like-pair correlation. We show that the imaging and Gaussian fits give the same result in the case of meson interferometry. We discuss the concept of an equivalent instantaneous source on which both techniques rely. We also discuss the phase-space occupancy and entropy per particle. Finally, we propose an improved formula for the phase-space occupancy that has a more controlled dependence on the uncertainty of the experimentally measured source functions.

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I. INTRODUCTION

The phase-space density of particles produced in an ultrarelativistic heavy-ion collision is a fundamental observable which is accessible, at least in part, via two-particle correlations. Measurements of this observable are interesting because they may either provide direct evidence for thermal phase-space distribution of particles at freeze-out or show evidence of deviations from such [1,2]. Also, if one can measure the average phase-space density, one can begin to look for effects of an overpopulation of phase-space or more exotic phenomena such as pion lasers, superradiance, etc. [3]. A quantity closely related to the phase-space density is the entropy per particle, a key thermodynamic property of high density matter. Indeed, a phase transition might not take place adiabatically and could generate entropy.

Several methods have been suggested for measuring the phase-space densities of the various particles in a heavy-ion collision. For the case of identical noninteracting pions, Bertsch [4] proposed a method that uses both the radius parameters from conventional Hanbury Brown–Twiss (HBT) analysis and the pion spectrum. He also suggested a physically intuitive definition of phase-space density: replace the single-particle sources in the Koonin-Pratt formalism [5,6] with an instantaneous source at a mean freeze-out time. This effective source is just the effective phase-space density at freeze-out. This replacement preserves the particle numbers at large times (after freeze-out) even though, by construction, it differs from the true density during the freeze-out process.

Another approach, based on the source imaging technique introduced by Brown and Danielewicz [7,8], allows one to extract the space-averaged phase-space density and entropy per particle from *any* like-pair correlations. This approach uses both the source function imaged from a correlation measurement and the single-particle spectrum. We will show that this approach is more versatile: it can be applied to any like-pair correlations and it reduces to the HBT result of [4] when applied to pions.

A third approach, introduced by Siemens and Kapusta [9], uses the ratio of total deuteron yield to total proton yield instead of the two-particle correlation function to estimate the total phase-space occupancy of nucleons. This approach is complimentary to the source imaging approach in the sense that both give access to the proton phase-space occupancy. However, while the imaging method requires one to measure the correlation function and one-particle spectrum for one species of particle, the deuteron-proton ratio approach requires one to measure the spectra of two different species. Usually these spectra have different acceptances and efficiencies of particle reconstruction and identification which may complicate experimental analysis. We will not discuss this approach in this paper.

The outline of the paper is as follows. First, we use the substitution of Ref. [4] to explain how the imaged sources can be used to extract the space-averaged phase-space density $\langle f(\vec{p})\rangle$ as discussed in [7]. We will show that these results for $\langle f(\vec{p})\rangle$ are a generalization of the result in Ref. [4] derived for identical mesons. Next, we will demonstrate that one does not change the source function by making this substitution. Within this framework, we will discuss the phase-space occupancy $\langle f\rangle$ and the entropy per particle $\mathcal{S}/N_{\rm part}$. We will find an expression for the phase-space occupancy that is an improvement over that in [7] as the formula here has a more controlled dependence on the uncertainty of the experimentally measured source functions.

II. THE CORRELATION AND SOURCE FUNCTIONS

We begin by defining a Lorentz-invariant single-particle source $D(r,\vec{p})$, which gives the rate for creating on-shell $(E=\sqrt{\vec{p}^2+m^2})$ particles (of all spin projections):

$$D(r,\vec{p})=\frac{Ed^7N}{d\vec{r}\,dt\,d\vec{p}}\,. \tag{1}$$

In our notation, $r=(t,\vec{r})$ is a four-vector and $\vec{p}$ is a three-vector. With our choice of normalization, the single-particle source transforms as a Lorentz scalar. The single-particle source may be computed directly in an event generator such as relativistic quantum molecular dynamics (RQMD) [10].

We define the two-particle source function as the probability density for producing a particle pair separated by $\vec{r}'$ in their center of mass (c.m.) frame. Following [5–8], we define this source function as the convolution of the single-particle sources:

$$S_{\vec{p}}(\vec{r}')=\frac{\int dt'\int d^4R\,D(R+r/2,\vec{p}_1)D(R-r/2,\vec{p}_2)}{\left(\int d^4R\,D(R,\vec{p}_1)\right)\left(\int d^4R\,D(R,\vec{p}_2)\right)}$$
$$\equiv\int dt'\int d^4R\,\tilde{D}(R+r/2,\vec{p}_1)\tilde{D}(R-r/2,\vec{p}_2). \qquad (2)$$

Here we denote the normalized single-particle sources with a tilde, coordinates taken in the pair c.m. frame with primes, and coordinates taken in the laboratory frame without primes. The average particle momentum is $\vec{p}=\frac{1}{2}(\vec{p}_1+\vec{p}_2)$ and the relative particle momentum is $\vec{q}=\frac{1}{2}(\vec{p}_1-\vec{p}_2)$. While all $\vec{q}$ pairs seem to enter in this definition (2), in practice only low relative momentum pairs contribute to correlations and hence to either the imaged sources or Gaussian fits to correlations. Typically, this is used to allow one to replace $\vec{p}_1$ and $\vec{p}_2$ with the average pair momentum $\vec{p}$ (see Ref. [6]). Following convention, we will not explicitly state the $\vec{q}$ dependence of the source function.

The two-particle source function may be extracted from the measured two-particle correlation function by inverting the Koonin-Pratt [5,6] equation:

$$C_{\vec{p}}(\vec{q}')=\frac{dN_{\rm pair}/d\vec{p}_1 d\vec{p}_2}{(dN/d\vec{p}_1)(dN/d\vec{p}_2)}=\int d^3r'\,|\Phi_{\vec{q}'}(\vec{r}')|^2 S_{\vec{p}}(\vec{r}'). \qquad (3)$$

Here $C_{\vec{p}}(\vec{q}')$ is the measured correlation function and $\Phi_{\vec{q}'}(\vec{r}')$ is the pair relative wave function in the pair c.m. frame. We comment that the correlation function can be measured and the source function extracted in any frame; however, relation between the two is simplest in the pair c.m. frame.

In order to extract the source function, we first discretize Eq. (3) to obtain the matrix equation $C_i=\Sigma_j K_{ij}S_j$. We then proceed as in [7,8] and find the set of source points S_j that minimize the χ^2. Here, $\chi^2=\Sigma_i(C_i-\Sigma_j K_{ij}S_j)^2/\Delta^2 C_i$. The χ^2 minimizing source is $S_j=\Sigma_i[(K^TBK)^{-1}K^TB]_{ji}(C_i-1)$ where K^T is the transpose of the kernel matrix and B is the inverse covariance matrix of the data $B_{ij}=\delta_{ij}/\Delta^2 C_i$. The error on the source is the square root of the diagonal elements of the covariance matrix of the source, $\Delta^2 S=(K^TBK)^{-1}$. Since this procedure works on any like-pair correlation, we can dispense with the correlation function and work directly with the source function.

III. THE PHASE-SPACE DENSITY

The phase-space density at time t is a Lorentz scalar; we write it as

$$f(t,\vec{r},\vec{p})=\frac{1}{\Gamma_0}\frac{d^6N}{d\vec{r}\,d\vec{p}}. \qquad (4)$$

Here we have defined a unit volume in phase space as $\Gamma_0=(2s+1)/(2\pi\hbar c)^3$ and a differential phase-space volume as $\Gamma_0 d\vec{r}\,d\vec{p}$. The $(2s+1)$ factor accounts for the spin of the particle of interest. Assuming that the particles propagate as free particles, the phase-space density at a specific time t_1 can be written in terms of the phase-space density of a different time t_0 as

$$f(t_1,\vec{r},\vec{p})=f(t_0,\vec{r}-\vec{v}(t_1-t_0),\vec{p})+\frac{1}{\Gamma_0 E}\int_{t_0}^{t_1}dt\,D(t,\vec{r}-\vec{v}(t_1-t),\vec{p}). \qquad (5)$$

This expression ignores processes that modify the particle's momentum (such as from a mean field, Coulomb forces, etc.). Since we want the particle at position $\vec{r}$, we must look backward (or forward) in time to where it was at its creation, e.g., $\vec{r}-\vec{v}(t_1-t)$.

A particle is said to have frozen out when it has undergone its last strong interaction and is now propagating to the detector. Since each particle can freeze out at a different time, the concept of a ''freeze-out phase-space density'' is ambiguous. One option to deal with this is to average over the creation times of the particles. However, then the question becomes how to perform this average and what the averaged density means. Reference [4] proposes a simple solution: replace the particle source $D(r,\vec{p})$ with an effective source $D^{\rm eff}(r,\vec{p})$ at an instantaneous freeze-out time t_f. In other words,

$$D(r,\vec{p})\to D^{\rm eff}(r,\vec{p})=\delta(t_f-t)E\Gamma_0 f^{\rm eff}(\vec{r},\vec{p}). \qquad (6)$$

The effective freeze-out phase-space density is related to the true single-particle source via

$$f^{\rm eff}(\vec{r},\vec{p})=\frac{1}{\Gamma_0 E}\int_{-\mathcal{T}}^{\mathcal{T}}dt\,D(t,\vec{r}-\vec{v}(t_f-t),\vec{p}). \qquad (7)$$

Here, $\mathcal{T}>t_f>-\mathcal{T}$ and $\pm\mathcal{T}$ is simply some large time after (before) which the source is turned off (on). It is clear from Eq. (7) that replacement in Eq. (6) gives the right phase-space density at large times $(t>\mathcal{T})$ as it must by Liouville's theorem. Furthermore, this can be easily checked by examining Eq. (5) for large times $t>\mathcal{T}$ with $f=0$ for times earlier than $-\mathcal{T}$. Finally, it is clear that this replacement does not alter the momentum-space density that one might calculate from Eq. (5).

Not only does this prescription give a reasonable way to define the freeze-out phase-space density, but it also is *entirely consistent with our definition of the source function in* Eq. (2). We will illustrate this by simply inserting the effective phase-space density into the definition of the source [Eq. (2)] and performing the required algebra. Special care must be taken in performing the various time integrals as some are performed in the laboratory frame while others are performed in the pair c.m. frame.

We now insert Eqs. (6) and (7) into the equation for the source function in Eq. (2) and perform one of the δ function integrals. We obtain

$$S_{\vec{p}}(\vec{r'})=\int dt'\ \delta(t)\int d^3R\int dT\int d\tau \times\tilde{D}(T+\tau/2,\vec{R}+\vec{r}/2-\vec{v}_1(t_f-T-\tau/2),\vec{p}_1) \times\tilde{D}(T-\tau/2,\vec{R}-\vec{r}/2-\vec{v}_2(t_f-T+\tau/2),\vec{p}_2). \quad (8)$$

Here $\vec{v}_i=\vec{p}_i/E_i$ are the velocities of the individual particles. Also, $T=(t_1+t_2)/2$ and $\tau=(t_1-t_2)$ are the average and relative time variables from the time integrals in the effective phase-space density. We can simplify Eq. (8) by introducing the average velocity of the pair: $\vec{v}\equiv\vec{p}/\sqrt{\vec{p}^2+m^2}$ so that $\vec{v}_i=\vec{v}+\delta\vec{v}_i$. With this, we make the change of variables $\vec{R}-\vec{v}(t_f-T)\to\vec{R}$ and remove much of the time dependence from the spatial arguments:

$$S_{\vec{p}}(\vec{r'})=\int dt'\ \delta(t)\int d^3R\int dT\int d\tau \times\tilde{D}(T+\tau/2,\vec{R}+(\vec{r}+\vec{v}\tau)/2-\delta\vec{v}_1(t_f-T-\tau/2),\vec{p}_1) \times\tilde{D}(T-\tau/2,\vec{R}-(\vec{r}+\vec{v}\tau)/2-\delta\vec{v}_2(t_f-T+\tau/2),\vec{p}_2). \quad (9)$$

Our next step is to do the t' integral. This appears to be a straightforward δ function integral; however, care must be taken due to the different reference frames involved. When we perform the t' integral, two things happen: $\int dt'\ \delta(t)\to 1/\gamma$ and everywhere in the integrand $t'\to -vr'_\parallel$. Here $r'_\parallel$ is the component of $\vec{r'}$ in the direction of the boost from the laboratory to the pair c.m. frame (this boost velocity is $\vec{v}$).

We make further progress by examining $\vec{r}+\vec{v}\tau$. First, define the four-vector $s=(\tau,\vec{s})=(\tau,\vec{r}+\vec{v}\tau)$. If we write this in the pair c.m. frame, we have $s=(\gamma(\tau'+vs'_\parallel),\gamma(s'_\parallel+v\tau'),\vec{r'}_\perp)$. So the parallel component of s is

$$s_\parallel=\gamma(s'_\parallel+v\tau')=\gamma\left(s'_\parallel+v\left(\frac{\tau}{\gamma}-vs'_\parallel\right)\right)=\frac{1}{\gamma}s'_\parallel+v\tau. \quad (10)$$

But note, from the definition of s and from the t' integral we already have

$$s_\parallel=r_\parallel+v\tau=\gamma(r'_\parallel+vt')+v\tau=\frac{1}{\gamma}r'_\parallel+v\tau. \quad (11)$$

Therefore, if we identify $s'_\parallel=r'_\parallel$ and change integration variables from τ to τ', we find the result

$$S_{\vec{p}}(\vec{r'})=\int d\tau'\int d^3R\int dT\,\tilde{D}(T+\tau/2,\vec{R}+\vec{r}/2-\delta\vec{v}_1(t_f-T-\tau/2),\vec{p}_1)\tilde{D}(T-\tau/2,\vec{R}-\vec{r}/2-\delta\vec{v}_2(t_f-T+\tau/2),\vec{p}_2). \quad (12)$$

In order to complete the connection between Eq. (12) and the source in Eq. (2), we must justify dropping the $\delta\vec{v}_i$'s in Eq. (12). Since $\Delta t=(t_f-T\mp t/2)$ is on the order of the freeze-out duration, if $\delta\vec{v}_i\Delta t$ is smaller than the characteristic length scale of the single-particle source, we can drop the $\delta\vec{v}_i$'s. Writing $\delta\vec{v}_i$ in terms of the relative momentum $\vec{q}$, we find

$$\delta\vec{v}_i=\vec{v}-\vec{v}_i=\pm\frac{1}{E}(\vec{q}-\vec{v}\cdot\vec{q}\vec{v})+O(\vec{q}^{\,2}). \quad (13)$$

Thus, the $\delta\vec{v}_i$ term shifts the spatial argument of $\tilde{D}$ in Eq. (13) $\sim\Delta tq/\gamma E$ in the direction parallel to $\vec{v}$ and $\sim\Delta tq/E$ perpendicular to it. For highly relativistic or massive pairs, this shift may be neglected. For low velocity or light pairs (such as π pairs) the shift is important, especially if the freeze-out duration is large or the system size is small.

A. Space-averaged phase-space density from the sources

Inserting the particle source with instantaneous freeze-out into Eq. (2), taking $\vec{p}_1,\vec{p}_2\approx\vec{p}$, taking the limit as $\vec{r'}\to 0$, and performing the integrals over time, we find Eq. (17) of Ref. [7]:

$$\langle f(\vec{p})\rangle\equiv\frac{\int d^3r\, f^2(\vec{r},\vec{p})}{\int d^3r\, f(\vec{r},\vec{p})}=\frac{1}{\Gamma_0}\frac{1}{m}\frac{Ed^3N}{d\vec{p}}\,S_{\vec{p}}(\vec{r'}\to 0). \quad (14)$$

For the sake of brevity, we drop the "eff" subscript on the density here and for the remainder of the paper.

For zero impact parameter collisions, we may exploit the azimuthal symmetry and average over the angle of the particle transverse momentum θ_{p_T} and use

$$\langle f(y,p_T)\rangle\equiv\frac{\int d\theta_{p_T}\int d^3r\, f^2(\vec{r},\vec{p})}{\int d\theta_{p_T}\int d^3r\, f(\vec{r},\vec{p})}=\frac{1}{\Gamma_0}\frac{1}{m}\frac{d^2N}{2\pi dy\, dp_T\, p_T}S_{\vec{p}}(\vec{r'}\to 0). \quad (15)$$

Note that, due to the azimuthal symmetry, the density is a function of the particle rapidity y and the magnitude of the particle transverse momentum, p_T.

As a side comment on using Eqs. (14) and (15), one can use the $\vec{r'}\to 0$ point either from a full three-dimensional reconstruction of the source function (such as in [11]) or from the angle-averaged source function. In practice, it is usually much easier to measure the angle-averaged two-particle correlation function (and hence source function) than the full three-dimensional correlation because one can sum over angles to increase statistics.

B. Gaussian meson sources

We now show that Eqs. (15) and (14) are direct generalizations of the results in [4] or [2] for pions. The correlation function for identical noninteracting spin-0 bosons can be written in terms of a matrix of radius parameters [12]:

$$C(\vec{Q}) = 1 + \lambda e^{-Q_i Q_j [R^2]_{ij}}. \quad (16)$$

Here we have dropped the average pair momentum label and all primes on the momenta. Here also $\vec{Q} = 2\vec{q}$ (the relative momentum variable used in the analysis of pion correlations) and λ is a fit parameter often called the chaoticity parameter. The matrix of radius parameters $[R^2]$ is the following real, symmetric matrix:

$$[R^2] = \begin{pmatrix} R_o^2 & R_{os}^2 & R_{ol}^2 \\ R_{os}^2 & R_s^2 & R_{sl}^2 \\ R_{ol}^2 & R_{sl}^2 & R_l^2 \end{pmatrix} \quad (17)$$

in the Bertsch-Pratt parametrization. For pions, the Koonin-Pratt equation is a Fourier cosine transform that may be inverted analytically [7] to give the source function directly in terms of the correlation function:

$$S(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3Q \cos(\vec{Q}\cdot\vec{r})[C(\vec{Q}) - 1]. \quad (18)$$

Inserting Eq. (16) into this expression yields a Gaussian source function:

$$S(\vec{r}) = \frac{\lambda}{(2\sqrt{\pi})^3 \sqrt{\det[R^2]}} \exp\left(-\frac{1}{4} r_i r_j [R^2]^{-1}_{ij}\right). \quad (19)$$

Taking the $r \to 0$ limit of this source,

$$S(\vec{r} \to 0) = \frac{\lambda}{(2\sqrt{\pi})^3 \sqrt{\det[R^2]}}, \quad (20)$$

and inserting this into the equation for the average phase-space density, Eq. (14) or (15) yields the result in Ref. [4]. Actually our result is more general than those in Refs. [2,4] as those results only apply either to diagonal $[R^2]$ or $[R^2]$ with $R_{ol}^2 \neq 0$. One should note that the $\vec{r} = 0$ intercept of the source function has units of an effective volume.

IV. PHASE-SPACE OCCUPANCY AND ENTROPY

We can now estimate $\langle f \rangle$ from our calculation of the space-averaged phase-space density $\langle f(y, p_T) \rangle$ or $\langle f(\vec{p}) \rangle$. In [7], the authors argue that $\langle f \rangle$ can be estimated via

$$\langle f \rangle = \frac{\int d^3p \, \langle f(\vec{p}) \rangle^2}{\int d^3p \, \langle f(\vec{p}) \rangle}. \quad (21)$$

Given the current state of available correlation data, the uncertainty in the extracted sources can be greater than 50% of the extracted source value. Therefore, in this expression the uncertainty in the result is dominated by the uncertainty in the underlying extracted source. Indeed, since the relative uncertainty in $\langle f(\vec{p}) \rangle$ is nearly $\delta S_{\vec{p}}(r \to 0)/S_{\vec{p}}(r \to 0)$, we find $\delta\langle f \rangle / \langle f \rangle \sim 3\,\delta S_{\vec{p}}(r \to 0)/S_{\vec{p}}(r \to 0)$ for sources with a strong $\vec{p}$ dependence. Thus, one easily finds uncertainties greater than the values themselves. Clearly an alternative is needed that has a smaller dependency on the error in the source.

Instead of Eq. (21), we propose the following method for evaluating $\langle f \rangle$:

$$\langle f \rangle = \frac{1}{N_{\text{part}}} \int d^3p \, \langle f(\vec{p}) \rangle \frac{d^3N}{d\vec{p}}, \quad (22)$$

and for azimuthally symmetric systems,

$$\langle f \rangle = \frac{2\pi}{N_{\text{part}}} \int dy \, dp_T \, p_T E \langle f(y, p_T) \rangle \frac{d^2N}{2\pi dy \, dp_T \, p_T}. \quad (23)$$

Using either of these expressions, the relative uncertainty varies like one factor of $\delta S_{\vec{p}}(r \to 0)/S_{\vec{p}}(r \to 0)$. One can see Eq. (22) [or Eq. (23)] by beginning with the definition of $\langle f \rangle$:

$$\begin{aligned} \langle f \rangle &= \frac{\int d^3r \, d^3p \, f^2(\vec{r}, \vec{p})}{\int d^3r \, d^3p \, f(\vec{r}, \vec{p})} \\ &= \frac{\int d^3p \left(\langle f(\vec{p}) \rangle \int d^3r \, f(\vec{r}, \vec{p}) \right)}{N_{\text{part}}/\Gamma_0} \\ &= \frac{\int d^3p \, [\langle f(\vec{p}) \rangle (1/\Gamma_0)(d^3N/d\vec{p})]}{N_{\text{part}}/\Gamma_0}, \end{aligned} \quad (24)$$

which immediately gives Eq. (22).

One might wonder if, through similar considerations, we may be able to improve on the calculation of the entropy per particle given in Refs. [7] or [9]. In short, we do not believe so. Consider the entropy for a gas of fermions (top) or bosons (bottom):

$$\mathcal{S}/N_{\text{part}} = -\frac{\int d^3r \, d^3p \, [f \ln f \pm (1 \mp f) \ln(1 \mp f)]}{\int d^3r \, d^3p \, f}. \quad (25)$$

To arrive at their expression for the entropy per particle, the authors of [7] neglected the spatial dependence of $f(\vec{r},\vec{p})$ and found

$$\mathcal{S}/N_{\rm part}=-\frac{\int d^3p\,\{\langle f(\vec{p})\rangle\ln\langle f(\vec{p})\rangle\pm[1\mp\langle f(\vec{p})\rangle]\ln[1\mp\langle f(\vec{p})\rangle]\}}{\int d^3p\,\langle f(\vec{p})\rangle}. \tag{26}$$

The authors of [9] go one step farther and neglect the momentum variation of $f(\vec{r},\vec{p})$ to obtain a result entirely dependent on the phase-space occupancy $\langle f\rangle$. Clearly, the entropy should get larger contributions from regions of coordinate space where the phase-space density is small; however, both Eq. (26) and the analogous result from Ref. [9] do not reflect this. The source function's $\vec{r}$ dependence does give us information about the spatial dependence of the phase-space density; however, it is not clear how one might use this information to obtain a better estimate of the entropy.

Now we present sample calculations of $\langle f\rangle$ and $\mathcal{S}/N_{\rm part}$ from two different experiments. The first calculation uses negative pion correlations measured from Pb+Pb collisions at 158 GeV/nucleon from the CERN-SPS experiment NA49 [13–15]. The second calculation uses proton correlations from the ^{14}N+^{27}Al reaction at 75 MeV/nucleon measured at the Michigan State University NSCL [16]. These calculations will show the general applicability of the source imaging and the superiority of Eq. (22) over Eq. (21).

For the first calculation, we use the space-averaged π^- phase-space densities of Ferenc *et al.* [2], extracted from experiment NA49 [13–15]. In this calculation we could have used an extracted source, but we can also get the $\vec{r}\,'\to 0$ source intercept from the radius parameters of a Gaussian fit to the pion correlations, as we saw in Sec. III B. Since this is exactly what is used in Ref. [2], we use their radii. We estimate the π^- spectrum as the product of the π^- rapidity and p_T distributions:

$$\frac{dN_{\pi^-}}{dy\,dp_T p_T}=0.9\frac{dn_{N^-}}{dy}\{T_{\rm eff}(y)[m_\pi+T_{\rm eff}(y)]\}^{-1}\times\exp\left(-\frac{\sqrt{p_T^2+m_\pi^2}-m_\pi}{T_{\rm eff}(y)}\right). \tag{27}$$

Here, dN_{h^-}/dy is the negative hadron rapidity distribution from Ref. [14] and the factor of 0.9 accounts for the fraction of negative hadrons that are actually pions. In Eq. (27), the p_T distribution is parametrized by a rapidity-dependent effective temperature $T_{\rm eff}(y)$ and the actual values of this effective temperature are obtained from [15]. Using Eq. (22), we find $\langle f\rangle=0.19\pm0.06$ while using Eq. (21), we find $\langle f\rangle=0.14\pm0.08$. Both results are consistent; however, the result from Eq. (22) has a 25% smaller uncertainty. Using Eq. (26), the entropy per pion can also be estimated as $\mathcal{S}/N_{\pi^-}=3.9\pm1.8$.

The numbers extracted from the sources from the NSCL pp data in [16] are more dramatic. Since the proton spectrum for this reaction is available at only a few angles, we follow Ref. [7] and approximate it with the thermal distribution

$$\frac{dN_p}{d\vec{p}}\propto\frac{1}{z^{-1}\exp(p^2/2mT)+1}. \tag{28}$$

Here the normalization constant is determined by normalizing to the number of participants (nine protons), T is the fitted temperature of 10.2 MeV, and z is set from the requirement of maximum entropy giving $z\sim1.1$. Since the source is given only at three fixed pair momenta, we follow Ref. [7] and average the results to obtain the zero intercept $S_{\rm av}(r\to0)=(15.5\pm2.5)\times10^{-4}\ {\rm fm}^{-3}$. Using Eq. (22), we find $\langle f\rangle=0.27\pm0.12$, while, using Eq. (21), we find $\langle f\rangle=0.25\pm0.04$—a factor of 3 improvement in the uncertainty. For the entropy per proton, we estimate $\mathcal{S}/N_p=2.7\pm0.7$. These results differ slightly from those in [7] simply because the authors of [7] set the error on the source to zero. Using the more accurately determined intercepts from Ref. [8], $S_{\rm av}(r\to0)=(18.7\pm1.1)\times10^{-4}\ {\rm fm}^{-3}$, we find $\langle f\rangle=0.30\pm0.02$ from Eq. (22) and $\langle f\rangle=0.30\pm0.05$ from Eq. (21). For the entropy we find $\mathcal{S}/N_p=2.45\pm0.21$. Note the substantial improvement from a better determination of the source intercept. Nevertheless, the result from Eq. (22) is still a factor of 2 improvement over Eq. (21).

V. CONCLUSIONS

The phase-space density is an important, fundamental observable that can provide insight into the underlying dynamics of the nuclear reactions and it may be extracted from sources imaged in heavy-ion reactions. This extraction relies on an appropriate definition of the effective phase-space density at freeze-out, since the true freeze-out density is not a uniquely defined concept. We have shown that the definition in Ref. [4] is entirely consistent with the source function obtained by imaging [7]. We have also shown how the extraction of the space-averaged phase-space density from imaged sources is a generalization of Bertsch's result for pion correlations. Finally, we have provided a formula for the calculation of the average phase-space occupancy. This formula is less sensitive to the uncertainties of the source functions than others in the literature. We believe that the source imaging method will be useful for extracting space-averaged phase-space densities from data of future relativistic heavy-ion experiments.

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