Solving relativistic integral equations for three body systems

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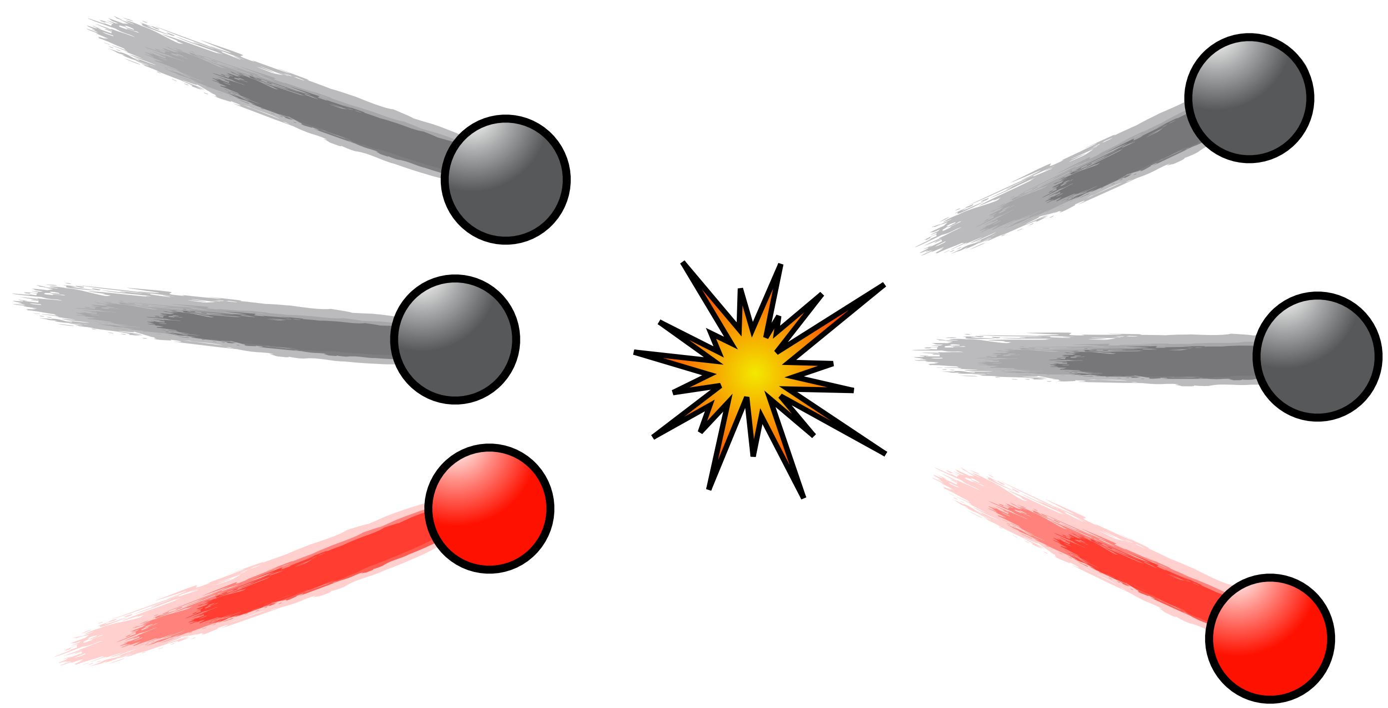


INT 20-2c — Accessing and Understanding the QCD Spectra
August 17 - September 4, 2020

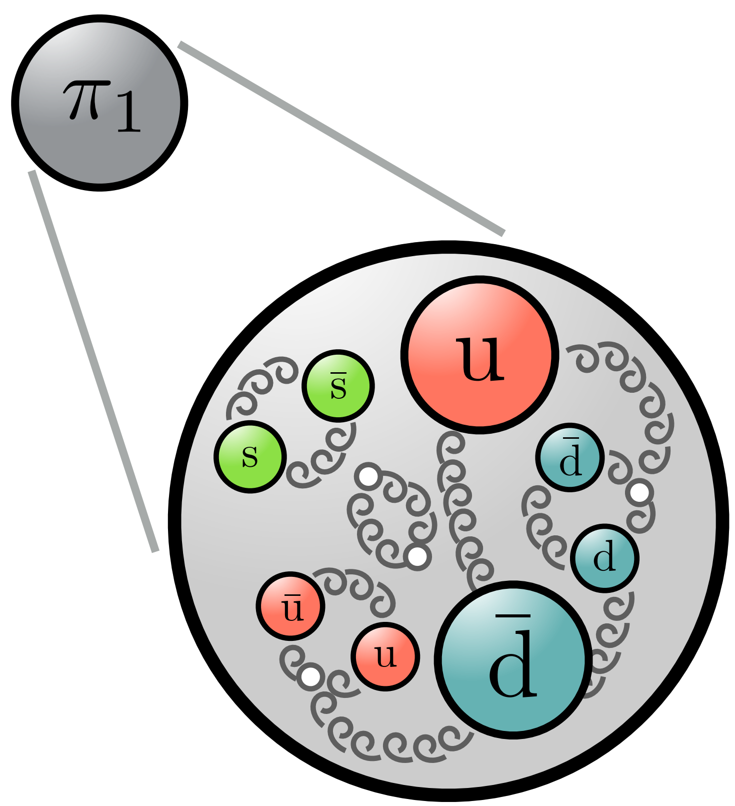
Why three body scattering?

Most excited states couple strongly to three (or more) particle channels

e.g. a different kind of exotic, the π_1 *hybrid candidate*

Understanding structure requires model independent determination of spectral properties

$$I^G(J^{PC}) = 1^-(1^{-+})$$



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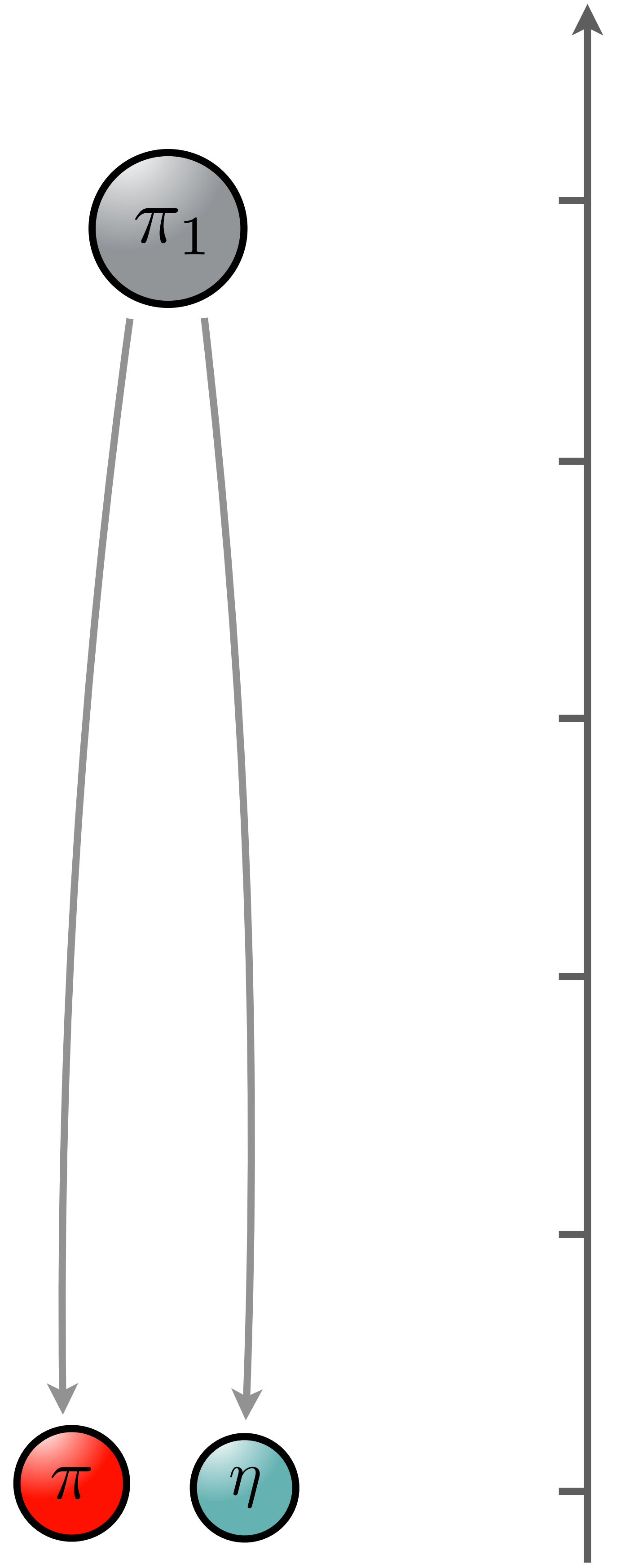

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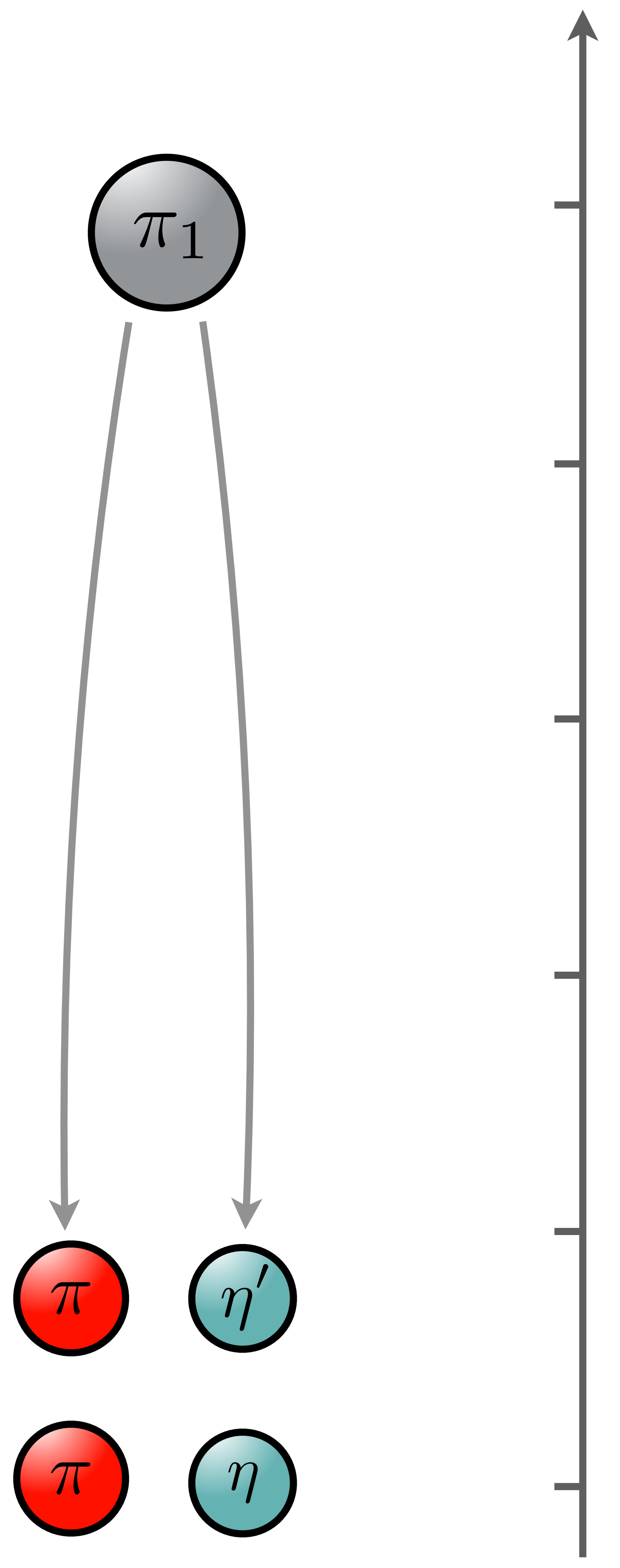

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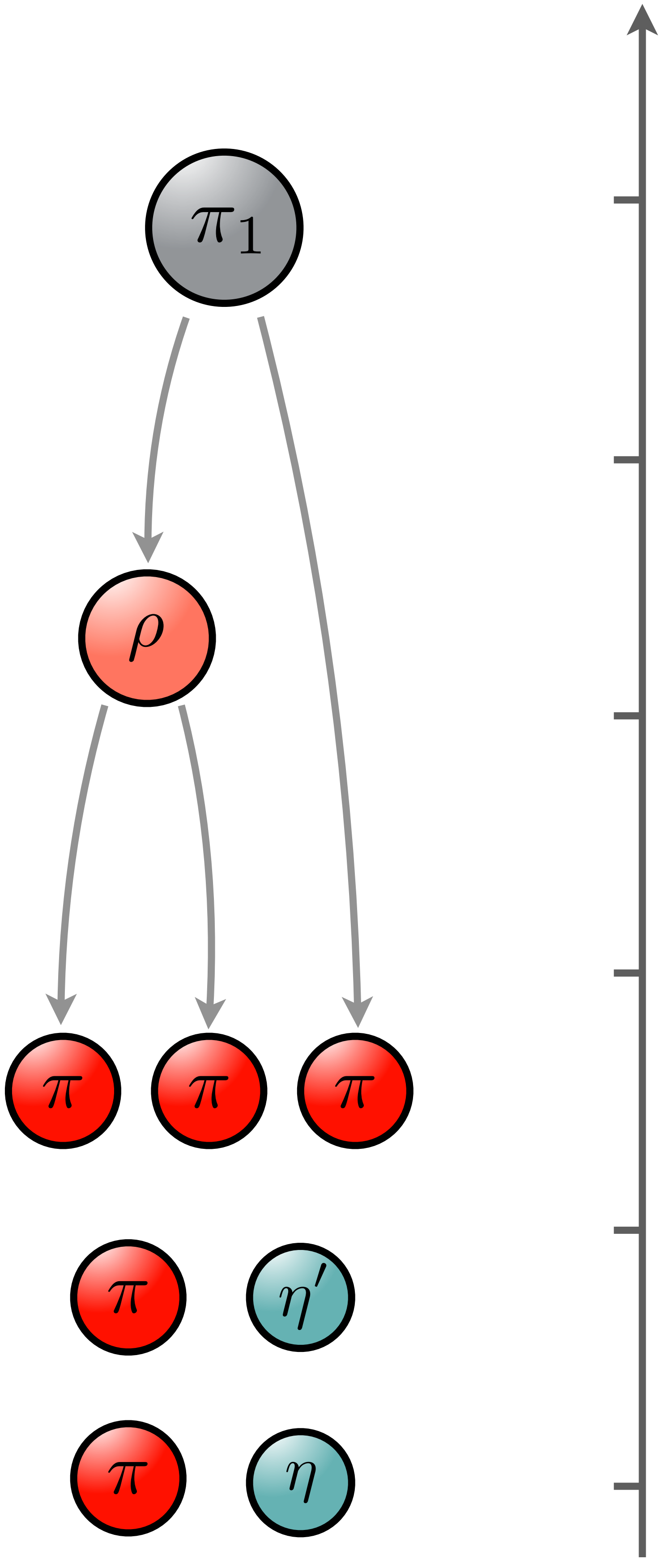

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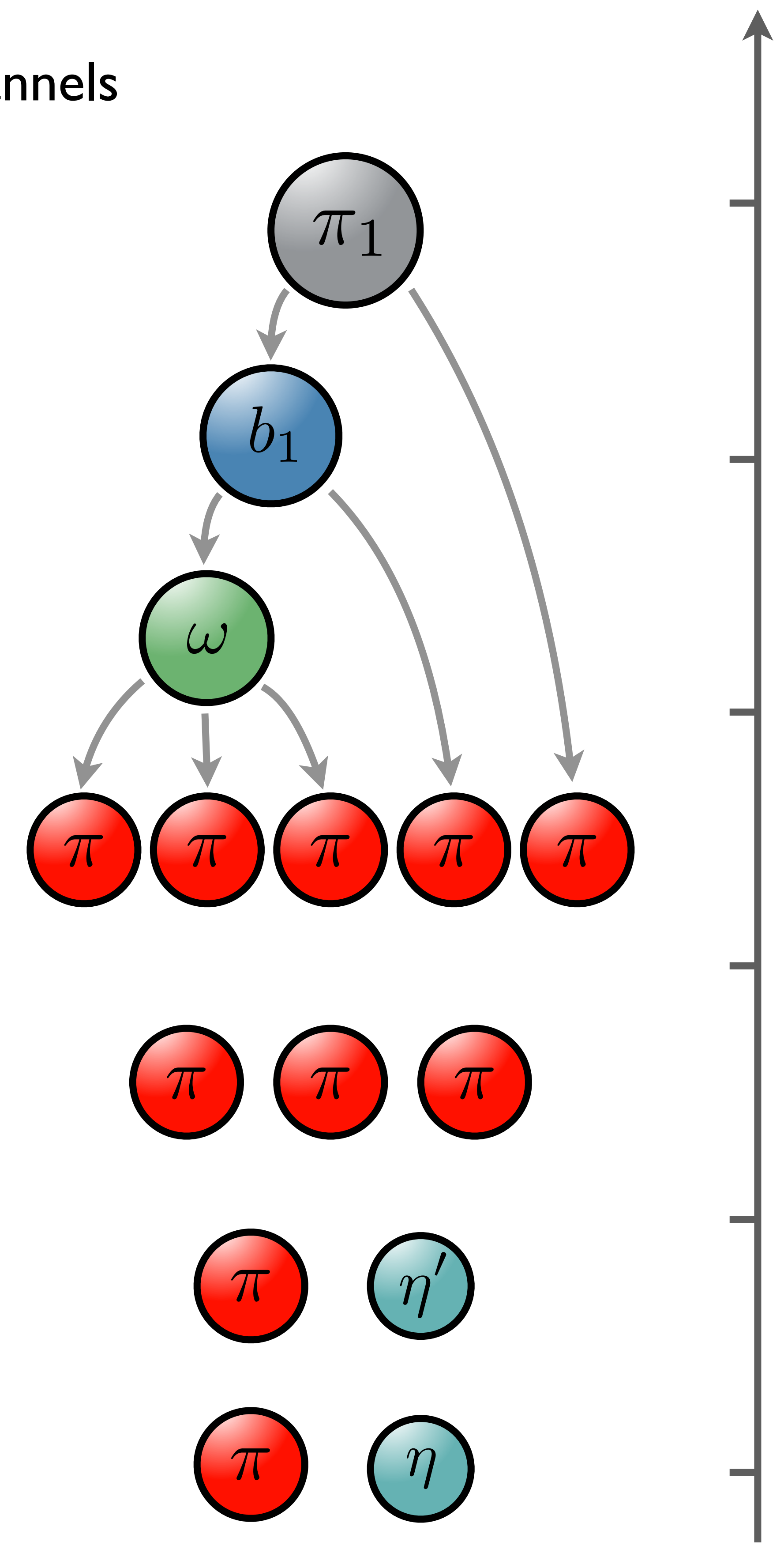

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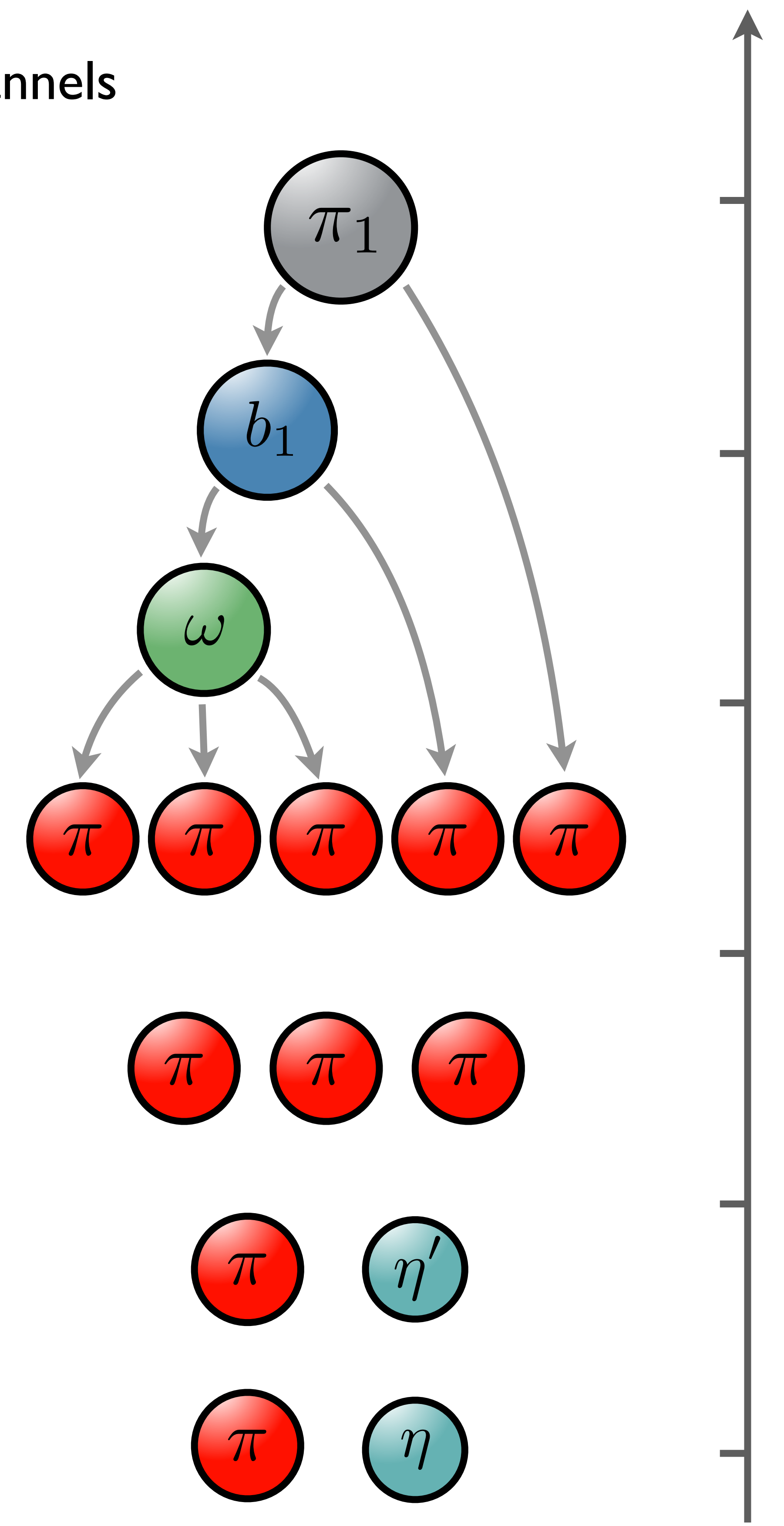


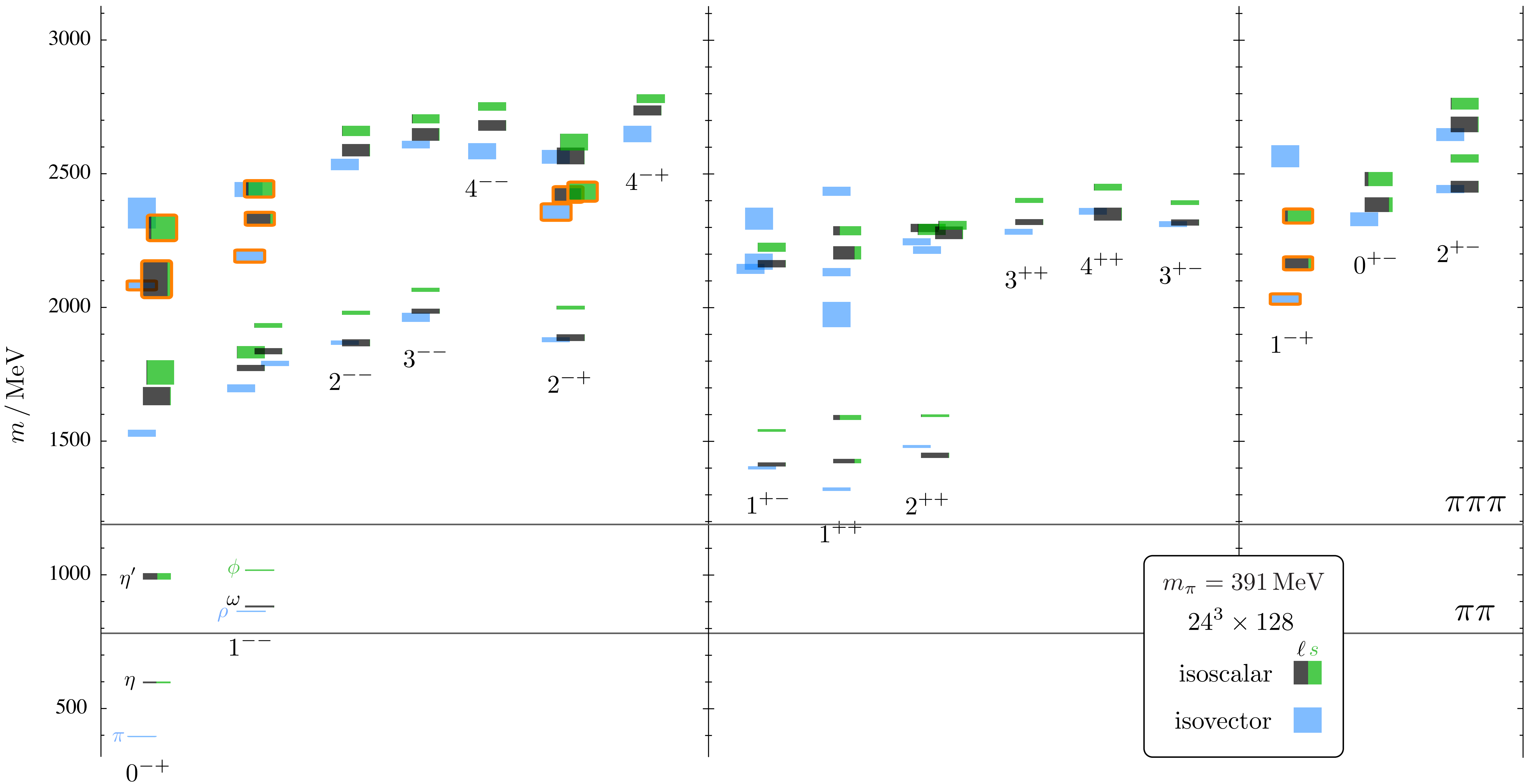
Multi-channel, multi-hadron scattering processes

Three body problem in lattice QCD

J. Dudek, R. Edwards, P. Guo, and C. Thomas, *Phys.Rev.D* 88, 094505 (2013)

Focus on three body physics from lattice QCD perspective

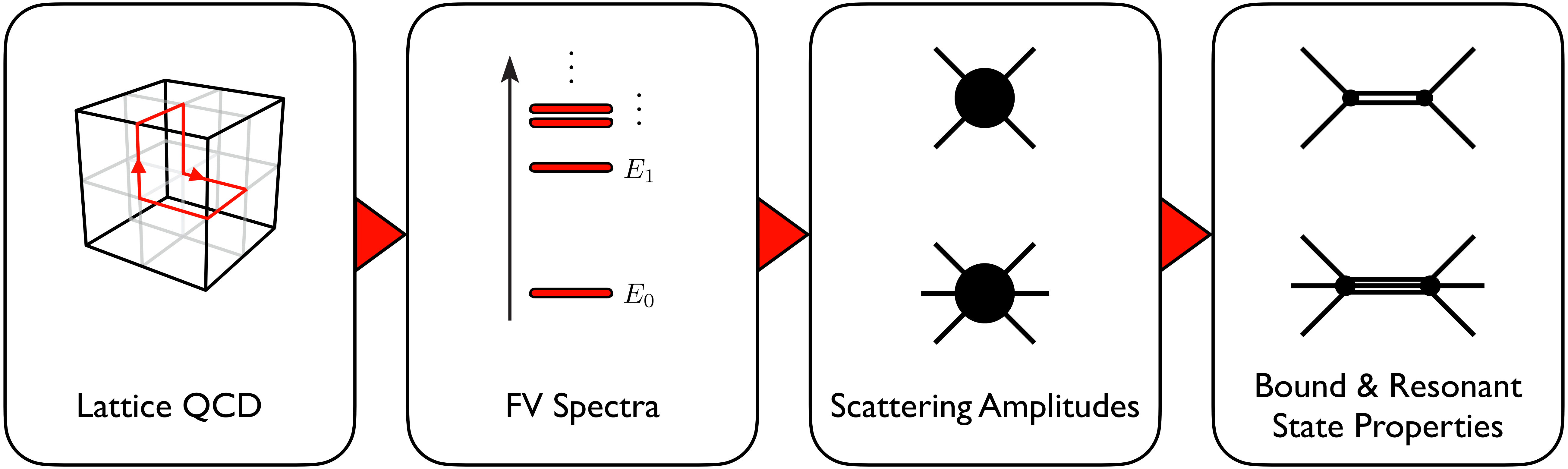
- Review formal aspects — Quantization conditions and integral equations
- Solving 2+1 scattering systems



Path to hadronic properties from QCD

Lattice QCD offers a systematic avenue to compute multi-hadron amplitudes

- Follow Lüscher methodology — connect spectra to amplitudes
- Much success in 2-body sector   See talk by C. Thomas on Monday, August 17



Three hadron resonances from QCD

Past decade has seen tremendous progress extending these ideas to three particles

Non-relativistic EFT (NREFT)

H. Hammer, J. Pang, and A. Rusetsky
JHEP 09, 109 (2017), JHEP 10, 115 (2017)

Finite-volume unitarity (FVU)

M. Mai and M. Döring
Eur. Phys. J. A 53, 240 (2017), Phys. Rev. Lett. 122, 062503 (2019)

M. Döring, H. Hammer, M. Mai, J. Pang, A. Rusetsky, and J. Wu
Phys. Rev. D 97, 114508 (2018)

M. Mai, M. Döring, C. Culver, and A. Alexandru
Phys.Rev.D 101, 054510 (2020)

All-orders relativistic field theory (RFT)

M. Hansen and S. Sharpe
Phys. Rev. D 90, 116003 (2014), Phys. Rev. D 92, 114509 (2015),
Phys. Rev. D 95, 034501 (2017)

R. Briceño, M. Hansen, and S. Sharpe
Phys. Rev. D 95, 074510 (2017), Phys. Rev. D 98, 014506 (2018),
Phys. Rev. D 99, 014516 (2019)

R. Briceño, M. Hansen, S. Sharpe, and A. Szczepaniak
Phys. Rev. D 100, 054508 (2019)

T. Blanton, F. Romero-Lopéz, and S. Sharpe
JHEP 03, 106 (2019)

F. Romero-Lopéz, S. Sharpe, T. Blanton,
R. Briceño, and M. Hansen
JHEP 10, 007 (2019)

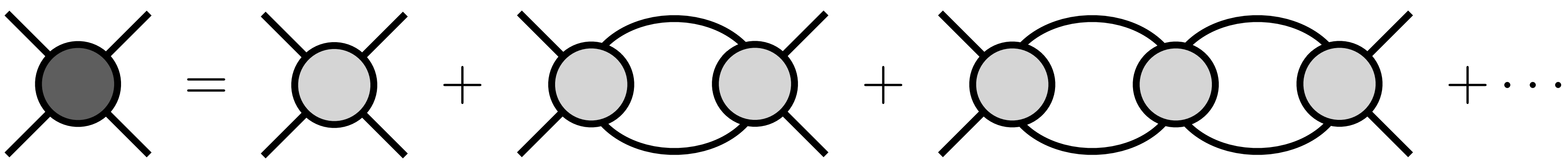
M. Hansen, F. Romero-Lopéz, and S. Sharpe
JHEP 07, 047 (2020)

T. Blanton, and S. Sharpe
arXiv:2007.16188, arXiv:2007.16190

On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

- e.g. $\mathbf{2} \to \mathbf{2}$



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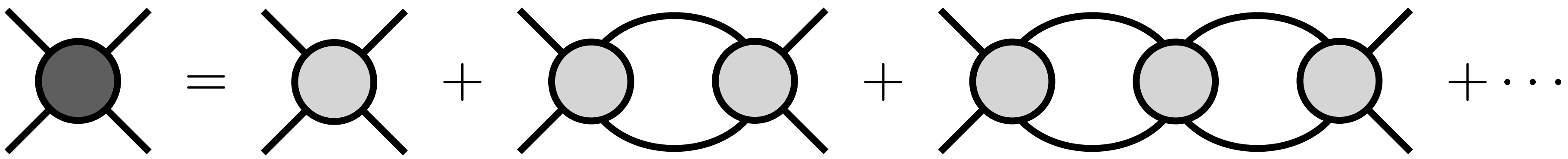

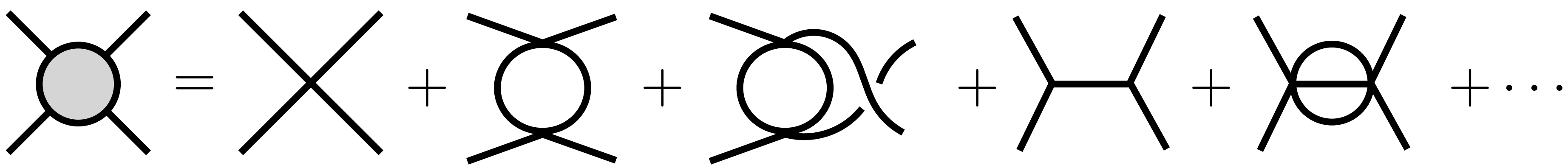

All 2PI diagrams - left hand cuts and higher multi-particle thresholds

On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

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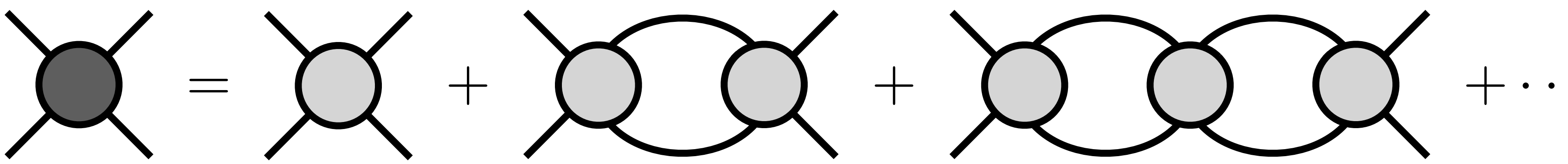

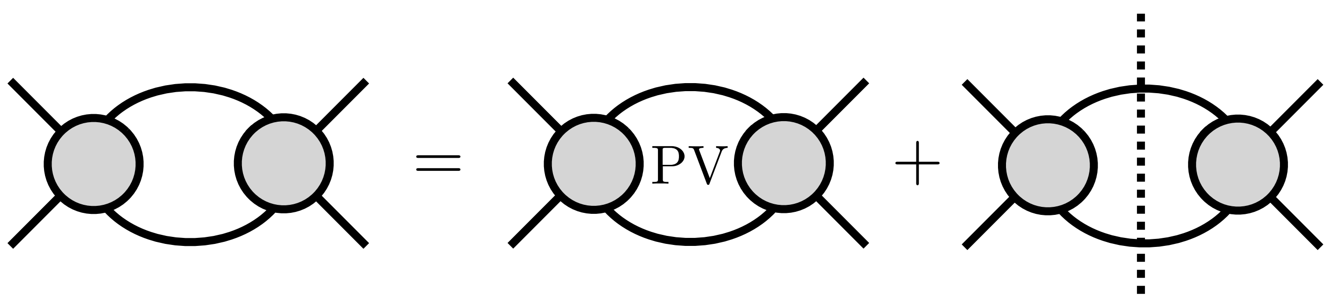

$$\rho = \frac{q}{8\pi E} \sim \sqrt{s - s_{\rm th}}$$

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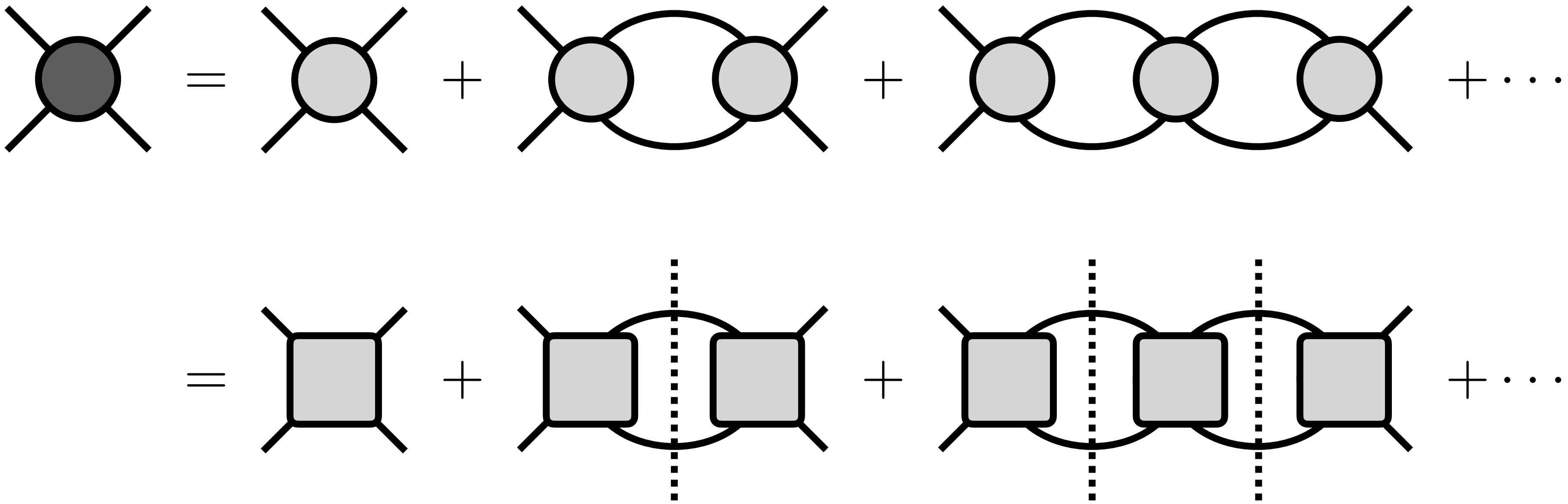
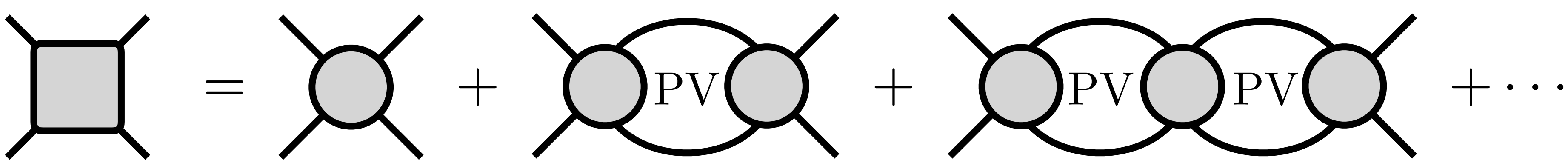


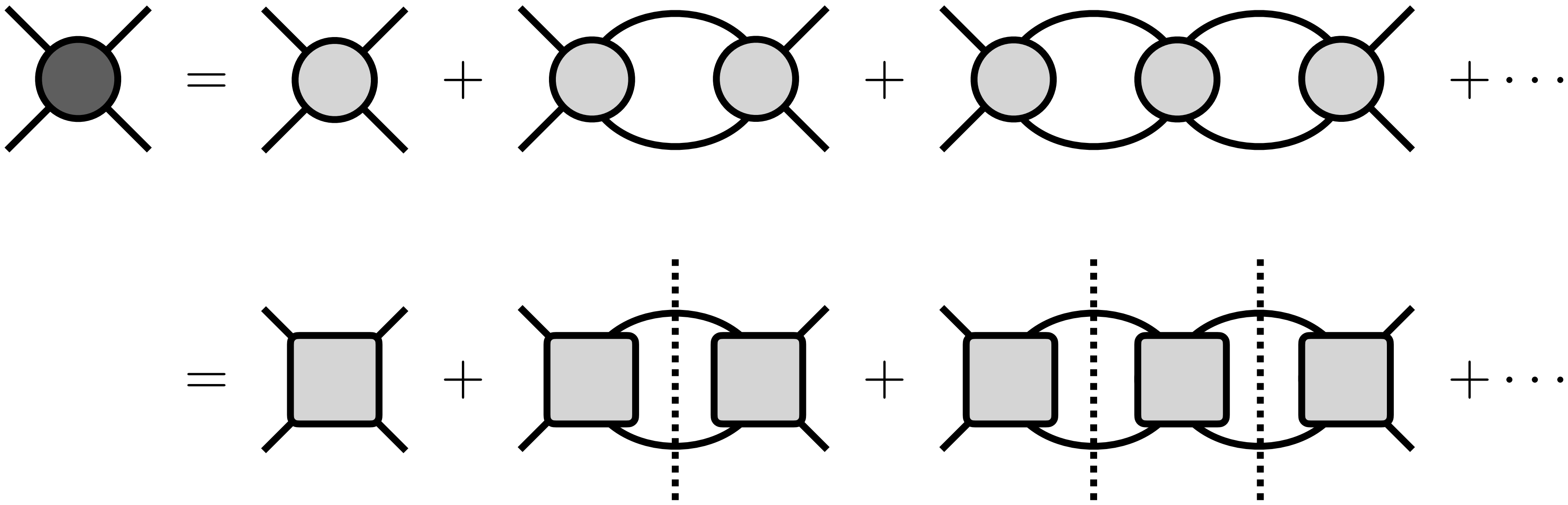


K matrix — unknown dynamical function unconstrained by unitarity

On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

- e.g. $\mathbf{2} \to \mathbf{2}$



$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 i\rho \mathcal{M}_2$$

For given K matrix, obtain on-shell solution for amplitude

On-shell scattering amplitudes from RFT

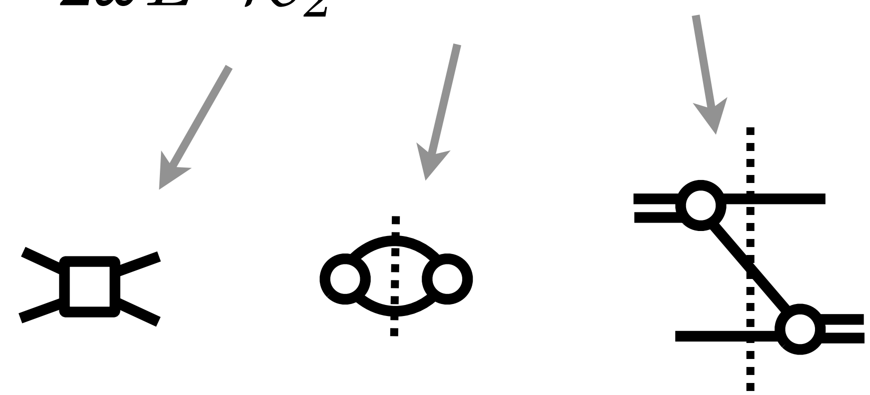
Repeat on-shell separation for $\mathbf{3} \rightarrow \mathbf{3}$

- For FV correlation functions — obtain condition relating spectrum to *K* matrix

$$\det\left[\, 1 + \mathcal{K}_{3,\mathrm{df}} F_{3,L} \,\right] = 0$$

$$F_{3,L} = F_{2,L}\left(\frac{1}{3} - \frac{1}{H_L} F_{2,L}\right)$$

$$H_L = \frac{1}{2\omega L^3\, \mathcal{K}_2} + F_{2,L} + G_L$$



Features

- *Relativistic*
- *Model independent*

Assumptions

- *Spinless particles*
- *No coupling to 2 particles*
- *Energies below coupled channel or 4 particles*

For details see literature

M. Hansen and S. Sharpe
Phys. Rev. D 90, 116003 (2014), Phys. Rev. D 92, 114509 (2015)

On-shell scattering amplitudes from RFT

Repeat on-shell separation for $\mathbf{3} \to \mathbf{3}$

- For FV correlation functions — obtain condition relating spectrum to *K* matrix
- For amplitudes — obtain integral equations for given *K* matrix

Features

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$$\mathcal{M}_3 = \mathcal{S}\{\, \mathcal{D} + \mathcal{L}\mathcal{T}\mathcal{L}^\top \,\}$$

$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \int \mathcal{M}_2 G \mathcal{D}$$

$$\mathcal{L} = \frac{1}{3} + \mathcal{M}_2 i\rho + \mathcal{D} i\rho$$

$$\mathcal{T} = \mathcal{K}_{3,\mathrm{df}} + \iint \mathcal{K}_{3,\mathrm{df}}\, i\rho\, \mathcal{L}\, \mathcal{T}$$

For details see literature

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On-shell scattering amplitudes from RFT

Repeat on-shell separation for $\mathbf{3} \to \mathbf{3}$

- For FV correlation functions — obtain condition relating spectrum to *K* matrix
- For amplitudes — obtain integral equations for given *K* matrix

$$\mathcal{M}_3 = \mathcal{S}\{\,\mathcal{D} + \mathcal{L}\mathcal{T}\mathcal{L}^\top\,\}$$

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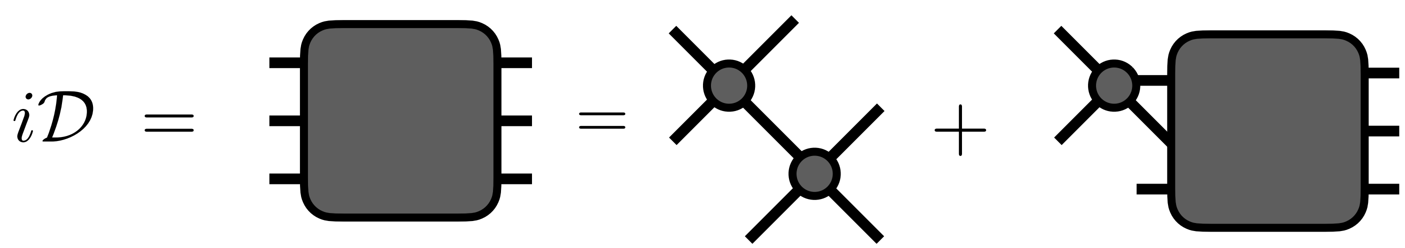
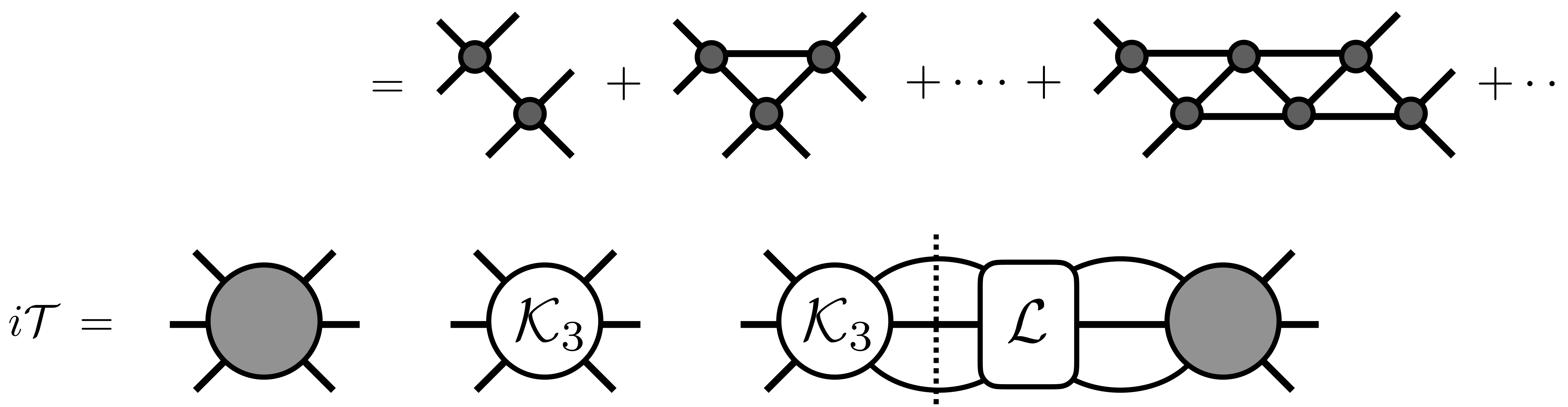




On-shell scattering amplitudes from RFT

Repeat on-shell separation for $\mathbf{3} \to \mathbf{3}$

- For correlation functions — obtain condition relating spectrum to K matrix
- For amplitudes — obtain integral equations for given K matrix
- Path to 3-body resonance parameters

$$\{E_{\mathfrak{n}}\} \longrightarrow \mathcal{K}_{3,\mathrm{df}} \longrightarrow \mathcal{M}_3 \longrightarrow s_{\text{res}}$$

Quantization Condition

Integral Equations

Analytic Continuation

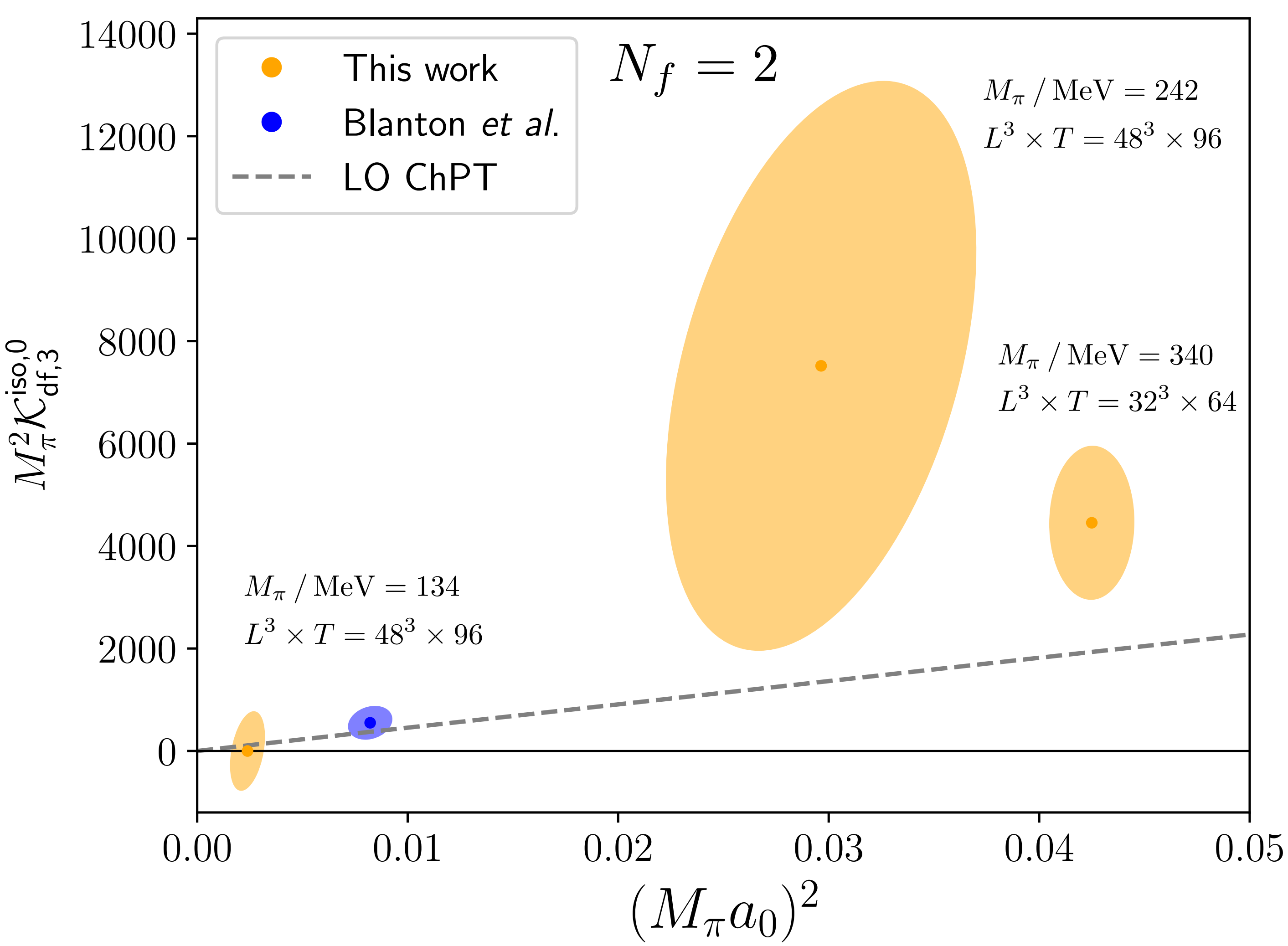
First lattice QCD spectra

First applications of this formalism being tested on $3\pi^+$

B. Hörz and A. Hanlon, Phys. Rev. Lett. 123, 142002 (2019) — See talk by B. Hörz on Monday, August 31

C. Culver et al., Phys. Rev. D 101, 114507 (2020) — See talk by M. Mai on Monday, August 31

M. Fischer et al. (ETMC), arXiv:2008.03035 (2020)

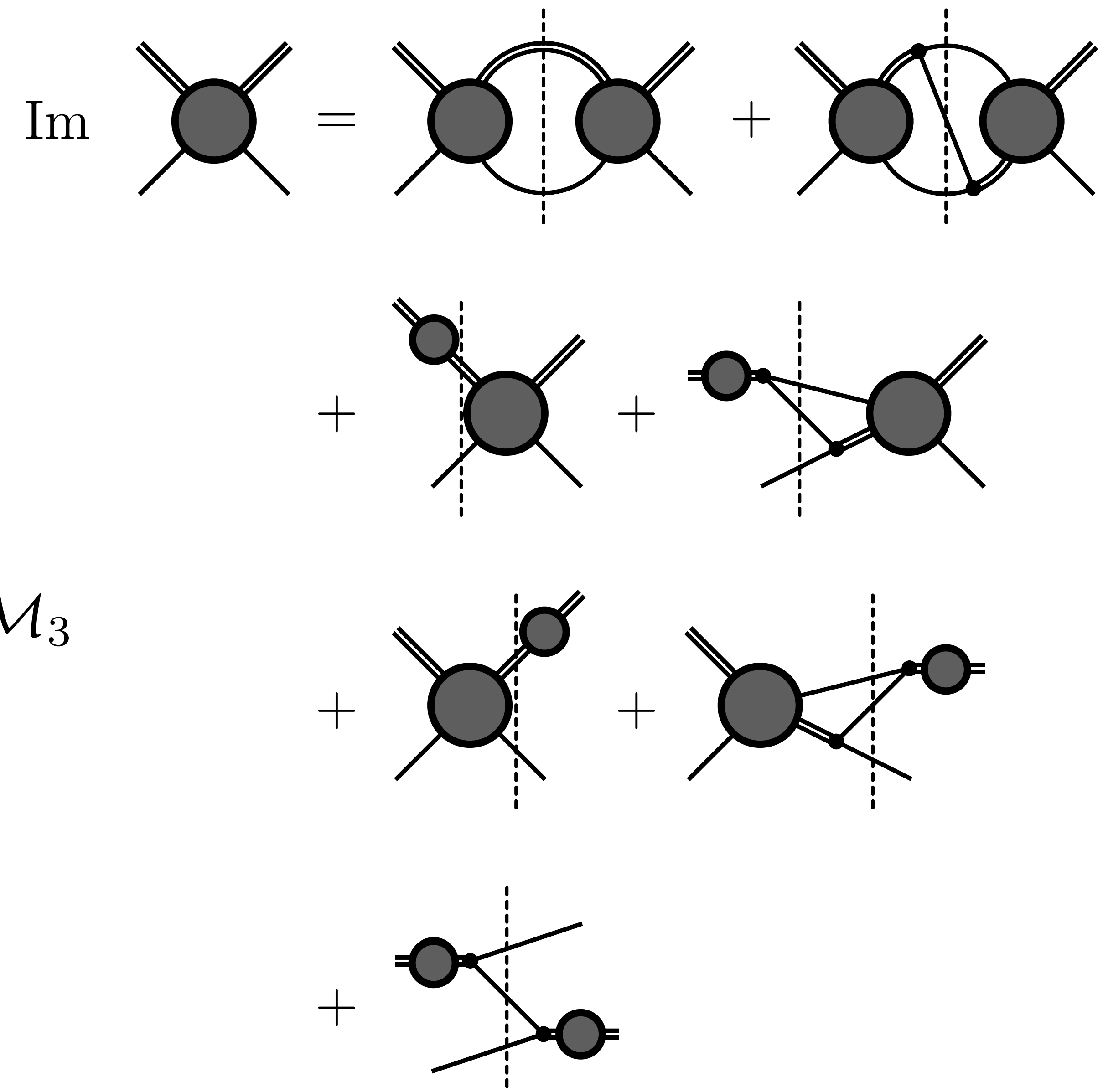


No complete amplitudes from LQCD to date — requires solving the integral equations for given K matrix

On-shell scattering amplitudes from unitarity

Start with *S* matrix unitarity

- Provides constrain for amplitude on real axis in physical region
- On-shell amplitude satisfies linear equation — check unitarity constraint



$$\mathcal{M}_3 = \mathcal{M}_2(\mathcal{R} - G)\mathcal{M}_2 + \int \mathcal{M}_2(\mathcal{R} - G)\mathcal{M}_3$$

R is a different short-distance function

M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak
Eur. Phys. J. A 53, 177 (2017)

AJ et al. [JPAC],
Eur. Phys. J. C79, no. 1, 56 (2019)

M. Mikhasenko, AJ et al. [JPAC],
Phys. Rev. D98, 096021 (2018)

M. Mikhasenko, AJ et al. [JPAC],
JHEP 08, 080 (2019)

Extension to FV

M. Mai and M. Döring
Eur. Phys. J. A 53, 240 (2017)

See talk by M. Mai on Monday, August 31

Equivalence of relativistic methods

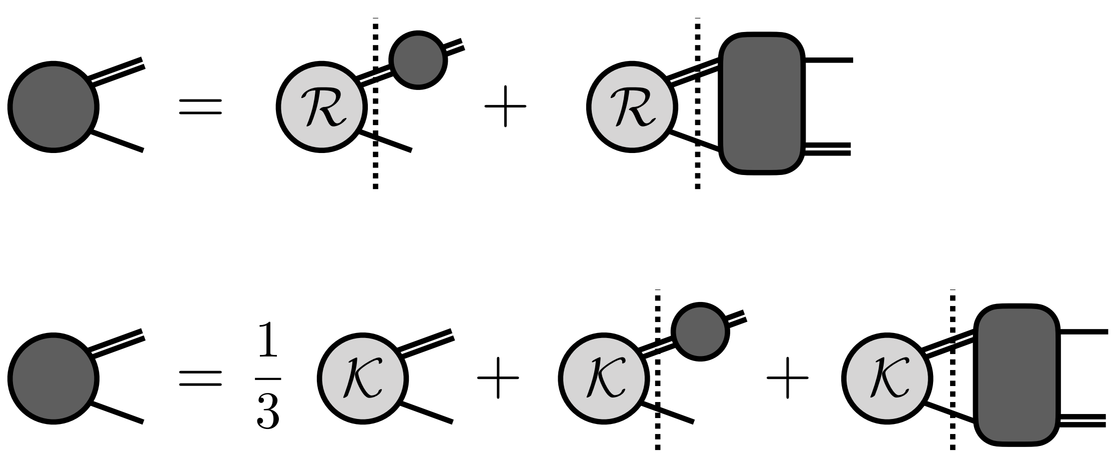
The RFT vs FVU methods can be summarized as "Bottom up" vs "Top down"

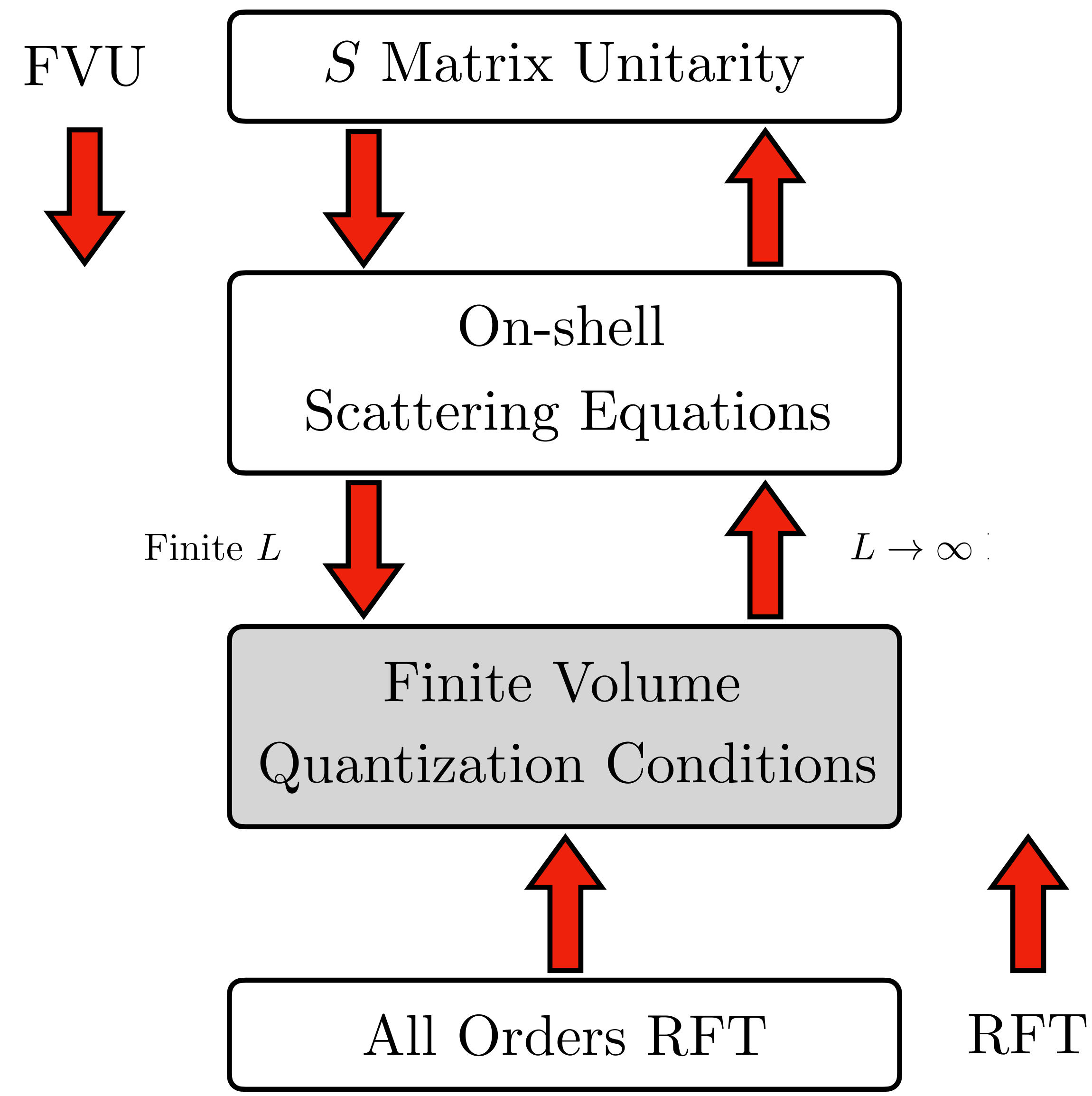
On-shell scattering equations are equivalent

AJ, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, S. Sharpe, and A. Szczepaniak et al., Phys. Rev. D100, 034508 (2019)

Quantization conditions are equivalent

T. Blanton and S. Sharpe, arXiv:2007.16190 (2020)





Small details, e.g. aspects of symmetrization — see literature for information

Solving the Integral Equations

Given a $\mathcal{K}_{3,\mathrm{df}}$, solve integral equations for scattering amplitude

$$\{E_{\mathrm{n}}\} \longrightarrow \mathcal{K}_{3,\mathrm{df}} \longrightarrow \mathcal{M}_3 \longrightarrow s_{\mathrm{res}}$$

Quantization Condition

Integral Equations

Analytic Continuation

Set foundation by studying simplest case

- Three identical scalar particles
- All angular momentum in *S* wave
- Bound state in 2-body system
- Assume $\mathcal{K}_{3,\mathrm{df}} = 0$

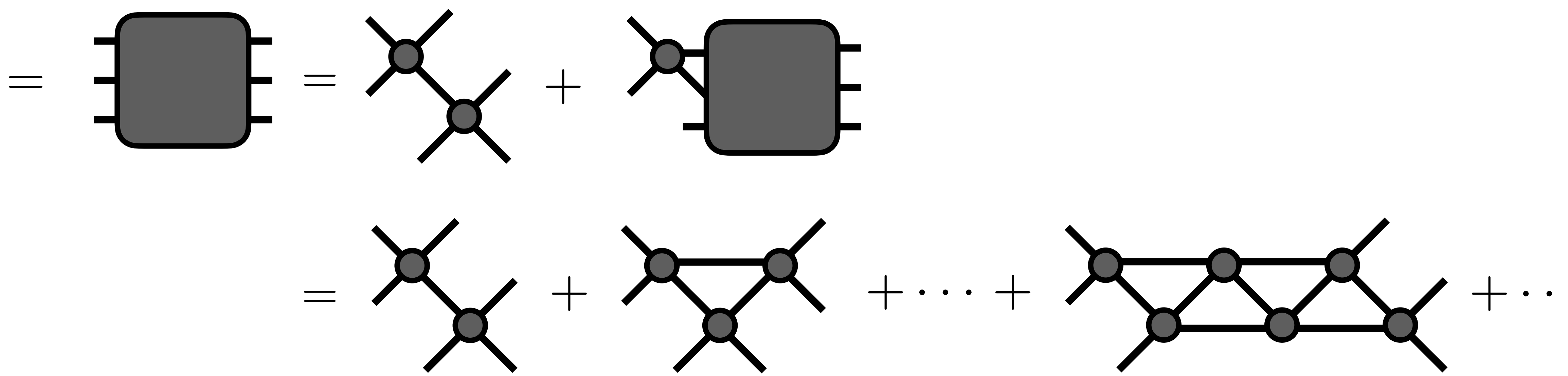
R. Briceño, S. Dawid, M.H. Islam, AJ, and C. McCarty
in progress

Solving the Integral Equations

Assume short-range *K* matrix is zero

- Base case for integral equations — non-zero *K* involves *K=0* case
- Series of increasing number of exchanges between 2-body systems

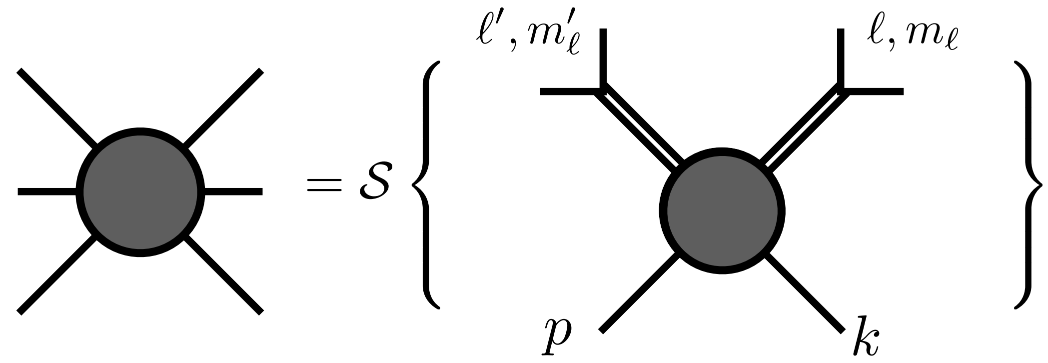
$$i\mathcal{M}_3 = i\mathcal{D}$$



Solving the Integral Equations

Amplitude in pair-spectator basis

- *S* wave dimers
- Remove initial and final state $\mathbf{2} \to \mathbf{2}$ amplitudes
- Project to total *S* wave



$$\mathcal{D} = \mathcal{M}_2 \, d \, \mathcal{M}_2$$

$$d \xrightarrow[J=0]{} d_s$$

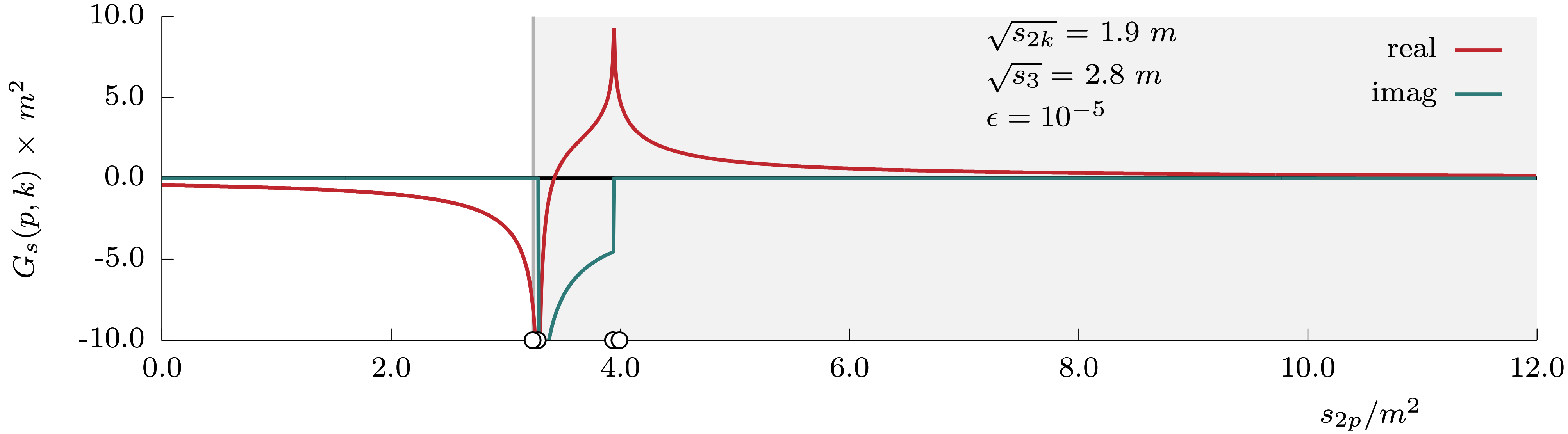
$$d_s(p,k) = -G_s(p,k) + \int \mathrm{d}k' \, K_s(p,k') \, d_s(k',k)$$

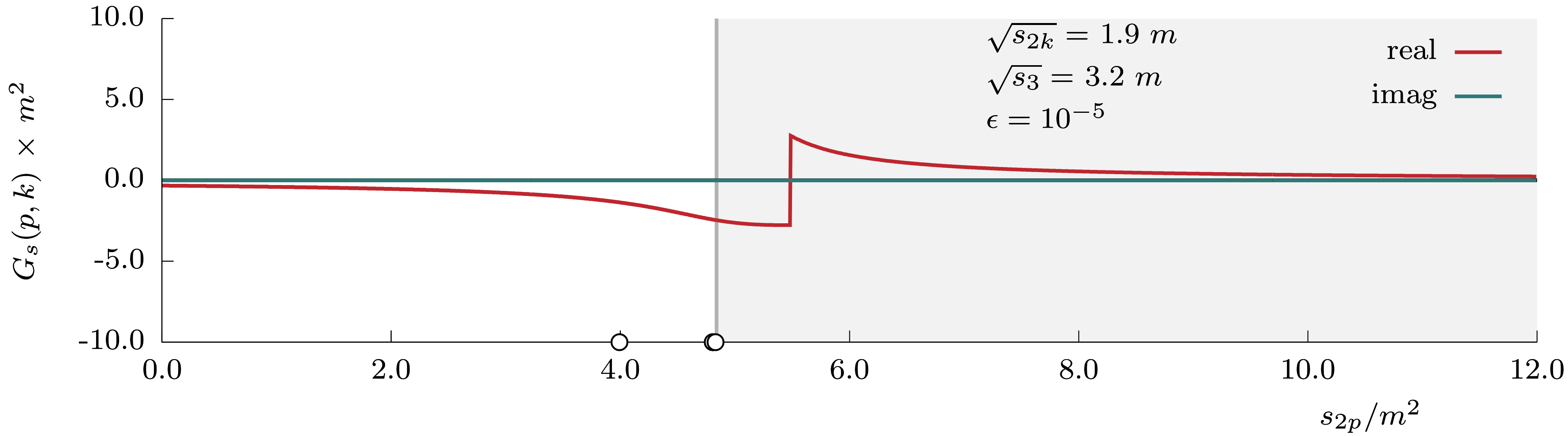
$$K_s(p,k) = -\frac{k^2}{(2\pi)^2 \omega_k} G_s(p,k) \mathcal{M}_2(k)$$

Solving the Integral Equations

Exchange amplitude yields logarithmic singularities

$$G = \frac{H(p,k)}{(P-k-p)^2 - m^2 + i\epsilon} \qquad\qquad G_s = -\frac{1}{4pk} H(p,k) \log\left(\frac{z(p,k)-1}{z(p,k)+1}\right)$$





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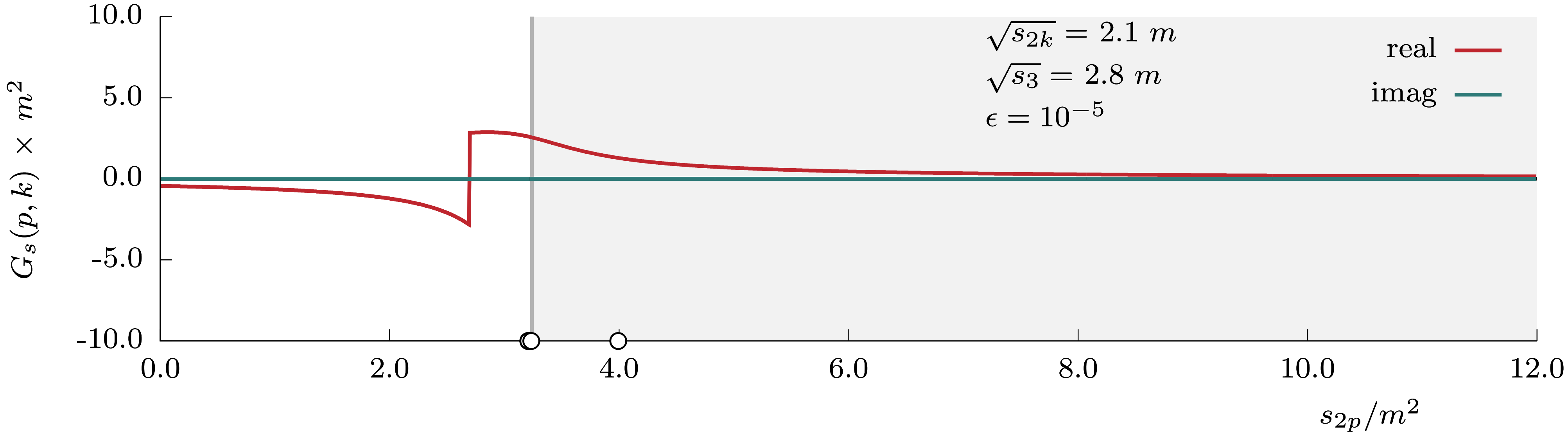
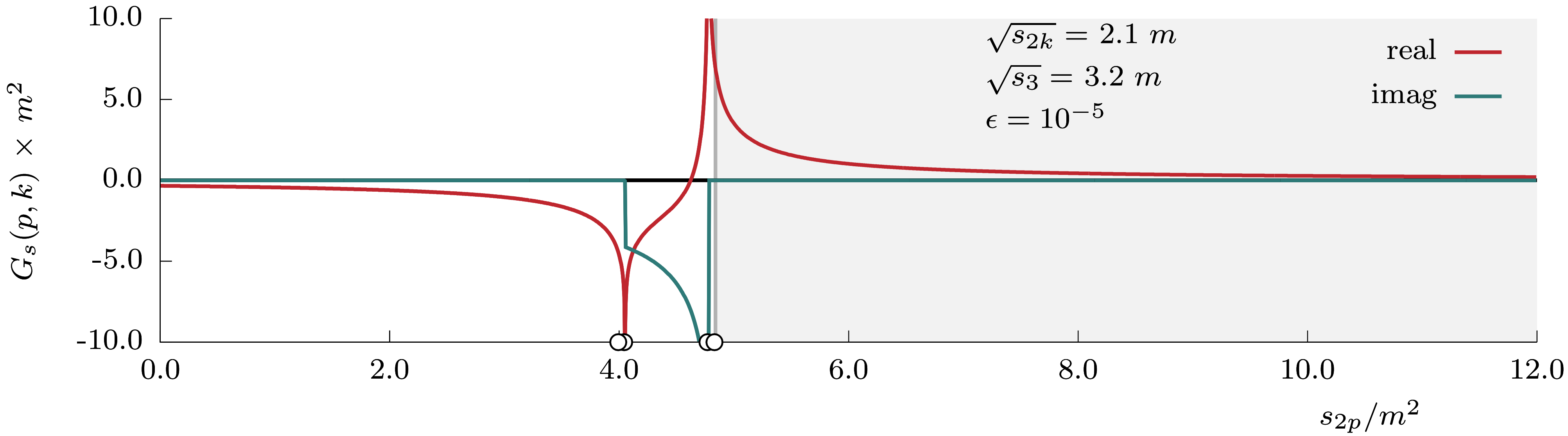
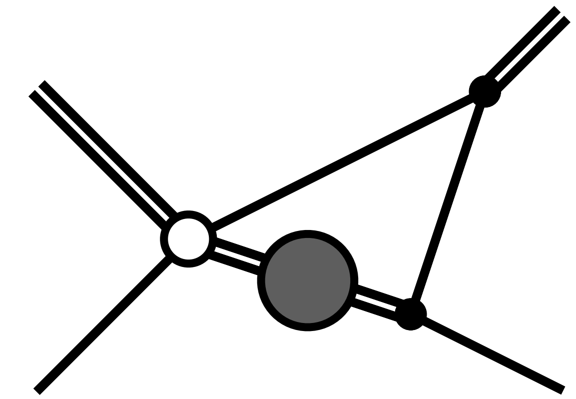


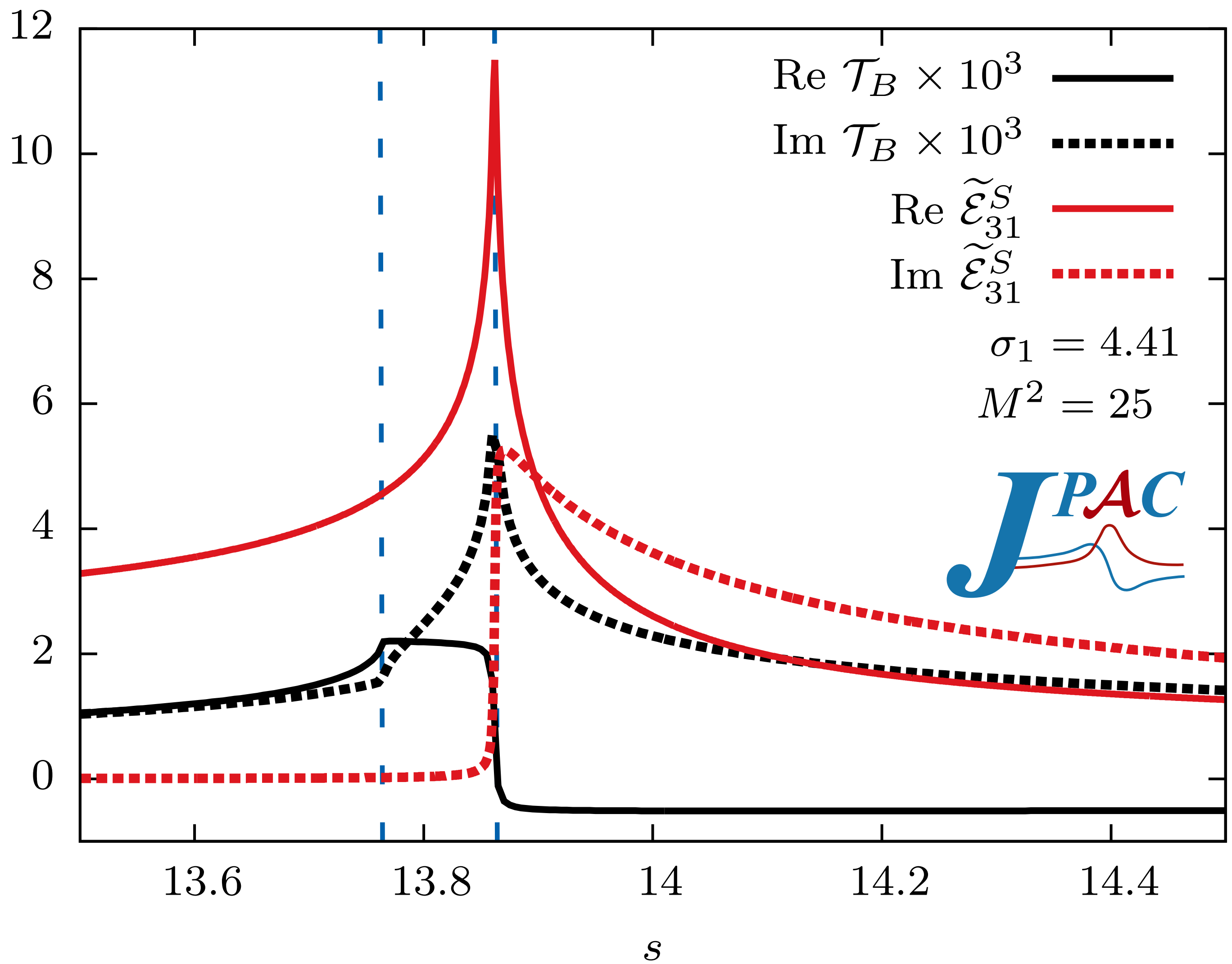


Solving the Integral Equations

Exchange amplitude yields logarithmic singularities

- Induces structures in amplitudes, e.g. triangle singularities

$\mathcal{T}_B =$ 



AJ et al. [JPAC],
Eur. Phys. J. C79, no. 1, 56 (2019)

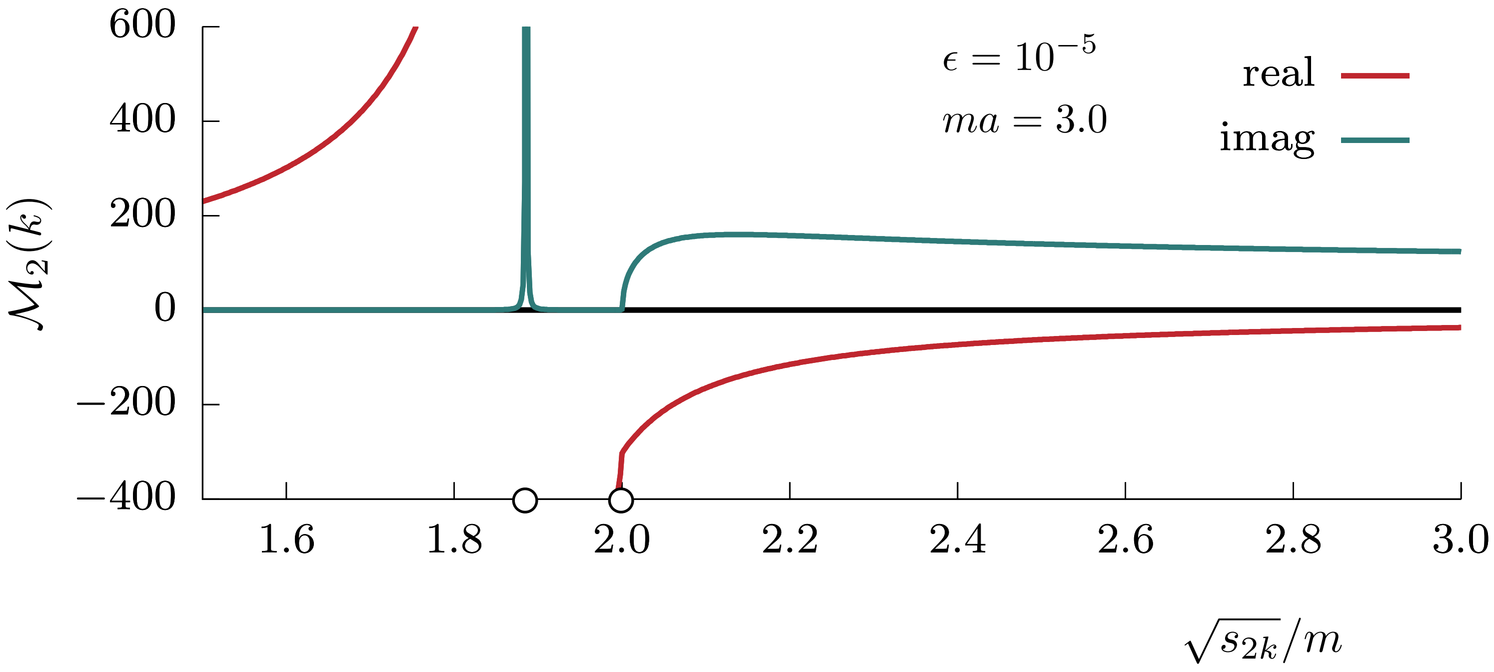
Solving the Integral Equations — 2+1 scattering

Focus on case where 2-body systems forms bound state

- Effective range expansion with scattering length a

$$\mathcal{M}_2(k) = 16\pi\sqrt{s_{2k}}\left(-\frac{1}{a} - \frac{i}{2}\sqrt{s_{2k} - 4m}\right)^{-1}$$

$$\sim \frac{-g^2}{s_{2k} - s_{2b}}$$



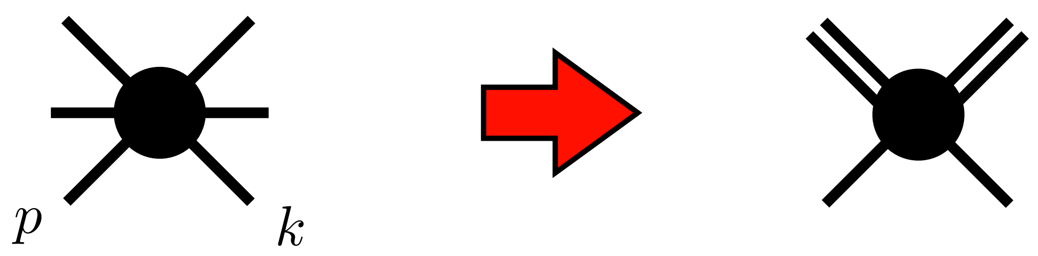
$$s_{2b} = 4\left(m^2 - \frac{1}{a^2}\right)$$

$$g^2 = 128\pi\sqrt{s_{2b}}/a$$

Solving the Integral Equations — 2+1 scattering

Factorize pole and residue for 2+1 scattering

$$\lim_{s_{2p}, s_{2k} \to s_{2b}} i\mathcal{M}_3(p,k) = ig \frac{i}{s_{2p} - s_{2b}} \, i\mathcal{M}_{\varphi b} \, \frac{i}{s_{2k} - s_{2b}} ig$$



$3\varphi \to 3\varphi$ $\qquad\qquad$ $\varphi b \to \varphi b$

- Relate to solution for d_s

$$\mathcal{M}_{\varphi b} = \lim_{s_{2k}, s_{2p} \to s_{2b}} g^2 \, d_s(p,k)$$

$$\mathcal{M}_{3\varphi \to \varphi b}(k) = \lim_{s_{2p} \to s_{2b}} g \, d_s(p,k)$$

Solving the Integral Equations — 2+1 scattering

Convert integral equation to linear equation

- Introduce regulators N (matrix size) and ϵ (pole shift)
- Recover amplitude in $N \to \infty,\ \epsilon \to 0^+$ limit

$$d_s = -G_s + \int K_s \, d_s \quad \Rightarrow \quad \mathbf{d}_s(k) = -\left[\mathbb{1} - \widetilde{\mathbb{K}}_s\right]^{-1} \mathbf{G}_s(k)$$

- Interpolate solution to bound state pole

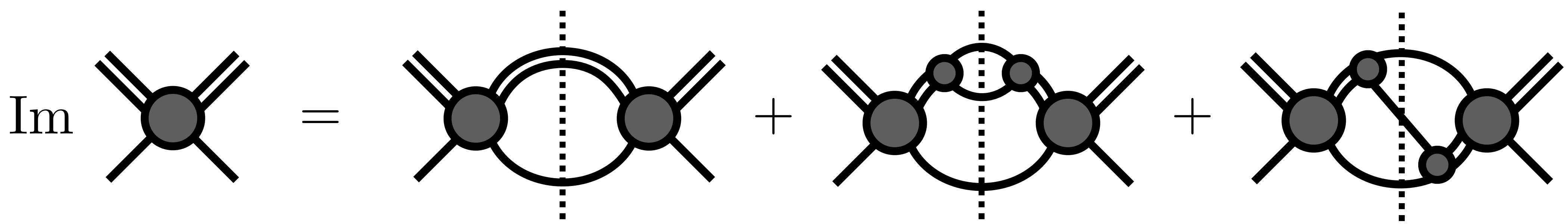
$$\mathcal{M}_{\varphi b}(N,\epsilon) = g^2 \, d_s(N,\epsilon)$$

$$\mathcal{M}_{\varphi b} = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} \mathcal{M}_{\varphi b}(N,\epsilon)$$

Solving the Integral Equations — 2+1 scattering

S matrix unitarity provides a way to check quality of solutions

- By construction, solutions are on-shell representations satisfying unitarity



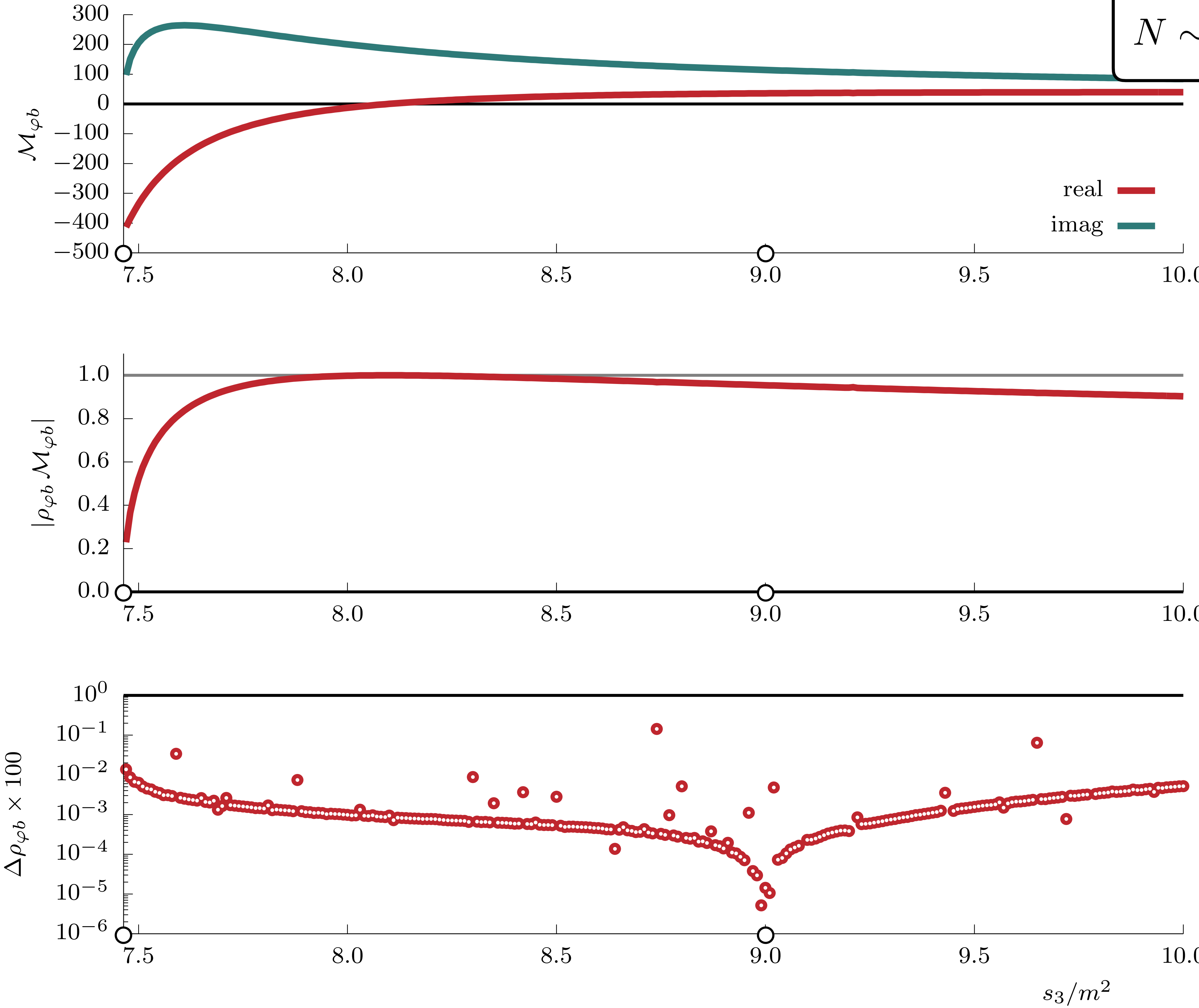
- Deviation from unitarity guides quality of solution
- Define deviation 'error'

$$\Delta\rho_{\varphi b} = 2 \left| \frac{\mathrm{Im}\,\mathcal{M}_{\varphi b}^{(\mathsf{LHS})} - \mathrm{Im}\,\mathcal{M}_{\varphi b}^{(\mathsf{RHS})}}{\mathrm{Im}\,\mathcal{M}_{\varphi b}^{(\mathsf{LHS})} + \mathrm{Im}\,\mathcal{M}_{\varphi b}^{(\mathsf{RHS})}} \right|$$

- For a given N and ϵ, accept solution if $\Delta\rho_{\varphi b}$ is small

Example of solution

$ma = 2$
$\epsilon/m^2 = 10^{-5}$
$N \sim 4500$

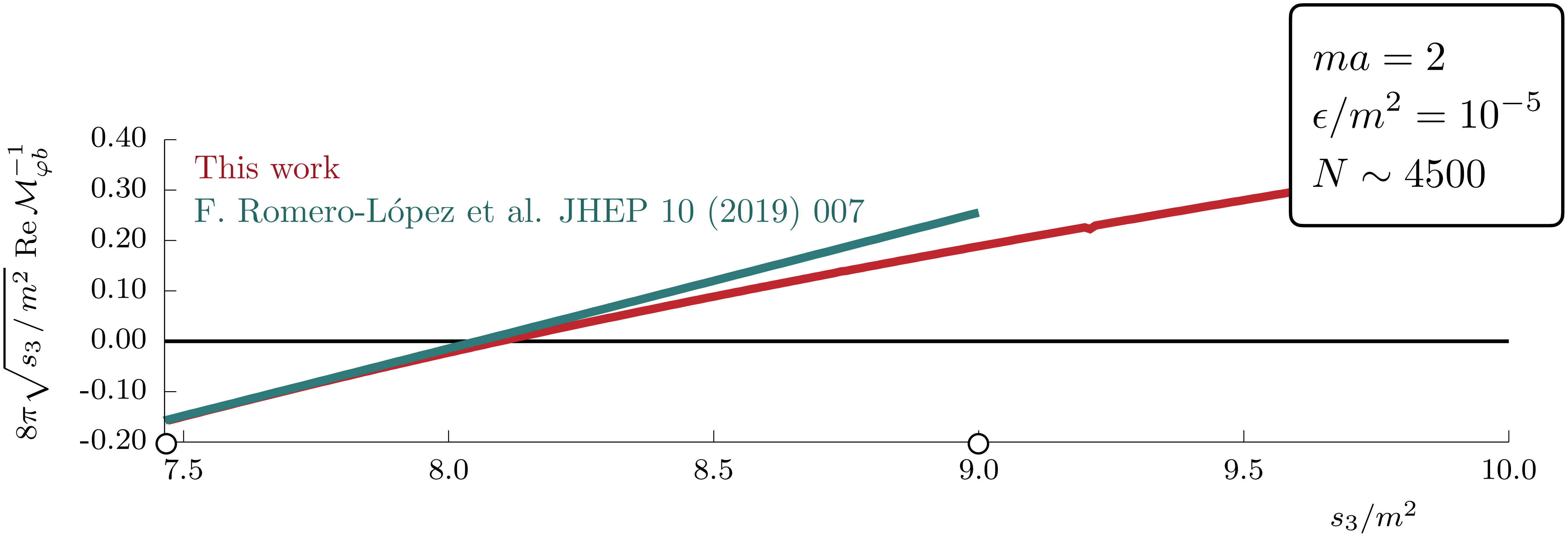


Example of solution

Agreement to FV studies of 2+1 system via QC3

$$8\pi\sqrt{s_3}\,\mathrm{Re}\,\mathcal{M}_{\varphi b}^{-1} = q\cot\delta_{\varphi b} = -\frac{1}{b_0} + \frac{1}{2}r_0 q^2$$

$$mb_0 = 6.357(4)$$
$$mr_0 = 2.292(2)$$



Example of solution

Agreement to FV studies of 2+1 system via QC3

$$8\pi\sqrt{s_3}\,\mathrm{Re}\,\mathcal{M}_{\varphi b}^{-1} = q\cot\delta_{\varphi b} = -\frac{1}{b_0} + \frac{1}{2}r_0 q^2$$

$$\Rightarrow \quad mb_0 = 6.357(4), \qquad mr_0 = 2.292(2)$$

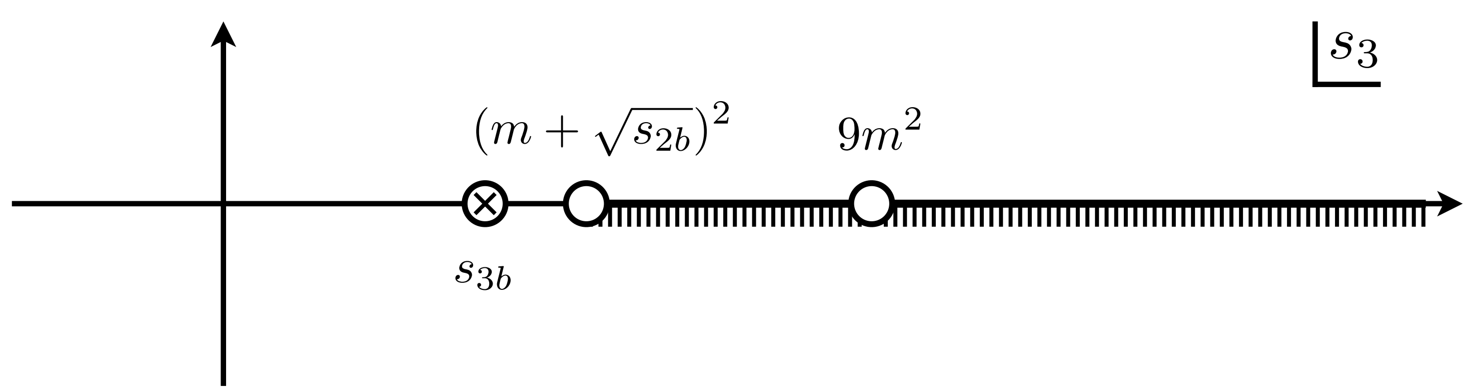
Suggestion of bound state in 2+1 system

$ma = 2$

$s_{2b}/m^2 = 3$

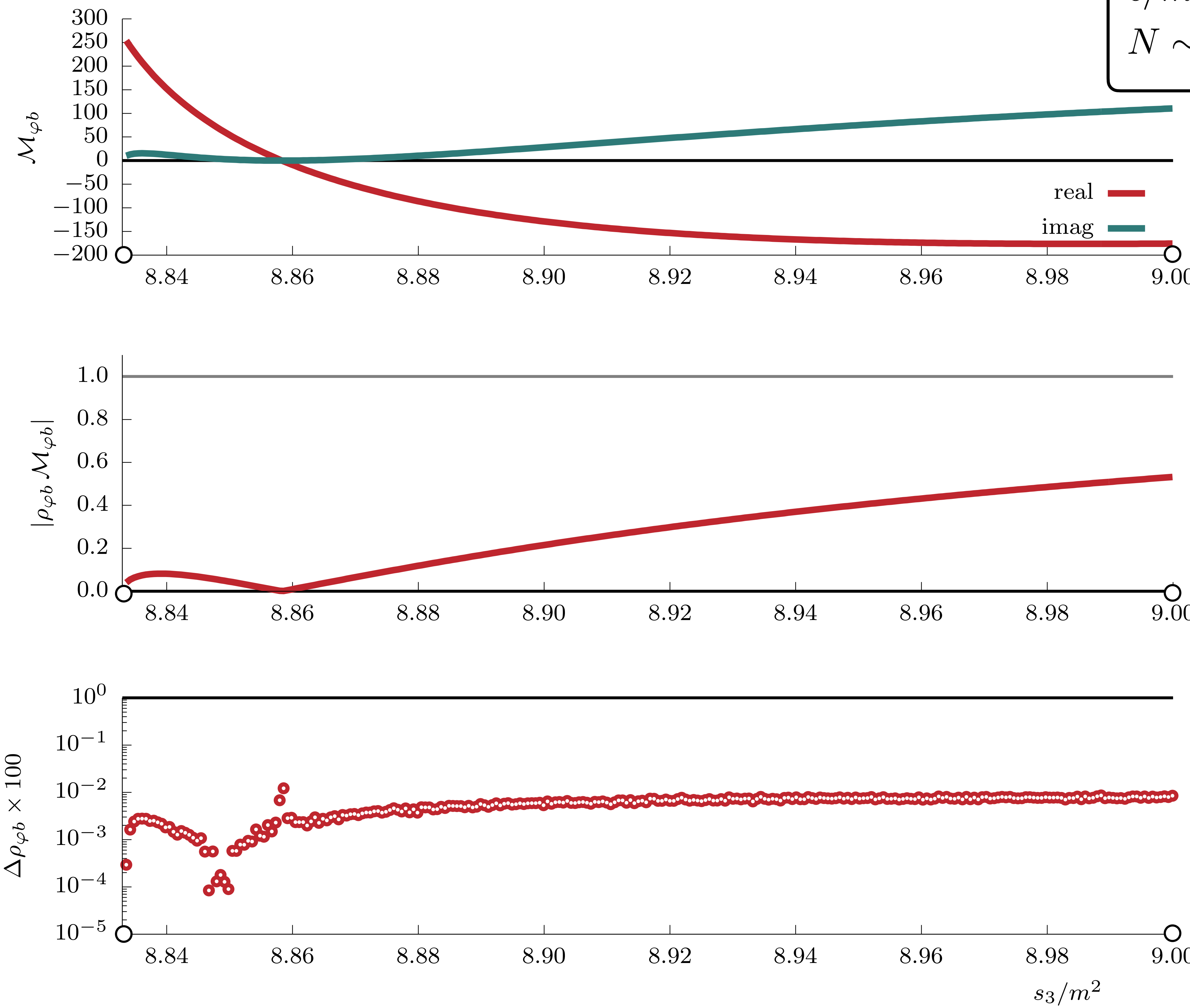
$(1+\sqrt{s_{2b}/m^2})^2 \sim 7.46$

$s_{3b}/m^2 \sim 7.28$



Assessing systematics

$ma = 6$
$\epsilon/m^2 = 10^{-11}$
$N \sim 9200$



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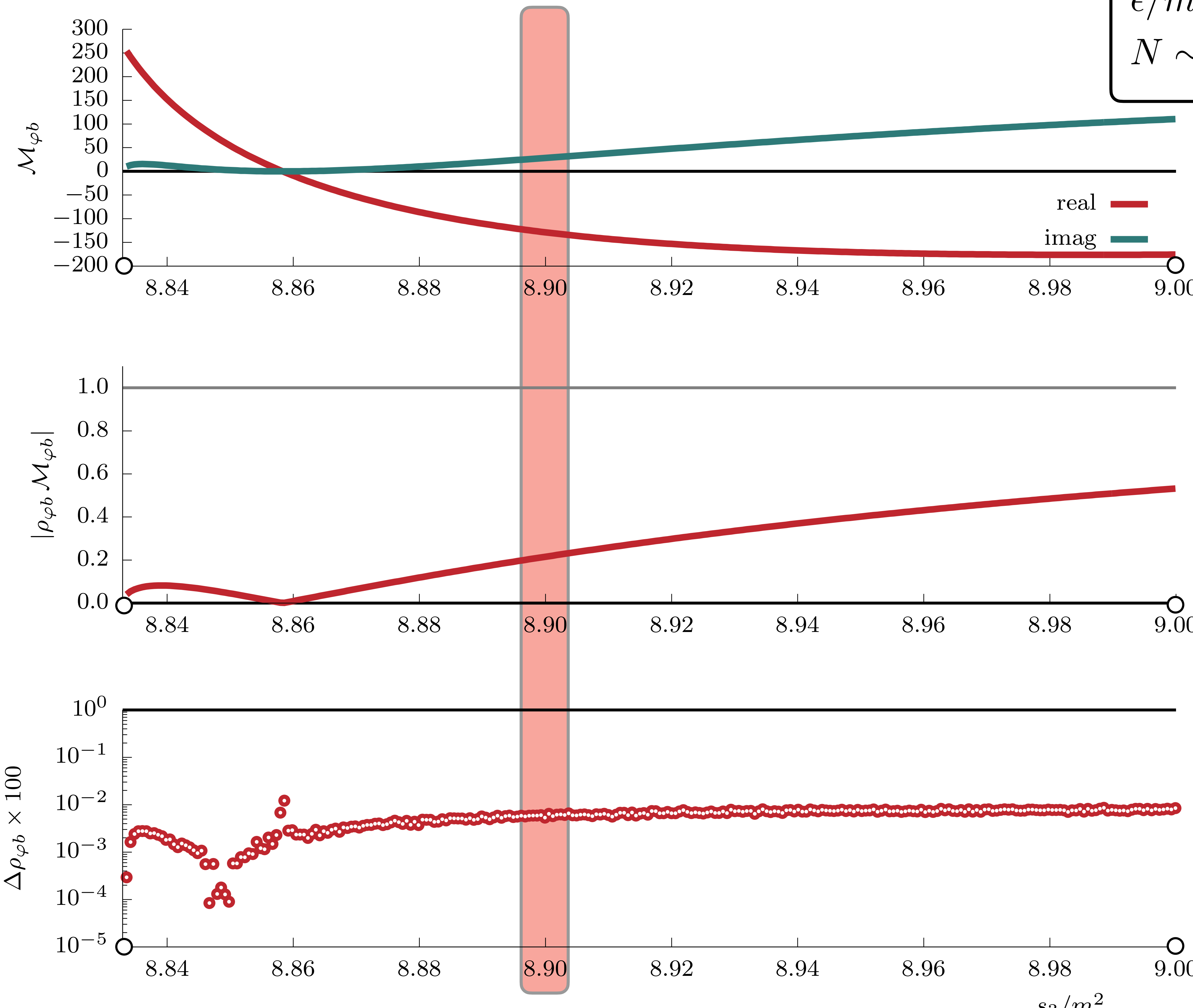
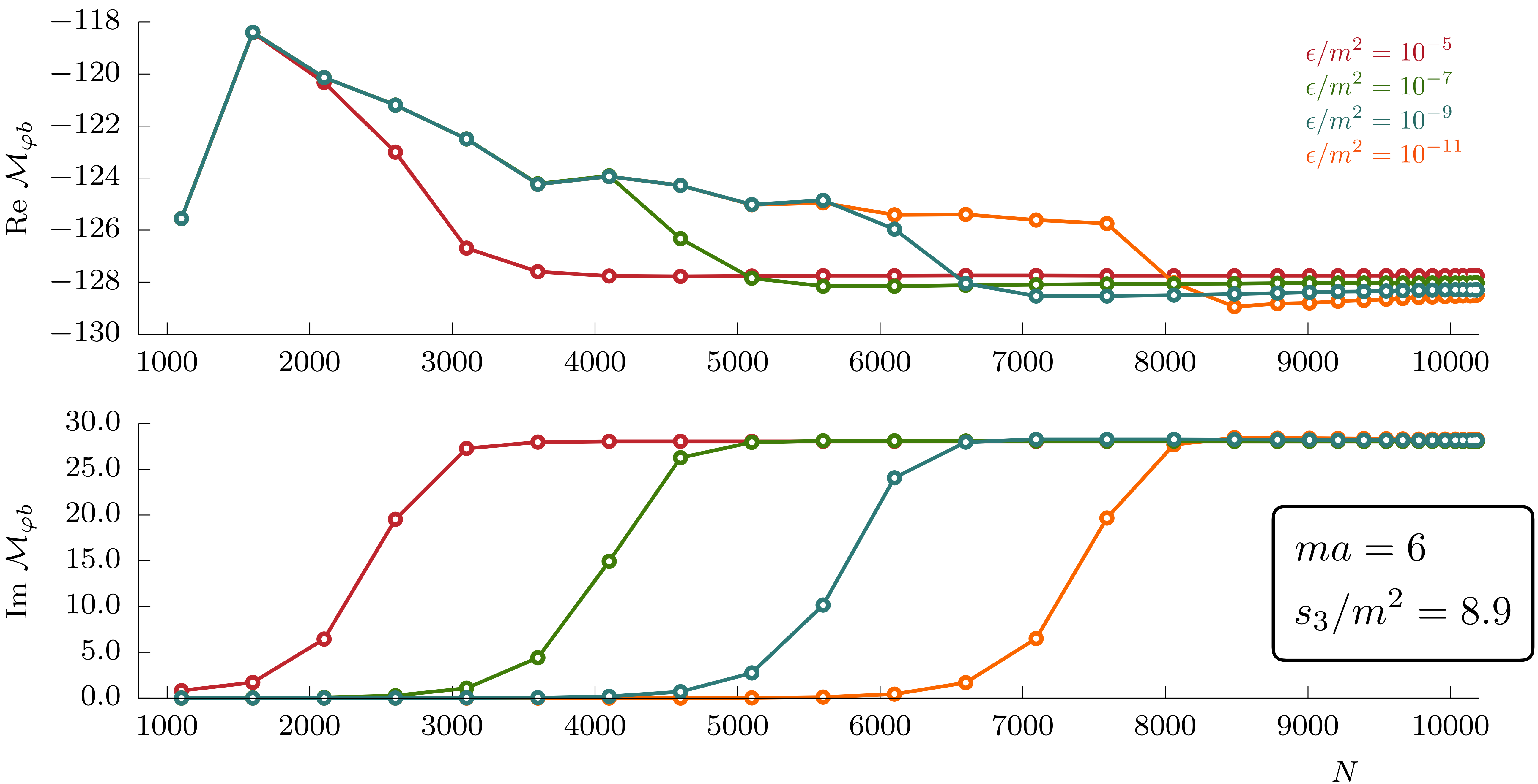


Assessing systematics

Physical scattering amplitude recovered in $N \to \infty,\ \epsilon \to 0^+$ limit

$$\mathcal{M}_{\varphi b} = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} \mathcal{M}_{\varphi b}(N, \epsilon)$$



Assessing systematics

Physical scattering amplitude recovered in $N \to \infty,\ \epsilon \to 0^+$ limit

- Poisson summation formula relates N and ϵ

$$\left[\sum_x \Delta x - \int \mathrm{d}x\right] \frac{1}{x^2 - x_0^2 + i\epsilon} \sim e^{-2\pi\epsilon/\Delta x}$$

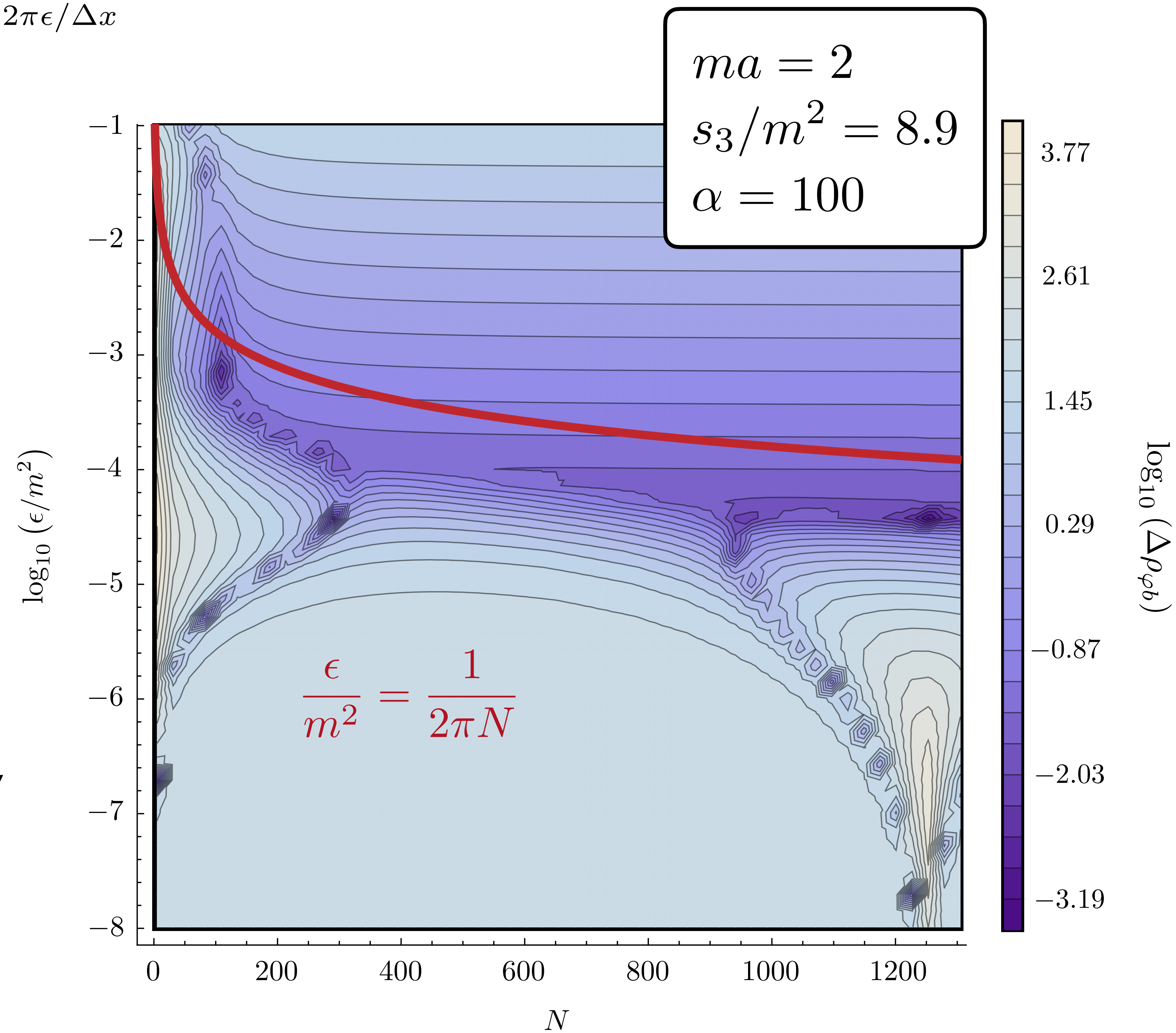
$$\implies \epsilon > \frac{1}{2\pi N} \qquad \Delta x = \frac{1}{N}$$

$$N = 10^3 \implies \epsilon > 10^{-4}$$

$$N = 10^4 \implies \epsilon > 10^{-5}$$

- Not all ϵ allowed for given N
- Ordered limit must be taken carefully

$$\mathcal{M}_{\varphi b} = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} \mathcal{M}_{\varphi b}(N, \epsilon)$$



Summary

First steps toward solving 3-body integral equations — Much more to explore

- Performing studies of stability of solutions as function of ϵ and N
- Exploring discretization schemes to improve convergence
- Estimates of systematic error R. Briceño, S. Dawid, M.H. Islam, AJ, and C. McCarty — *in progress*

Possible (immediate) future extensions

- Non-zero J — trivial
- Higher spin dimers — not so trivial, but doable
- 2-body resonances

Applications to lattice QCD data

- Extensions for more interesting channels, e.g. 3π in $I = 0,1$

Outstanding issues

- Analytic continuation below threshold / complex energy plane

Solution strategies

Large values near poles

Brute Force

- Include finite $+i\epsilon$ prescription for BS pole

Semi-Analytic

$$\mathcal{M}_2(s_{2k'}) = g^2\, i\pi\delta(s_{2k'} - s_b) + \lim_{\epsilon\to 0} \Delta\mathcal{M}_2(s_{2k'}, \epsilon)$$

- Remove contribution from pole

- Residual ϵ-dependence from removing imaginary part

$$\delta_\epsilon(s_{2k'} - s_b) = \frac{\epsilon}{\pi\left((s_{2k'} - s_b)^2 + \epsilon^2\right)}$$

Find consistent solutions — some improvement on unitarity checks, but not always

Solving the Integral Equations

Discretize integral equation $\Longrightarrow$ Obtain matrix equation

$$\int_0^{k_{\mathsf{max}}} \mathrm{d}k\, f(k) \longrightarrow \sum_{n=0}^{N-1} \Delta k_n\, f(k_n)$$

$$k_{\mathsf{mesh}} = \{\, k_0 = 0,\, k_1,\, k_2,\, \ldots,\, k_{N-1},\, k_N = k_{\mathsf{max}} \,\}$$

$$\Delta k_n = k_{n+1} - k_n$$

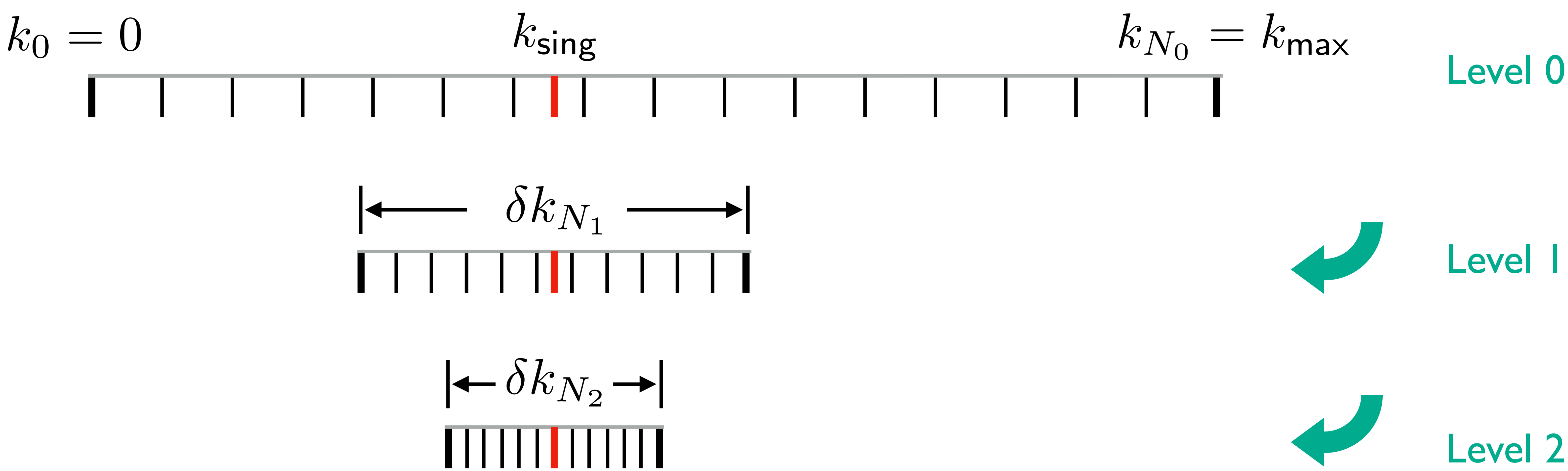
- Integration can run over two-body bound-state pole — Poses numerical challenges
- Construct variable mesh to allow finer sampling near pole

Refinement meshing

Discretize integral equation $\Longrightarrow$ Obtain matrix equation $\int_0^{k_{\text{max}}} \mathrm{d}k\, f(k) \longrightarrow \sum_{n=0}^{N-1} \Delta k_n\, f(k_n)$

- Integration can run over two-body bound-state pole — Poses numerical challenges

- Construct variable mesh to allow finer sampling near pole



- Alternative meshing schemes being studied — more later

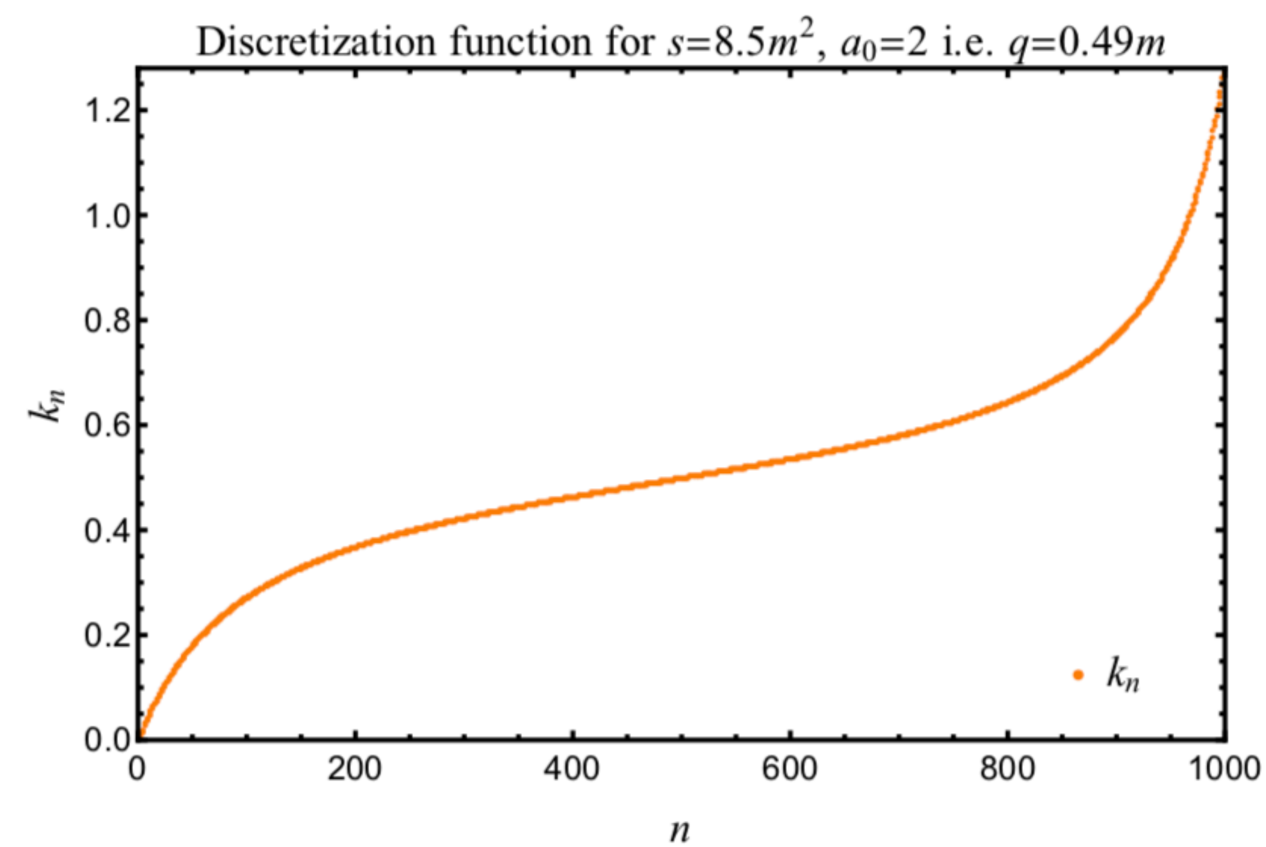
Meshing techniques

Refinement scheme may not be most efficient meshing technique

- Requires a priori knowledge of exact location of pole
- Does not capture other features of kernel, e.g. logarithmic enhancements
- Not flexible for other non-bound systems
- For different parameter sets, may need drastically different tunings

Testing variety of alternative methods

- Mapping functions
- Dynamical
- Adaptive



(a) $\alpha = 10$.

Assessing systematics

We have qualitatively good solutions — need to quantify systematic errors

- Physical scattering amplitude recovered in $N \to \infty,\ \epsilon \to 0^+$ limit
- What are size of errors due to finite N and ϵ
- Notice real part more volatile than imaginary part — manifestation of unitarity?

Workflow

- For given energy and ϵ, compute solutions for various N,

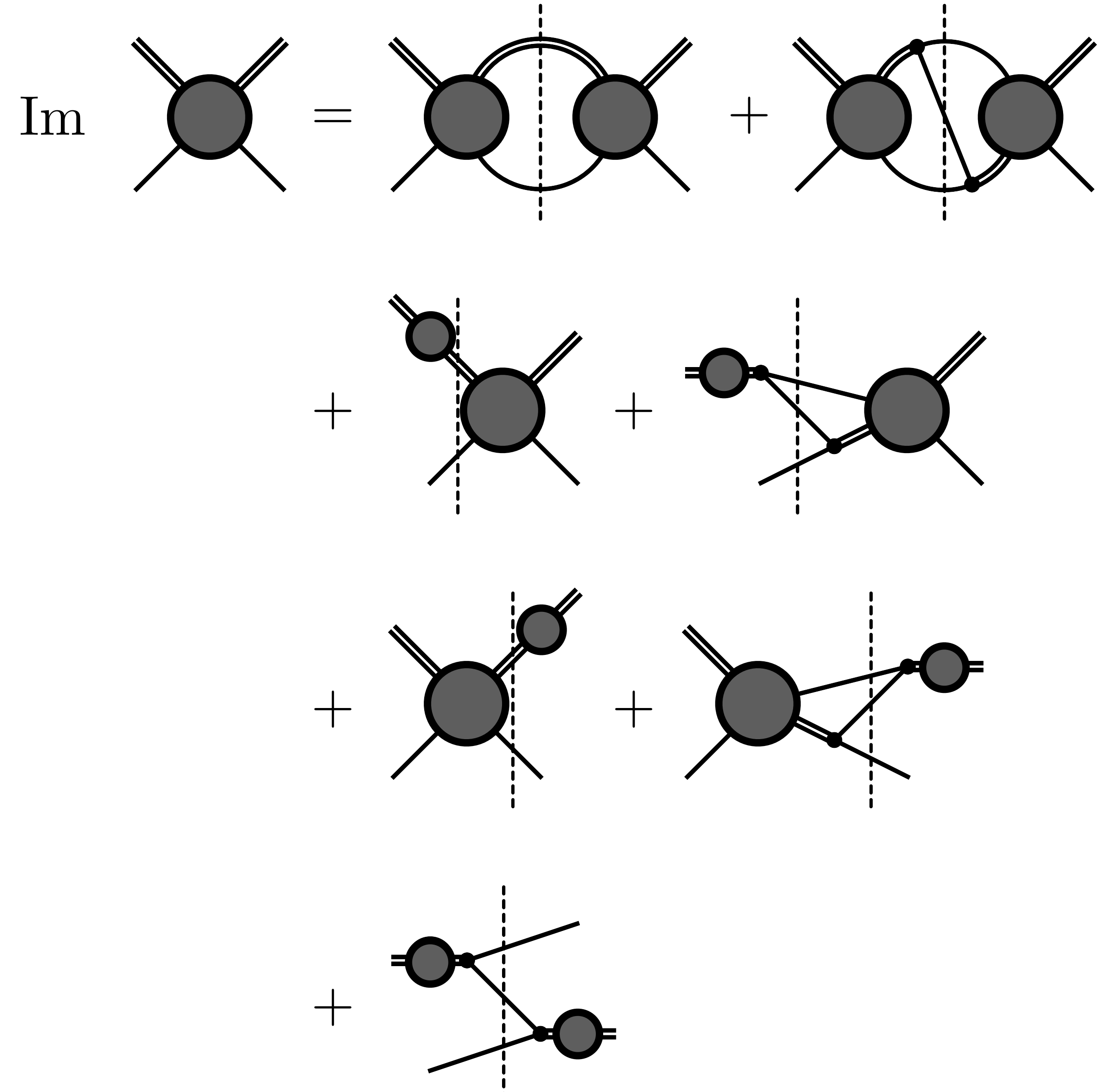
  e.g. $s_3/m^2 = 8.9, \epsilon = 10^{-5}, 10^3 \leq N \leq 10^4$.

- Solutions qualitatively pass unitarity check, e.g. $\Delta\rho_{\varphi b} \lesssim 1\,\%$
- Repeat for same energy at various ϵ, e.g. $10^{-7} \leq \epsilon \leq 10^{-3}$
- For a given energy, extract a nominal value for amplitude as well as associated error
- Repeat for all energies desired

On-shell scattering amplitudes from unitarity

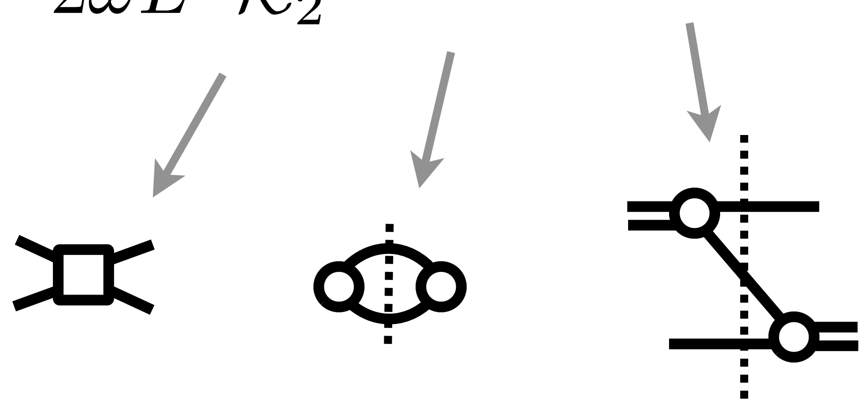
Start with S matrix unitarity

- Provides constrain for amplitude on real axis in physical region
- Make *ansatz* for linear equation for amplitude — check unitarity constraint
- FVU — take on-shell rep, go to finite volume



$$\det\left[H_L - \frac{1}{2\omega L^3}\mathcal{R}\frac{1}{2\omega L^3} \right] = 0$$

$$H_L = \frac{1}{2\omega L^3\,\mathcal{K}_2} + F_{2,L} + G_L$$



M. Mai and M. Döring
Eur. Phys. J. A 53, 240 (2017)

See talk by M. Mai on Monday, August 31