Khuri-Treiman equations: review and applications

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Accessing and Understanding the QCD Spectra
(INT-20-2c)
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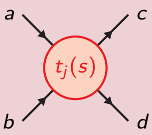
Outline

1. Introduction
2. First example of KT equations: $\omega \to 3\pi, \gamma^*\pi^0$
3. Generalization of KT equations to coupled channels: $\eta \to 3\pi$
4. Using KT equations for scattering: $\pi\pi$
5. Generalization of KT equations for (decaying) spin

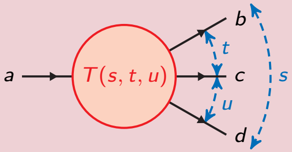
Introduction: Khuri-Treiman equations in a nutshell



- Partial wave expansion **in the s-channel**:

$$T(s,t,u) = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z_s) t_\ell(s)$$

- Two main (connected) problems:
  - Infinite number of PW
  - PW have RHC and LHC
- Only RHC: BS equation, K-matrix, DR,...
- Problem with "truncation": $t_\ell(s)$ only depends on s, so singularities in the t-, u-channel can only appear suming an infinite number of PW.



- In many decay processes one wants to take into account unitarity/FSI interactions in the three possible channels.

Introduction: Khuri-Treiman equations in a nutshell

- Khuri-Treiman equations are a tool to achieve this **two-body unitarity** in the **three channels**
- Consider three (s-, t-, u-channels) **truncated** "isobar" expansions.
- Isobars $f_\ell^{(s)}(s)$ have only RHC: amenable for **dispersion relations**.

$$\begin{aligned} T(s,t,u) &= \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z_s)\; t_\ell(s) \\ &= \sum_{\ell=0}^{n_s} (2\ell+1) P_\ell(z_s) f_\ell^{(s)}(s) + \sum_{\ell=0}^{n_t} (2\ell+1) P_\ell(z_t) f_\ell^{(t)}(t) + \sum_{\ell=0}^{n_u} (2\ell+1) P_\ell(z_u) f_\ell^{(u)}(u) \end{aligned}$$

- s-channel singularities appear in the s-channel isobar, $t_\ell^{(s)}(s)$.
- Singularities in the t-, u-channel are recovered!
- The LHC of the partial waves are given by the RHC of the crossed channel isobars

$$t_\ell(s) = \frac{1}{2} \int \mathrm{d}z\, P_\ell(z) T(s,t',u') = f_\ell^{(s)}(s) + \frac{1}{2} \int \mathrm{d}z\, Q_{\ell\ell'}(s,t') f_{\ell'}^{(t)}(t') \;.$$

$\omega \to 3\pi$ amplitude. Phenomenology

MA *et al.* (JPAC Collab.), arXiv:2006.01058

- Amplitude:

$$\mathcal{M}_+(s,t,u) = \frac{\sqrt{\phi(s,t,u)}}{2} F(s,t,u) \;. \qquad \left(\phi(s,t,u) = 4sp^2(s)q^2(s)\sin^2\theta_s\right)$$

- Decay width: $d^2\Gamma \sim \phi(s,t,u)\,|F(s,t,u)|^2$
- Dalitz plot parameters $(\alpha,\ \beta,\ \gamma)$ “equivalent” to bins... $(X,Y) \leftrightarrow (Z,\varphi) \leftrightarrow (s,t,u)$

$$|F(s,t,u)|^2 = |\mathcal{N}|^2 \left(1 + 2\alpha Z + 2\beta Z^{\frac{3}{2}} \sin 3\varphi + 2\gamma Z^2 + \cdots\right)$$

- **Why revisit $\omega \to 3\pi$?**

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	Bonn (2012)		JPAC (2015)		
	Eur. Phys. J., C72, 2014 (2012)		Phys. Rev., D91, 094029 (2015)		
	w/o KT	w KT	w/o KT	w KT	
α	130 ± 5	79 ± 5	125	84	
β	31 ± 2	26 ± 2	30	28	

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	Bonn (2012)		**JPAC (2015)**		**BESIII (2018)**
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	w/o KT	w KT	w/o KT	w KT	Exp.
α	130 ± 5	79 ± 5	125	84	$120.2 \pm 7.1 \pm 3.8$
β	31 ± 2	26 ± 2	30	28	$29.5 \pm 8.0 \pm 5.3$

- One (or more) out of three is wrong...
  1) Experiment?
  2) KT eqs., in general?
  3) Something particular?

KT equations: DR, subtractions, solutions, and all that...

- PW decomposition: $F(s,t,u) = \sum_{j\mathrm{odd}} \dot{P}_j(\cos\theta_s)(p(s)q(s))^{j-1} f_j(s) = f_1(s) + \cdots$
- KT/isobar decomposition: consider only $j = 1$ (ρ) isobar, $F(s)$:

$$\boxed{F(s,t,u) = F(s) + F(t) + F(u)}$$

- PW projection of the KT decomposition:

$$f_1(s) = F(s) + \hat{F}(s)\ , \qquad \hat{F}(s) = \frac{3}{2}\int_{-1}^{1} \mathrm{d}z_s (1 - z_s^2) F(t(s,z_s))$$

- Discontinuity:

$$\Delta F(s) = \Delta f_1(s) = \rho(s) t_{11}^*(s) f_1(s) = \rho(s) t_{11}^*(s) \left(F(s) + \hat{F}(s) \right)$$

- Unsubtracted DR:

$$F(s) = a\, F_a'(s),$$

$$F_a'(s) = \Omega(s) \left[1 + \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\sin\delta(s') \hat{F}_a'(s')}{|\Omega(s')|(s'-s)} \right] \ .$$

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- Once-subctracted DR:

$$F(s) = a\left(F_a'(s) + b\,F_b(s)\right)\ ,$$

$$F_a'(s) = \Omega(s)\left[1 + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\hat{F}_a'(s')}{|\Omega(s')|(s'-s)}\right]\ ,$$

$$F_b(s) = \Omega(s)\left[s + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\hat{F}_b(s')}{|\Omega(s')|(s'-s)}\right]\ .$$

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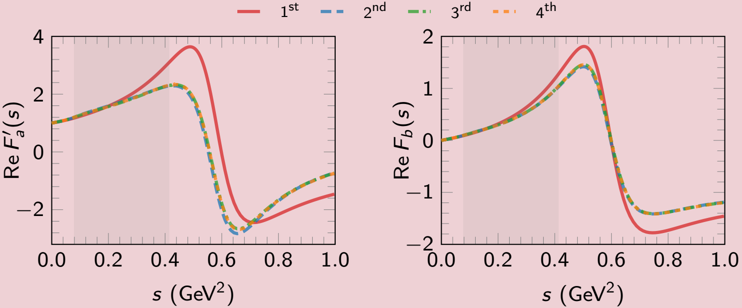

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$$F(s) = a\left(F'_a(s) + b\,F_b(s)\right)\ ,$$

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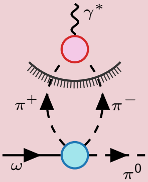
$\omega \to \pi^0$ transition form factor

- The decays $\omega(\to \pi^0\gamma^*) \to \pi^0 l^+ l^-$ and $\omega \to \pi^0\gamma$ governed by the TFF $f_{\omega\pi^0}(s)$.

$$\mathcal{M}(\omega \to \pi^0\ell^+\ell^-) = f_{\omega\pi^0}(s)\, \epsilon_{\mu\nu\alpha\beta}\epsilon^\mu(p_\omega,\lambda)p^\nu q^\alpha \frac{ie^2}{s}\bar{u}(p_-)\gamma^\beta v(p_+) \;,$$

$$\Gamma(\omega \to \pi^0\gamma) = |f_{\omega\pi^0}(0)|^2 \frac{e^2(m_\omega^2 - m_{\pi^0}^2)^3}{96\pi m_\omega^3} \;,$$

- Dispersive representation:



$$f_{\omega\pi^0}(s) = f_{\omega\pi^0}(0) + \frac{s}{12\pi^2}\int_{4m_\pi^2}^{\infty} \mathrm{d}s' \frac{q_\pi(s')^3}{s'^{\frac{3}{2}}(s'-s)}\Big(F(s') + \hat{F}(s')\Big)F_\pi^V(s')^*$$

- $f_{\omega\pi^0}(0) = |f_{\omega\pi^0}(0)|\, e^{i\phi_{\omega\pi^0}(0)}$
- Experimental information: $F_{\omega\pi^0}(s) = \frac{f_{\omega\pi^0}(s)}{f_{\omega\pi^0}(0)}$
- Only the relative phase $\frac{a}{f_{\omega\pi^0}(0)} = \frac{|a|}{\left|f_{\omega\pi^0}(0)\right|}\frac{1}{e^{i(\phi_{\omega\pi^0}(0)-\phi_a)}}$.

Summary of amplitudes/free parameters/exp. input

$\omega \to 3\pi$ amplitude $[F(s,t,u)]$

Free parameters: $|a|$, $|b|$, ϕ_b

Experimental input:

- $\Gamma_{3\pi}$
- Dalitz plot parameters

$\omega \to \gamma^{(*)}\pi^0$ TFF $[f_{\omega\pi^0}(s)]$

Free parameters: $|f_{\omega\pi^0}(0)|$, $\phi_{\omega\pi^0}(0)$ $(\bigoplus |a|, |b|, \phi_b)$

Experimental input:

- $\Gamma_{\gamma\pi^0}$
- $|F_{\omega\pi^0}(s)|^2$

First analysis in three steps

MA *et al.* (JPAC Collab.), arXiv:2006.01058

1. Fix $|b| \simeq 2.9$, $\phi_b \simeq 1.9$ with the **DP parameters**.
2. Fix $|a| \simeq 280\ \text{GeV}^{-3}$, $|f_{\omega\pi^0}(0)| \simeq 2.3\ \text{GeV}^{-1}$ from $\Gamma_{\omega\to 3\pi}$, $\Gamma_{\omega\to\gamma\pi}$.
3. You are left with $\phi_{\omega\pi^0}(0)$ and the TFF Data.

1. $\chi^2_{\text{DP}} = \left(\frac{\alpha^{(t)}-\alpha^{(e)}}{\sigma_\alpha}\right)^2 + \cdots$
2. $\chi^2_{\Gamma} = \left(\frac{\Gamma^{(t)}_{3\pi}-\Gamma^{(e)}_{3\pi}}{\sigma_{\Gamma_{3\pi}}}\right)^2 + \left(\frac{\Gamma^{(t)}_{\gamma\pi}-\Gamma^{(e)}_{\gamma\pi}}{\sigma_{\Gamma_{\gamma\pi}}}\right)^2$
3. $\chi^2_{\text{A2,NA60}} = \sum_i \left(\frac{|F_{\omega\pi}(s_i)|^2-\left|F^{(i)}_{\omega\pi}\right|^2}{\sigma_{F^{(i)}_{\omega\pi}}}\right)^2$

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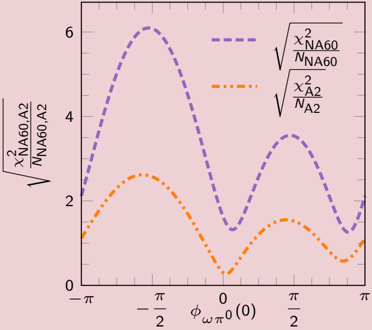


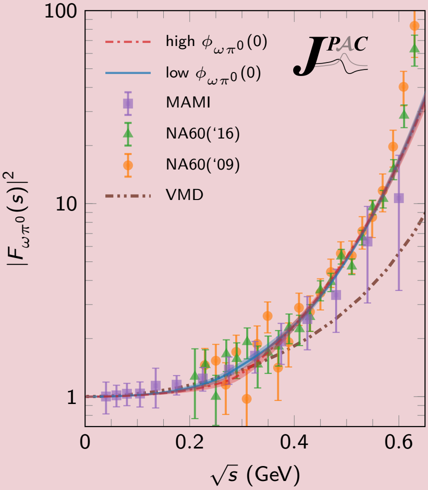
- Two different minima (low and high $\phi_{\omega\pi^0}(0)$) are found.
- Both have similar χ^2 of the TFF.

Make a **global, simultaneous** analysis

$$\overline{\chi}^2 = N\left(\frac{\chi^2_{\text{DP}}}{N_{\text{DP}}} + \frac{\chi^2_{\Gamma}}{N_{\Gamma}} + \frac{\chi^2_{\text{NA60}}}{N_{\text{NA60}}} + \frac{\chi^2_{\text{A2}}}{N_{\text{A2}}}\right)$$

Results

MA *et al.* (JPAC Collab.), arXiv:2006.01058



	α	β	γ
BESIII	111(18)	25(10)	22(29)
low	112(15)	23(6)	29(6)
high	109(14)	26(6)	19(5)

Results (2)

MA *et al.* (JPAC Collab.), arXiv:2006.01058

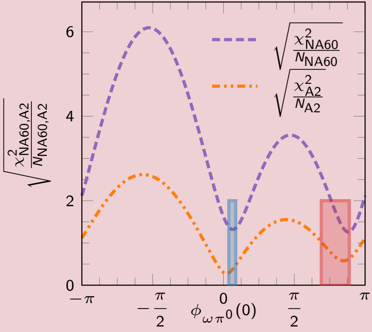
	2 par.		3 par.	
	low $\phi_{\omega\pi^0}(0)$	high $\phi_{\omega\pi^0}(0)$	low $\phi_{\omega\pi^0}(0)$	high $\phi_{\omega\pi^0}(0)$
$10^{-2}\,\lvert a\rvert\ \left[\text{GeV}^{-3}\right]$	3.14(25)	2.63(25)	3.11(28)	2.70(30)
$\lvert b\rvert$	3.15(22)	2.59(35)	3.25(26)	2.65(35)
ϕ_b	2.03(14)	1.61(38)	2.03(13)	1.70(27)
$\lvert f_{\omega\pi^0}(0)\rvert\ \left[\text{GeV}^{-1}\right]$	2.314(32)	2.314(32)	2.314(32)	2.315(32)
$\phi_{\omega\pi^0}(0)$	0.207(60)	2.39(46)	0.195(76)	2.48(31)
χ^2_{DP}	0.19	< 0.01	0.10	0.03
$10^4\chi^2_{\Gamma}$	2.4	2.4	1.1	3.5
χ^2_{A2}	2.3	3.6	2.4	3.7
χ^2_{NA60}	31	35	31	35

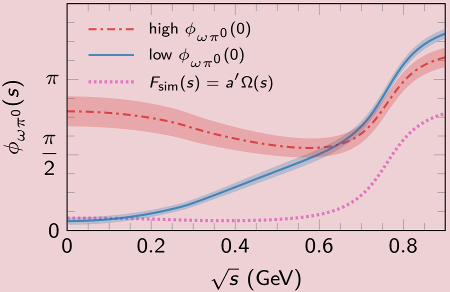
Meaning of the phase?

- Original solutions around $\phi_{\omega\pi^0}(0) \sim 0, \pi$
- Global fits remain near the original ones...

If $f_{\omega\pi^0}(0)$, a are considered as part of a microscopic (lagrangian) calculation, they would be real (hermiticity), and their relative phase would be ± 1 .

On the other hand, we find 2σ deviation: almost real, but not exactly.....



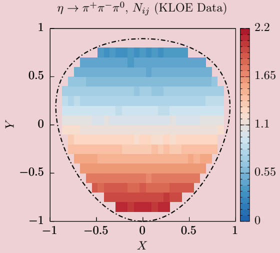


Generalities about $\eta \to 3\pi$

- In **QCD** isospin-breaking phenomena are driven by

$$H_{IB} = -(m_u - m_d)\bar{\psi}\frac{\lambda_3}{2}\psi$$

- Isospin-breaking induced by EM & strong interactions are **similar** in size, but
- **$\eta \to 3\pi$ is special**, since EM effects are smaller
- $\Gamma_{\eta\to 3\pi} \propto Q^4$, with $$Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - (m_u + m_d)^2/2}$$



- **Experimental situation:** Several high-statistics studies; $|T|^2$ well known across the Dalitz plot ⇒ stringent tests for the amplitudes (before getting Q!)

$\eta \to 3\pi^0$	$\eta \to \pi^+\pi^-\pi^0$
Crys. Ball, PRL**87**,192001('01)	KLOE, JHEP**0805**,006('08)
Crys. Ball@MAMI, A2, PRC**79**,035204('09)	WASA-at-COSY, PRC**90**,045207('14)
Crys. Ball@MAMI, TAPS, A2, EPJA**39**,169('09)	BESIII, PRD**92**,012014('15)
WASA-at-COSY, PLB**677**,24('09)	KLOE-2, JHEP**1605**,019('16)
KLOE, PLB**694**,16('11)	

Dispersive approaches to $\eta \to 3\pi$

- **Chiral $\mathcal{O}(p^4)$** amplitude fails in describing experiments.

  Gasser, Leutwyler, Nucl. Phys. B**250**, 539 (1985)

- Several attemps to include **unitarity/FSI/rescattering** effects.

  Neveu, Scherk, AP**57**, 39('70); Roiesnel, Truong, NPB**187**, 293('81); Kambor, Wiesendanger, Wyler, NPB**465**, 215('96); Anisovich, Leutwyler, PLB**375**, 335('96); Borasoy, R. Nißler, EPJA**26**, 383('05); Schneider, Kubis, Ditsche, JHEP**1102**, 028('11); Kampf, Knecht, Novotný, Zdráhal, PRD**84**, 114015('11); Colangelo, Lanz, Leutwyler, Passemar, PRL**118**, 022001('17); Guo, Danilkin, Fernández-Ramírez, Mathieu, Szczepaniak, PL**771**, 497('17).

- Here we reconsider the **KT approach**. N. Khuri, S. Treiman, Phys. Rev. **119**, 1115 (1960)

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  - $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980)$, $(K\bar{K})_0$
  - Double resonance effect $\eta\pi$ ISI, $a_0(980)$, $(K\bar{K})_1$

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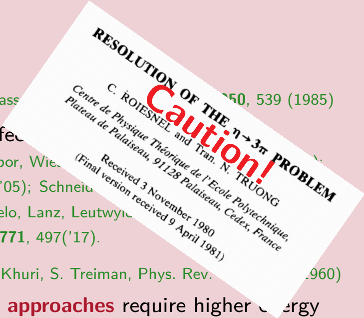
RESOLUTION OF THE $\eta \to 3\pi$ PROBLEM

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Received 3 November 1980 (Final version received 9 April 1981)

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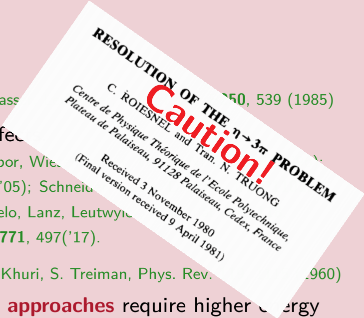

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Isospin amplitudes

- Start with well-defined **isospin amplitudes:**

$$\mathcal{M}^{I,I_z}(s,t,u) = \langle \eta\pi; 1, I_z | \hat{T}^{(1)}_0 | \pi\pi; I, I_z \rangle = \langle I, I_z; 1, 0 | \, 10 \rangle \, \langle \eta\pi \| \hat{T}^{(1)} \| \pi\pi; I \rangle$$

- They can be written in terms of a **single amplitude** ($\eta\pi^0 \to \pi^+\pi^-$), $A(s,t,u)$ (like in $\pi\pi$ scattering):

$$\begin{bmatrix} -\sqrt{3}\mathcal{M}^0(s,t,u) \\ \sqrt{2}\mathcal{M}^1(s,t,u) \\ \sqrt{2}\mathcal{M}^2(s,t,u) \end{bmatrix} = \begin{bmatrix} -\sqrt{3}\mathcal{M}^{0,0}(s,t,u) \\ \sqrt{2}\mathcal{M}^{1,1}(s,t,u) \\ \sqrt{2}\mathcal{M}^{2,1}(s,t,u) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} A(s,t,u) \\ A(t,s,u) \\ A(u,t,s) \end{bmatrix}$$

- Reconstruction theorem (for Goldstone bosons):

J. Stern, H. Sazdjian, N. Fuchs, Phys. Rev. D**47**, 3814 (1993)

$$A(s,t,u) = -\epsilon_L \Big[M_0(s) - \frac{2}{3} M_2(s) + M_2(t) + M_2(u) + (s-u)M_1(t) + (s-t)M_1(u) \Big] \qquad \epsilon_L = \frac{1}{Q^2} \frac{m_K^2 - m_\pi^2}{3\sqrt{3} f_\pi^2} \frac{m_K^2}{m_\pi^2}$$

- Or in general, "the" KT approximation:

Infinite sum of s-channel PW $\to$ Truncated sums of s-, t-, and u-channels PWs

- **Single variable functions:** amenable for dispersion relations.

Partial wave amplitudes

- Summary of previous slide: $\mathcal{M}^I(s,t,u)$ is written in terms of $A(s,t,u)$ (and permutations), and $A(s,t,u)$ is written in terms of $M_I(w)$.
- Now, define **partial waves:** $\mathcal{M}^I(s,t,u) = 16\pi\sqrt{2}\sum_j (2j+1)\mathcal{M}^I_j(s) P_j(z)$

$$\mathcal{M}^0_0(s) = \epsilon_L \frac{\sqrt{6}}{32\pi}[M_0(s) + \hat{M}_0(s)] \,, \quad \mathcal{M}^2_0(s) = \epsilon_L \frac{-1}{32\pi}[M_2(s) + \hat{M}_2(s)] \,,$$
$$\mathcal{M}^1_1(s) = \epsilon_L \frac{\kappa(s)}{32\pi}[M_1(s) + \hat{M}_1(s)] \,,$$

LHC $[\hat{M}_I(s)]$

$\hat{M}_I(s)$ written as angular averages.
Take $M_0(s)$ as an example:

$$\hat{M}_0(s) = \frac{2}{3}\langle M_0\rangle + \frac{20}{9}\langle M_2\rangle$$
$$+ 2(s-s_0)\langle M_1\rangle + \frac{2}{3}\kappa(s)\langle zM_1\rangle$$
$$\langle z^n M_I\rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z))$$
$$\kappa(s) = \sqrt{(1-4m_\pi^2/s)\lambda(s, m_\eta^2, m_\pi^2)}$$

RHC $[M_I(s)]$

$\hat{M}(s)$ no discontinuity along the RHC:

$$\text{disc } M_I(s) = \text{disc}\mathcal{M}^I_j(s) =$$
$$= \sigma_\pi(s) t^I(s)^* \mathcal{M}^I_j(s)$$
$$= \sigma_\pi(s) t^I(s)^* \left(M_I(s) + \hat{M}_I(s)\right)$$
$$\sigma_\pi(s) = \sqrt{1-4m_\pi^2/s}$$
$$\sigma_\pi(s) t^I(s) = \sin\delta_I(s)\, e^{i\delta_I(s)}$$

Muskhelisvili-Omnès representation

$$\mathrm{disc} M_I(s) = \sigma_\pi(s) t_I^*(s) \left[M_I(s) + \hat{M}_I(s) \right]$$

- MO (dispersive) representation of $M_I(s)$:

$$M_0(s) = \Omega_0(s) \left[\alpha_0 + \beta_0 s + \gamma_0 s^2 + \hat{I}_0(s) s^2 \right] ,$$
$$M_1(s) = \Omega_1(s) \left[\beta_1 s + \hat{I}_1(s) s \right] ,$$
$$M_2(s) = \Omega_2(s) \left[\hat{I}_2(s) s^2 \right] .$$

$$\Omega_I(s) = \exp \left[\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s)} \right] \text{ (Omnès function/matrix)}$$

$$\hat{I}_I(s) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\sin \delta_I(s') \hat{M}_I(s')}{|\Omega_I(s')| (s')^{m_I} (s'-s)} , \quad (m_{0,2} = 2, \; m_1 = 1)$$

- $m_\eta^2 + i\varepsilon$ prescription needed. Integral equations solved iteratively.
- Subtraction constants: Most natural way is to match with **ChPT:**

$$\mathcal{M}(s,t,u) - \overline{\mathcal{M}}_\chi(s,t,u) = \mathcal{O}(p^6)$$

Descotes-Genon, Moussallam, EPJ,C74,2946(2014)

- **Matching conditions:** fix α_0, β_0, β_1, γ_0 in terms of ChPT amplitudes (no free parameters).

Coupled channels

MA, B. Moussallam, EPJ,C77,508('17)

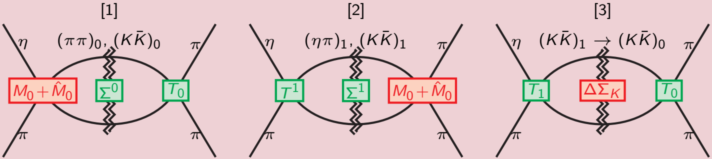
Coupled channels: take into account **intermediate states** other than $(\pi\pi)_I$.

$$\boldsymbol{M}_0 = \begin{bmatrix} M_0 & G_{10} \\ N_0 & H_{10} \end{bmatrix} = \begin{bmatrix} (\eta\pi)_1 \to (\pi\pi)_0 & (K\bar{K})_1 \to (\pi\pi)_0 \\ (\eta\pi)_1 \to (K\bar{K})_0 & (K\bar{K})_1 \to (K\bar{K})_0 \end{bmatrix},$$

$$\boldsymbol{T}_0 = \begin{bmatrix} t_{(\pi\pi)_0\to(\pi\pi)_0} & t_{(\pi\pi)_0\to(K\bar{K})_0} \\ t_{(\pi\pi)_0\to(K\bar{K})_0} & t_{(K\bar{K})_0\to(K\bar{K})_0} \end{bmatrix}, \quad \boldsymbol{T}_1 = \begin{bmatrix} t_{(\eta\pi)_1\to(\eta\pi)_1} & t_{(\eta\pi)_1\to(K\bar{K})_1} \\ t_{(\eta\pi)_1\to(K\bar{K})_1} & t_{(K\bar{K})_1\to(K\bar{K})_1} \end{bmatrix}$$

$$\begin{aligned} \text{disc}\, \boldsymbol{M}_0(s) &= \boldsymbol{T}^{0*}(s)\Sigma^0(s)\left[\boldsymbol{M}_0(s+i\epsilon) + \hat{\boldsymbol{M}}_0(s)\right] && \to [1] \\ &+ \left[(\boldsymbol{M}_0(s-i\epsilon) + \hat{\boldsymbol{M}}_0(s)\right]\Sigma^1(s)\,\boldsymbol{T}^1(s) && \to [2] \\ &+ \boldsymbol{T}^{0*}(s)\Delta\Sigma_K(s)\boldsymbol{T}^1(s) && \to [3] \end{aligned}$$

Schematically:



Coupled channels: MO representations

$$\begin{aligned} \text{disc } \boldsymbol{M}_0(s) &= \boldsymbol{T}^{0^*}(s)\Sigma^0(s)\left[\boldsymbol{M}_0(s+i\epsilon)+\hat{\boldsymbol{M}}_0(s)\right] && \to [1] \\ &+ \left[(\boldsymbol{M}_0(s-i\epsilon)+\hat{\boldsymbol{M}}_0(s)\right]\Sigma^1(s)\,\boldsymbol{T}^1(s) && \to [2] \\ &+ \boldsymbol{T}^{0^*}(s)\Delta\Sigma_K(s)\boldsymbol{T}^1(s) && \to [3] \end{aligned}$$

- MO representation for $\boldsymbol{M}_0(s)$:

$$\begin{bmatrix} M_0(s) & G_{10}(s) \\ N_0(s) & H_{10}(s) \end{bmatrix} = \boldsymbol{\Omega}_0(s)\left[\boldsymbol{P}_0(s)+s^2\left(\hat{\boldsymbol{I}}_a(s)+\hat{\boldsymbol{I}}_b(s)\right)\right]{}^t\boldsymbol{\Omega}_1(s)$$

- $\boldsymbol{P}_0(s)$ is a matrix of polynomials.
- The $\hat{\boldsymbol{I}}(s)$ functions are:

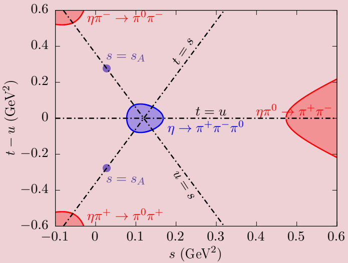
$$\hat{\boldsymbol{I}}_{a,b}(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{(s')^2(s'-s)}\,\Delta\boldsymbol{X}_{a,b}(s')\,,$$

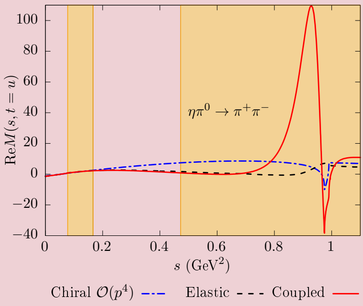
$$\Delta\boldsymbol{X}_a = \boldsymbol{\Omega}_0^{-1}(s-i\epsilon)\left[\underbrace{\boldsymbol{T}^{0^*}(s)\,\Sigma^0(s)\,\hat{\boldsymbol{M}}_0(s)}_{[1]}+\underbrace{\hat{\boldsymbol{M}}_0(s)\,\Sigma^1(s)\,\boldsymbol{T}^1(s)}_{[2]}\right]{}^t\boldsymbol{\Omega}_1^{-1}(s+i\epsilon)\,,$$

$$\Delta\boldsymbol{X}_b = \underbrace{\boldsymbol{\Omega}_0^{-1}(s-i\epsilon)\boldsymbol{T}^{0^*}(s)\,\Delta\Sigma_K(s)\,\boldsymbol{T}^1(s)\,{}^t\boldsymbol{\Omega}_1^{-1}(s+i\epsilon)}_{[3]}$$

Results

MA, B. Moussallam, EPJ,C77,508('17)



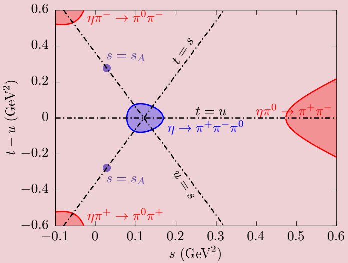


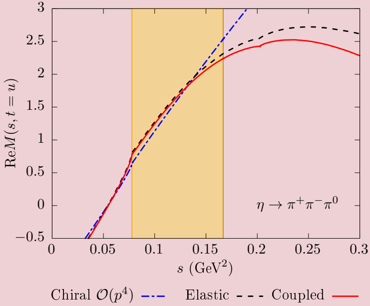
Behaviour in different regions:

- $s \sim 1$ GeV^2 Very sharp energy variation,
  - $a_0(980)$ and $f_0(980)$ interference,
  - K^+K^- and $K^0\bar{K}^0$ thresholds.
- $0.7 \lesssim s \lesssim 0.97$ GeV^2 Coupled channel largely **enhanced** compared with elastic amplitude.
- $s \lesssim 0.7$ GeV^2 Effect of coupled channels is to reduce the amplitude.
- $s \lesssim s_{\text{th}}$ Elastic and inelastic amplitudes indistinguishable.

Results

MA, B. Moussallam, EPJ,C77,508('17)





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Dalitz plot

MA, B. Moussallam, EPJ,C77,508('17)

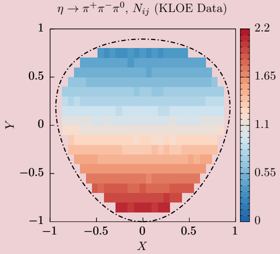
- DP variables X,Y: $X = \frac{\sqrt{3}}{2m_\eta Q_c}(u-t)$, $Y = \frac{3}{2m_\eta Q_c}\left((m_\eta - m_{\pi^0})^2 - s\right) - 1$

- Charged mode amplitude written as:

$$\frac{|M_c(X,Y)|^2}{|M_c(0,0)|^2} = 1 + a\,Y + b\,Y^2 + d\,X^2 + f\,Y^3 + g\,X^2Y + \cdots$$

- Neutral decay mode amplitude $[Q_c \to Q_n]$:

$$\frac{|M_n(X,Y)|^2}{|M_n(0,0)|^2} = 1 + 2\alpha\,|z|^2 + 2\beta\,\mathrm{Im}(z^3) + \cdots$$



		$O(p^4)$	elastic	coupled	KLOE	BESIII
charged	a	-1.328	-1.156	-1.142(45)	-1.095(4)	-1.128(15)
	b	0.429	0.200	0.172(16)	0.145(6)	0.153(17)
	d	0.090	0.095	0.097(13)	0.081(7)	0.085(16)
	f	0.017	0.109	0.122(16)	0.141(10)	0.173(28)
	g	-0.081	-0.088	-0.089(10)	-0.044(16)	–
neutral					PDG	
	α	+0.0142	-0.0268	-0.0319(34)	-0.0318(15)	
	β	-0.0007	-0.0046	-0.0056	–	

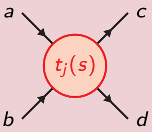
BESIII Collab., Phys. Rev. D**92**,012014 (2015)
KLOE-2 Collab., JHEP **1605**, 019 (2016)

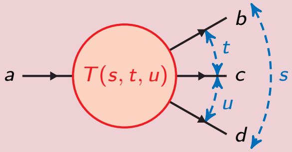
- (Theory) **uncertainty estimation**:
  1. $\eta\pi$ interaction put to zero or to "large"
  2. $10^3 L_3^r = -3.82 \to -2.65$
- General trend: **improve agreement** $[\mathcal{O}(p^4) \to$ elastic $\to$ coupled$]$
- Particularly relevant: $\boldsymbol{\alpha}$.

Khuri-Treiman equations for $\pi\pi$ scattering

MA *et al.* (JPAC Collab.), EPJ,C78,574('18)

- KT equations for 3-body decays. Crossing: 2-to-2 scattering. Test: $\pi\pi$ scattering.

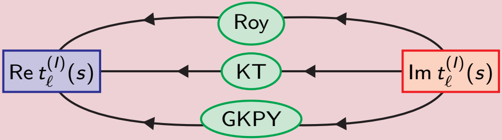




What happens if you apply KT equations to $\pi\pi$ scattering?

- KT equations for $\pi\pi$ scattering can be written as Roy-like equations:

$$t_\ell^{(I)}(s) = k_\ell^{(I)}(s) + \sum_{\ell',I'} \int_{s_{\text{th}}}^{\infty} dt' \, K_{\ell\ell'}^{II'}(s,t') \, \text{Im}\, t_{\ell'}^{(I')}(t')$$



Results: Comparison with Roy equations

MA *et al.* (JPAC Collab.), EPJ,C78,574('18)

- Roy equations [PL,36B,353(1971)] and KT equations written as:

$$t_\ell^{(I)}(s) = k_\ell^{(I)}(s) + \sum_{\ell',I'} \int_{s_{\rm th}}^\infty dt'\, K_{\ell\ell'}^{II'}(s,t')\, \mathrm{Im}\, t_{\ell'}^{(I')}(t')$$

They differ in the expressions for the polynomial ($k_\ell^{(I)}(s)$) and the kernel ($K_{\ell\ell'}^{II'}(s,t')$).

- Restrict KT to
  1. S, P-waves ($t_0^{(0)}$, $t_0^{(2)}$, $t_1^{(1)}$),
  2. one subtraction in each channel: only two subtraction constants.
- Difference between KT and Roy equations amplitudes:

$$(t_{\rm KT})_\ell^{(I)}(s) - (t_{\rm Roy})_\ell^{(I)}(s) = \tilde{k}_\ell^{(I)}(s) - k_\ell^{(I)}(s) + \sum_{\ell',I'} \int_{s_{\rm th}}^\infty \mathrm{d}t'\, \Delta_{\ell\ell'}^{II'}(4m^2,t')\, \mathrm{Im}\, t_{\ell'}^{(I')}(t')$$

- $\Delta_{\ell\ell'}^{II'}(s,t')$: Difference of kernels is polynomial (logarithmic terms cancel).
- Five conditions that can be fulfilled with the two subtraction constants.

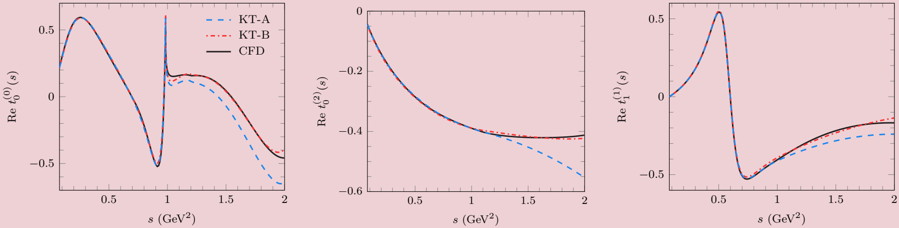
> KT equations and Roy equations are equal.

Results: Comparison with GKPY

MA *et al.* (JPAC Collab.), EPJ,C78,574('18)

- Take a succesful parameterization of the amplitude as **input for $\mathrm{Im}t_\ell^{(I)}(s)$**, and compare the **output $\mathrm{Re}t_\ell^{(I)}(s)$** Madrid group, PR,D83,074004(2011)

  A: one subtraction ($\times$ 6), but only 5 free constants. $s_{\text{max}} = 1.0$ GeV^2
  B: two subtractions ($\times$ 6), but only 7 free constants. $s_{\text{max}} = 1.9$ GeV^2



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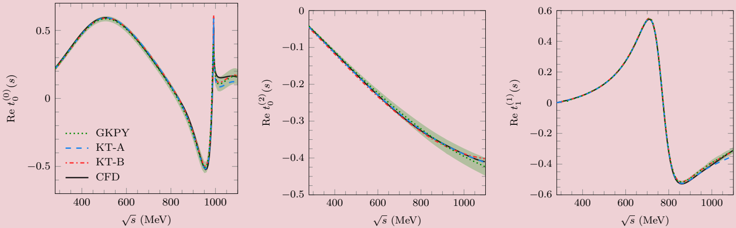

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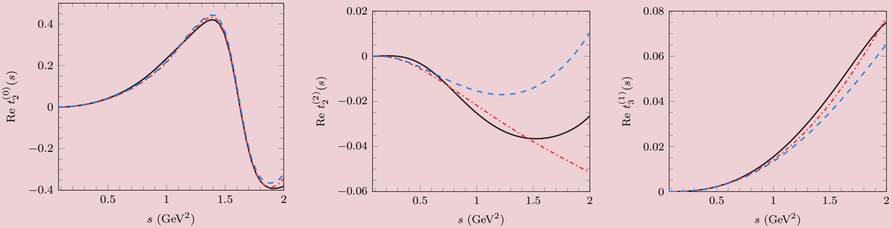

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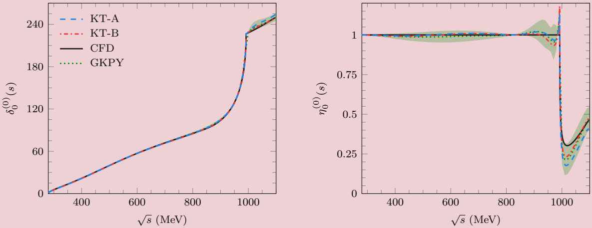

Results: Comparison with GKPY (II)

MA *et al.* (JPAC Collab.), EPJ,C78,574('18)

- Threshold parameters (right):

$$\frac{m^{2\ell}}{p^{2\ell}(s)} \mathrm{Re}\, t_\ell^{(I)}(s) = a_\ell^{(I)} + b_\ell^{(I)} \frac{p^2(s)}{m^2} + \cdots .$$

- Poles and residues (bottom):

$$t_{II}^{-1}(s) = t_I^{-1}(s) + 2i\sigma(s) ,$$

$$t_{II}(s) \simeq \frac{\tilde{g}_p^2}{s - s_p} + \cdots$$

PR,D83,074004('11); PRL,107,072001('11); PL,B749,399('15)

	KT-A	KT-B	GKPY—CFD
$a_0^{(0)}$	0.217	0.213	0.221(9)
$b_0^{(0)}$	0.274	0.275	0.278(7)
$a_0^{(2)}$	−0.044	−0.047	−0.043(8)
$b_0^{(2)}$	−0.078	−0.079	−0.080(9)
$10^3 \cdot a_1^{(1)}$	37.5	37.9	38.5(1.2)
$10^3 \cdot b_1^{(1)}$	5.6	5.7	5.1(3)
$10^4 \cdot a_2^{(0)}$	17.8	17.8	18.8(4)
$10^4 \cdot b_2^{(0)}$	−3.4	−3.4	−4.2(1.0)
$10^4 \cdot a_2^{(2)}$	1.9	1.8	2.8(1.0)
$10^4 \cdot b_2^{(2)}$	−3.2	−3.2	−2.8(8)
$10^5 \cdot a_3^{(1)}$	5.7	5.7	5.1(1.3)
$10^5 \cdot b_3^{(1)}$	−4.0	−4.0	−4.6(2.5)

	KT-A	KT-B	GKPY—CFD
$\sqrt{s_\sigma}$ (MeV)	(448, 270)	(448, 269)	$(457^{+14}_{-13}, 279^{+11}_{-7})$
$\lvert g_\sigma \rvert$ GeV	3.36	3.37	$3.59^{+0.11}_{-0.13}$
$\sqrt{s_\rho}$ (MeV)	(762.2, 72.4)	(763.4, 73.5)	$(763.7^{+1.7}_{-1.5}, 73.2^{+1.0}_{-1.1})$
$\lvert g_\rho \rvert$	5.95	6.01	$6.01^{+0.04}_{-0.07}$
$\sqrt{s_{f_0}}$ (MeV)	(1000, 24)	(995, 26)	$(996 \pm 7, 25^{+10}_{-6})$
$\lvert g_{f_0} \rvert$ (GeV)	2.4	2.3	2.3 ± 0.2
$\sqrt{s_{f_2}}$ (MeV)	(1275.1, 89.5)	(1268.9, 86.4)	$(1267.3^{+0.8}_{-0.9}, 87 \pm 9)$
$\lvert g_{f_2} \rvert$ (GeV^{-1})	5.6	5.5	5.0 ± 0.3

Khuri-Treiman equations for spin

MA *et al.* (JPAC Collab.), PR,D101,054018('20)

- $\eta \to 3\pi$, $\pi\pi \to \pi\pi$: $J = 0$, no spin complications.
- $\omega \to 3\pi$: single amplitude, $F(s,t,u) = F(s,u,t) = F(t,s,u)$. $J = 1$ particular case.
- For general $J \neq 0$, there are more than a single amplitude, and the t-, u-isobar amplitudes related with s-isobar through **crossing**.

PC	J_{min}	I	notation (for $I = 0, 1$)
$++$	1	odd	a_J
$+-$	1	even	h_J
$-+$	0	odd	π_J
$--$	0	even	ω_J/ϕ_J

Crossing symmetry

MA *et al.* (JPAC Collab.), PR,D101,054018('20)

$$\mathcal{A}^{abcd}(\epsilon(p_X), p_3; p_1, p_2) = \langle \pi^c(p_1)\pi^d(p_2) \,|\, \hat{T} \,|\, X_J^a(\epsilon(p_X))\, \pi^b(p_3)\rangle$$

- Definition of s- and t-channel helicity amplitudes:

$$\mathcal{A}_\lambda^{(s)abcd}(s,t,u) \equiv \mathcal{A}^{abcd}(\epsilon_\lambda^{(s)}(p_X), p_3; p_1, p_2)$$

$$\mathcal{A}_{\lambda'}^{(t)acbd}(t,s,u) \equiv \mathcal{A}^{abcd}(\epsilon'^{(t)}_{\lambda'}(p'_X), -p'_1, p'_2, -p'_3)$$

- Crossing, helicity amplitudes: $\mathcal{A}_{\lambda'}^{(t)acbd}(t,s,u) = \sum_\lambda d^J_{\lambda\lambda'}(\omega_t)\, \mathcal{A}_\lambda^{(s)abcd}(s,t,u)$
  Jacob, Wick, Ann.Phys.,7,404('59); Trueman, Wick, Ann.Phys.,26,322('64);
  Hara, PTP,45,584('71); Martin & Spearman ('70);
- Crossing, Isospin: $\mathcal{A}_{\lambda'}^{(t)acbd}(t,s,u) = (-1)^{\lambda'}\, \mathcal{A}_{\lambda'}^{(s)acbd}(t,s,u)$
- Combining both results:

$$\mathcal{A}_\lambda^{(s)abcd}(s,t,u) = \sum_{\lambda'} (-1)^\lambda\, d^J_{\lambda'\lambda}(\omega_t)\, \mathcal{A}_{\lambda'}^{(s)acbd}(t,s,u)$$

Why is this relation so important?

It allows the relation between the same one-variable functions (helicity partial waves or helicity isobars) for s and t.

KT decomposition & equations

MA *et al.* (JPAC Collab.), PR,D101,054018('20)

- Isospin projection:

$$\mathcal{A}_{\lambda\, I}(s,t,u) \equiv \frac{1}{(2I+1)} \sum_{a,b,c,d} P^{(I)}_{abcd}\, \mathcal{A}^{(s)abcd}_{\lambda}(s,t,u)$$

- KT decomposition in terms of isobars:

$$\begin{aligned}\mathcal{A}_{\lambda\, I}(s,t,u) = &\sum_{j\geqslant|\lambda|}^{j_{\max}} (2j+1)\, d^{j}_{\lambda 0}(\theta_s)\, a_{j\lambda\, I}(s) \\ &+ \sum_{\lambda' j' I'} (-1)^{\lambda}\, (2j'+1)\, d^{J}_{\lambda'\lambda}(\omega_t)\, d^{j'}_{\lambda' 0}(\theta_t)\, a_{j'\lambda'\, I'}(t)\, \frac{1}{2} C_{II'} \\ &+ \sum_{\lambda' j' I'} (-1)^{\lambda'} (2j'+1)\, d^{J}_{\lambda'\lambda}(\omega_u)\, d^{j'}_{\lambda' 0}(\theta_u)\, a_{j'\lambda'\, I'}(u)\, \frac{1}{2} C_{II'}\, (-1)^{I+I'}\end{aligned}$$

- Discontinuity:

$$\Delta a_{j\lambda I}(s) = \rho(s) t^*_{jI}(s) \left(a_{j\lambda I}(s) + \overline{a}_{j\lambda I}(s) \right) \ ,$$

- Inhomogeneity:

$$\overline{a}_{j\lambda I}(s) = (-1)^{\lambda} \sum_{I' j' \lambda'} \frac{1}{2} C_{II'} \int \mathrm{d}\cos\theta'\, d^{j}_{\lambda 0}(\theta') d^{J}_{\lambda'\lambda}(\omega_{t'}) d^{j'}_{\lambda' 0}(\theta'_t) a_{j\lambda' I'}(t')$$

- One last point: kinematical singularities and constraints fully taken into account in the paper.

Summary

- KT equations are a powerful tool to study **3-body decays**.
- They allow to implement **two-body unitarity** in all the **three channels** (s, t, u).
- Iterative solution converges fast, linear in subtraction constants.
- For $\omega \to 3\pi$ decays: MA *et al.* (JPAC Collab.), arXiv:2006.01058
  - Using once-subtracted DRs, we are able to reproduce the $\omega \to 3\pi$ DP parameters,
  - and the $\omega \to \pi^0\gamma^*$ transition form factor data.
- For $\eta \to 3\pi$: MA, B. Moussallam, EPJ,C77,508('17)
  - Not well described by the perturbative chiral amplitudes.
  - We have presented an **extension** of this approach to **coupled channels**. The extension is quite **general**.
  - Effects of $K\bar{K}$ and $\eta\pi$ amplitudes [$f_0(980)$, $a_0(980)$] play some role in the DP parameters, tend to improve.
- For $\pi\pi$ scattering: MA *et al.* (JPAC Collab.), EPJ,C78,574('18)
  - We have applied KT equations to $\pi\pi$ scattering as benchmark.
  - Restricted to S- and P-waves, KT equations are equal to Roy equations.
  - When other waves are included, good comparison is obtained with GKPY equations.
- We have presented a **generalization** of the KT equations for arbitrary quantum numbers of the decaying particle. MA *et al.* (JPAC Collab.), PR,D101,054018('20)
  - Not trivial, because of spin/crossing.

Thanks!

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- D. Winney (IU)

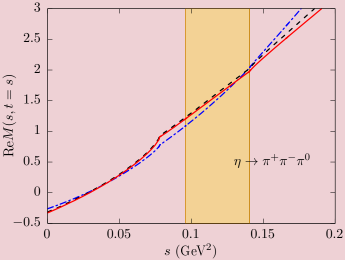
Khuri-Treiman equations: review and applications

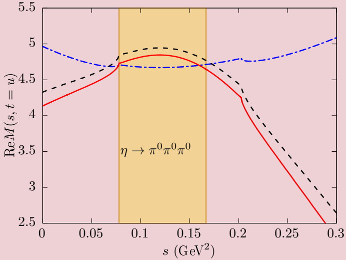
Jefferson Lab

Miguel Albaladejo

Accessing and Understanding the QCD Spectra
(INT-20-2c)
August 19th, 2020

Results





- Subthreshold region: chiral, elastic, and coupled amplitudes **very close.**
- Adler zero ($s_A \simeq 0.03$ GeV^2):

	NLO	el.	cou.
$s_A/m^2_{\pi^+} =$	1.42	1.45	1.49

- Substantial influence of coupled channels in the whole region,
- and there is no region in which dispersive and chiral amplitudes agree.
- $T_{\eta\to 3\pi^0} = M_0(s) + M_0(t) + M_0(u) + \ldots$

Quark mass ratio

From the amplitudes $M_I(s)$ one can compute the width up to the unknown factor Q^2:

$$\Gamma = \epsilon_L^2 \int_{4m_\pi^2}^{m_-^2} \mathrm{d}s \int_{t_-(s)}^{t_+(s)} \mathrm{d}t \, |M_0(s) + \cdots|^2$$

$$\epsilon_L = Q^{-2} \frac{m_K^2 - m_\pi^2}{3\sqrt{3} f_\pi^2} \frac{m_K^2}{m_\pi^2} \,, \quad Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

$\Gamma(\eta \to 3\pi^0)/\Gamma(\eta \to \pi^+\pi^-\pi^0)$	
PDG (fit)	1.426(26)
PDG (average)	1.48(5)
CLEO	1.496(43)(35)
chiral $\mathcal{O}(p^4)$	1.425
elastic	1.449
coupled	1.451

Decay	Q elastic	Q coupled
$\Gamma^{(\mathrm{exp})}_{(\mathrm{neu.})} = 299(11)$ eV	21.9(2)	21.7(2)
$\Gamma^{(\mathrm{exp})}_{(\mathrm{cha.})} = 427(15)$ eV	21.8(2)	21.6(2)

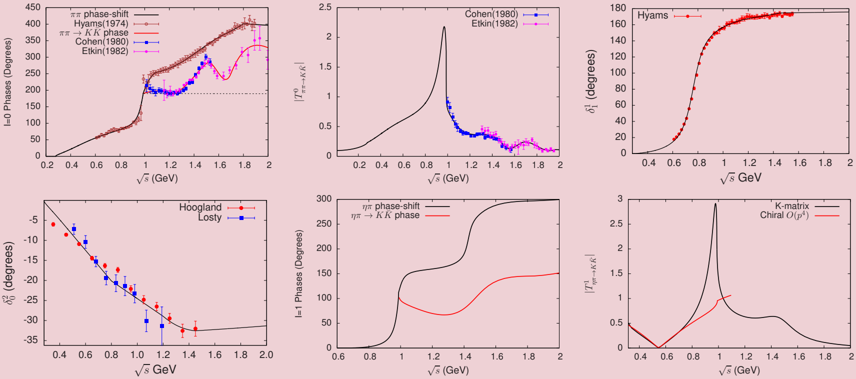
- Effect of inelastic channels $\sim 1\%$ (decreasing)
- Theoretical error on Q:
  - Phase shifts $[s \leqslant 1\ \mathrm{GeV}^2]$: $\sim 1\%$
  - $\mathcal{O}(p^4)$ ampl. $[L_3]$: $\sim 1\%$
  - NNLO ampl.: $\Delta Q_{\mathrm{th.}} = \pm 2.2$

$$Q = 21.6 \pm 0.2 \pm 2.2$$

- Fitted (not matched) polynomial parameters:

$$Q_{\mathrm{fit}} = 21.50 \pm 0.67 \pm 0.70$$

Isospin conserving T-matrices



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R. García-Martín, B. Moussallam, Eur. Phys. J. C**70**, 155 (2010);

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Amplitudes M_1 and M_2

An analogous analysis can be done with $M_1(s)$ and $M_2(s)$ amplitudes:

$M_1(s)$ [P-wave]

$$\boldsymbol{M}_1(s) = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} = \begin{bmatrix} (\eta\pi)_{1-} \to (\pi\pi)_{1+} \\ (\eta\pi)_{1-} \to (K\bar{K})_{1+} \end{bmatrix}$$

$$\boldsymbol{T}_1^1(s) = \begin{bmatrix} (\pi\pi)_1 \to (\pi\pi)_1 & (\pi\pi)_1 \to (K\bar{K})_1 \\ (\pi\pi)_1 \to (K\bar{K})_1 & (K\bar{K})_1 \to (K\bar{K})_1 \end{bmatrix}$$

$$\Delta \boldsymbol{M}_1(s) = \boldsymbol{T}_1^{1*}(s)\Sigma^0(s) \times \left[\boldsymbol{M}_1(s+i\epsilon) + \hat{\boldsymbol{M}}_1(s)\right]$$

$M_2(s)$ [S-wave]

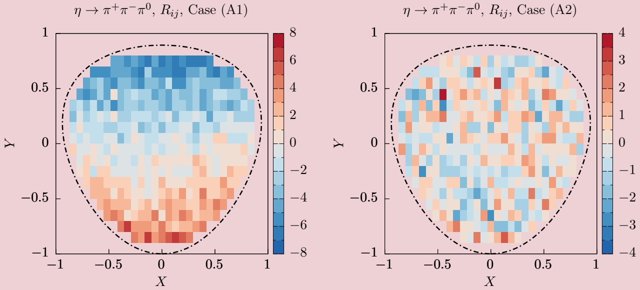
$$\boldsymbol{M}_2(s) = \begin{bmatrix} M_2 \\ G_{12} \end{bmatrix} = \begin{bmatrix} (\eta\pi)_1 \to (\pi\pi)_2 \\ (K\bar{K})_1 \to (\pi\pi)_2 \end{bmatrix}$$

$$t_0^2(s) = t_{(\pi\pi)_2 \to (\pi\pi)_2}$$

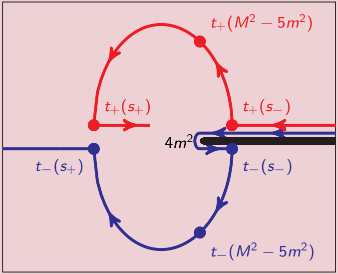
$$\text{disc } \boldsymbol{M}_2(s) = \boldsymbol{T}^1(s)\Sigma^1(s) \times \left(\boldsymbol{M}_2(s-i\epsilon) + \hat{\boldsymbol{M}}_2(s)\right) + \sigma_\pi(s)(t_0^2(s))^*\left(\boldsymbol{M}_2(s+i\epsilon) + \hat{\boldsymbol{M}}_2(s)\right)$$

- **Consistent approximation:** $\hat{N}_0(s)$, $\hat{G}_{10}(s)$, $\hat{H}_{10}(s)$, $\hat{G}_{12}(s)$: we neglect these LHC functions (would require all the related cross channels amplitudes...).
- Further approximation: For $I = J = 1$, we consider elastic $\pi\pi$.

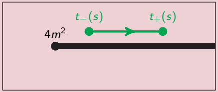
Fitting



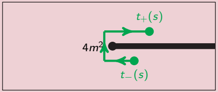
Endpoints



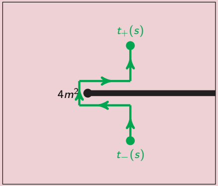
$4m^2 \leqslant s \leqslant \frac{M^2-m^2}{2}$



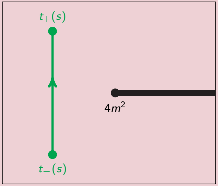
$\frac{M^2-m^2}{2} \leqslant s \leqslant s_-$



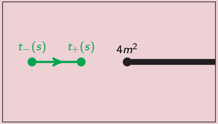
$s_- \leqslant s \leqslant M^2-5m^2$



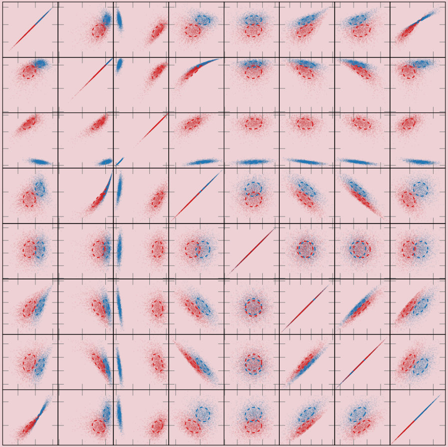
$M^2-5m^2 \leqslant s \leqslant s_+$



$s_+ \leqslant s$



MC and correlations



KT and phase space