Semi-inclusive Electroweak Interactions with Nuclei

T. W. Donnelly, M.I.T.

Study being done in collaboration with J. W. Van Orden (see Wally's talk later)
and O. Moreno

… based on the paper by O. Moreno, TWD, J. W. Van Orden and W. P. Ford, *Phys. Rev.* **D90** (2014) 013014 [MDVF]

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1) Basics of kinematics for semi-inclusive electroweak reactions
2) General form of the nuclear response; dynamical variables
3) Trajectories in the missing energy – missing momentum plane
4) Discussion of implications

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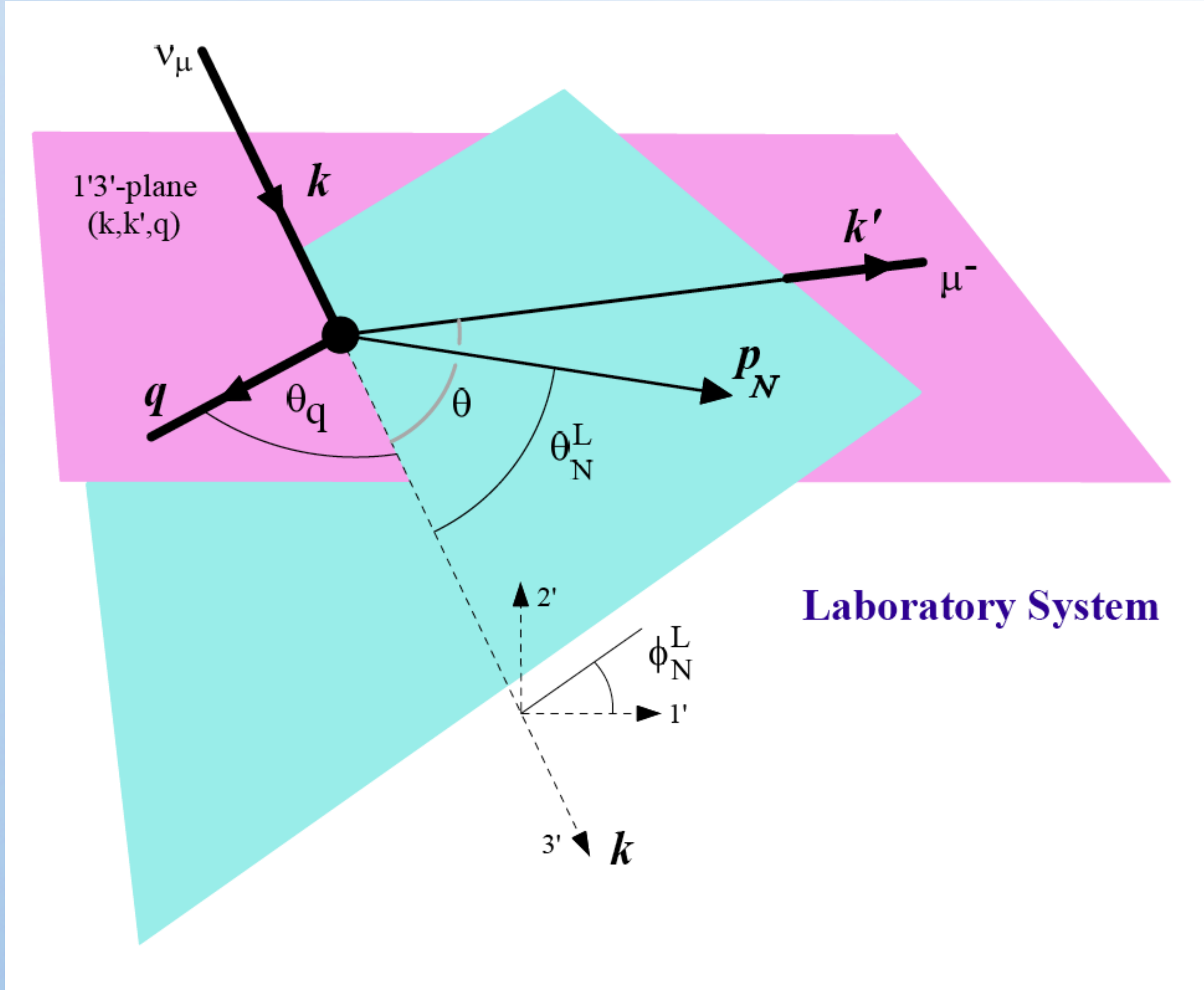
4) Discussion of implications

Note: Only very basic assumptions here (Lorentz covariance, parity properties, etc.), that is, no detailed model assumptions


ν_μ
1'3'-plane
(k,k',q)
k
k'
μ^-
q
θ_q
θ
p_N
θ_N^L
2'
ϕ_N^L
1'
3'
k
Laboratory System

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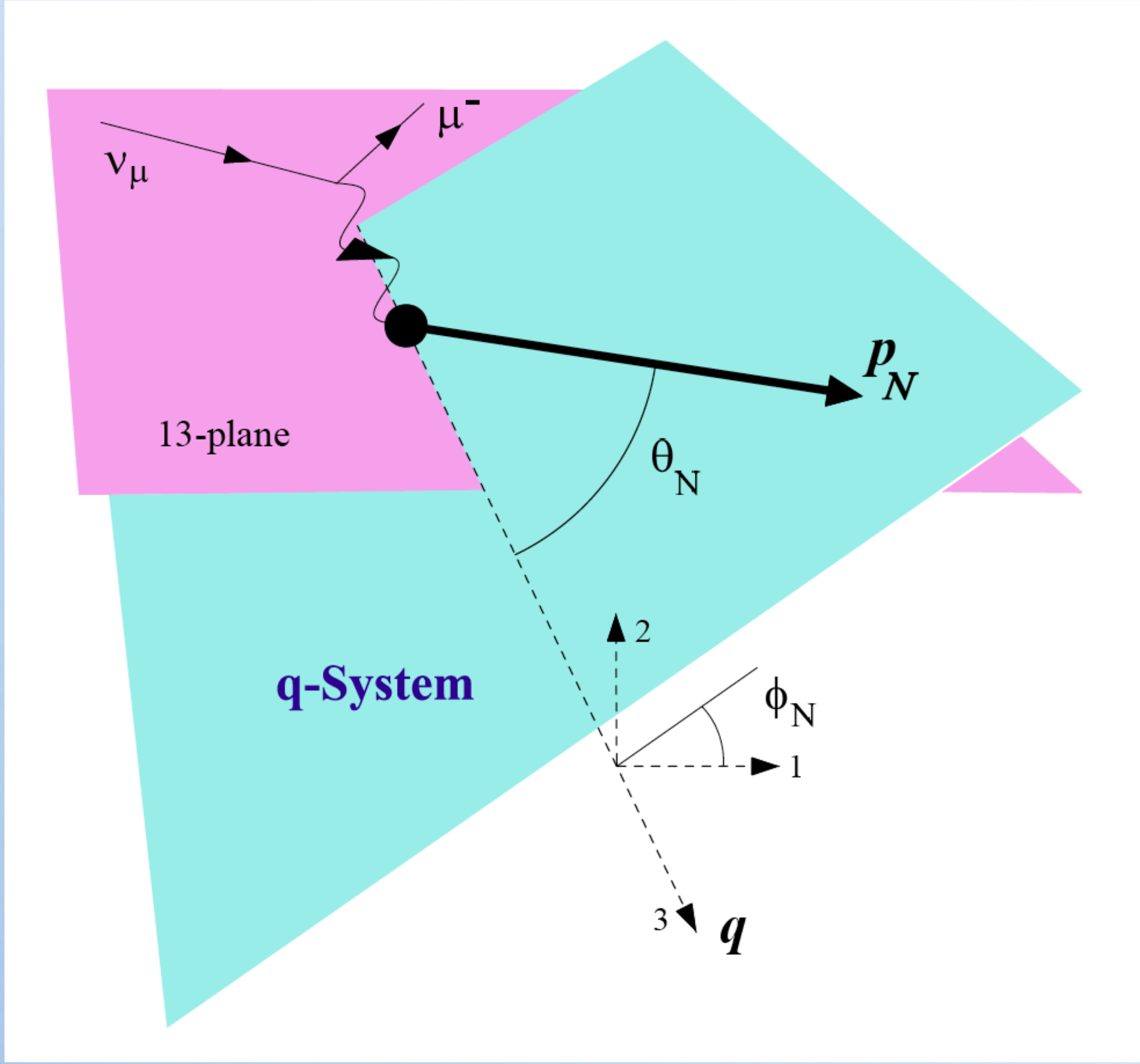

ν_μ
μ^-
p_N
13-plane
θ_N
2
q-System
ϕ_N
1
3
q

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Note that as the neutrino momentum k changes the direction of q also changes, and thus (θ_N,ϕ_N) change

… however, these variables are better suited to treating the general form of the semi-inclusive cross section

The general form of the cross section involves the contraction of the leptonic and hadronic tensors:

$$\sigma \sim \eta_{\mu\nu} W^{\mu\nu} = \eta^{s}_{\mu\nu} W^{\mu\nu}_{s} + \chi \, \eta^{a}_{\mu\nu} W^{\mu\nu}_{a},$$

where $\chi = 1$ for incident neutrinos and $\chi = -1$ for antineutrinos.

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For inclusive scattering one has

$$\begin{aligned} \eta^s_{\mu\nu} W^{\mu\nu}_s &\sim \widehat{V}_{CC} W^{CC}_{incl} + \widehat{V}_{CL} W^{CL}_{incl} + \widehat{V}_{LL} W^{LL}_{incl} + \widehat{V}_T W^T_{incl} \\ \eta^a_{\mu\nu} W^{\mu\nu}_a &\sim \widehat{V}_{T'} W^{T'}_{incl}, \end{aligned}$$

where each of the 5 responses is a function of the neutrino momentum k and 2 other variables, for instance (k', θ), the muon momentum and the lepton scattering angle, or (q, ω), the 3-momentum transfer and energy transfer:

$$\begin{aligned} W^K_{incl} &= W^K_{incl}(k; k', \theta) \\ &= W^K_{incl}(k; q, \omega), \end{aligned}$$

where $K = CC, CL, LL, T$ and T'. The factors V_K are the leptonic kinematic factors ("Rosenbluth factors") which can be found, for instance, in MDVF.

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In contrast, for semi-inclusive reactions one has more terms. Specifically, for CCν reactions (see MDVF), one has the following completely general structure:

$$\begin{aligned} \eta^s_{\mu\nu} W^{\mu\nu}_s \quad &\sim \quad \widehat{V}_{CC} W^{CC}_{semi} + \widehat{V}_{CL} W^{CL}_{semi} + \widehat{V}_{LL} W^{LL}_{semi} \\ &\quad + \widehat{V}_T W^{T}_{semi} + \widehat{V}_{TT} W^{TT}_{semi} + \widehat{V}_{TC} W^{TC}_{semi} + \widehat{V}_{TL} W^{TL}_{semi} \end{aligned}$$

$$\eta^a_{\mu\nu} W^{\mu\nu}_a \sim \widehat{V}_{T'} W^{T'}_{semi} + \widehat{V}_{TC'} W^{TC'}_{semi} + \widehat{V}_{TL'} W^{TL'}_{semi}$$

$$\begin{aligned} W^K_{semi} \quad &= \quad W^K_{semi}(k; k', \theta; p_N, \theta^L_N, \phi^L_N) \\ &= \quad W^K_{semi}(k; q, \omega; p, \mathcal{E}, \phi_N). \end{aligned}$$

Total: 10 response functions, each a function of 6 variables. (Actually, in general there are 16 classes of response for electroweak reactions of all types; for CCν reactions 6 cases do not have the corresponding leptonic factors, labeled $\underline{TT}, \underline{TC}.\underline{TL}, \underline{CL'}, \underline{TC'}, \underline{TL'}$.)

The ϕ_N dependence can be made explicit, leaving 6 individual responses, each a function of 5 variables, say $(k; q, \omega; p, \mathcal{E})$:

$$
\begin{aligned}
W^{CC}_{semi} &= \frac{1}{\rho^2}\left\{\rho^2 X_1 + \rho\nu^2 X_2 + X_3 + 2\sqrt{\rho}\nu X_4\right. \\
&\quad \left. + H^2 X_5 + 2\sqrt{\rho}\nu H X_6 + 2H X_7\right\} \\
W^{CL}_{semi} &= \frac{\nu}{\rho^2}\left\{\rho X_2 + X_3 + \sqrt{\rho}(\frac{1}{\nu}+\nu) X_4\right. \\
&\quad \left. + H^2 X_5 + \sqrt{\rho}(\frac{1}{\nu}+\nu) H X_6 + 2H X_7\right\} \\
W^{LL}_{semi} &= \frac{1}{\rho^2}\left\{-\rho^2 X_1 + \rho X_2 + \nu^2 X_3 + 2\sqrt{\rho}\nu X_4\right. \\
&\quad \left. + \nu^2 H^2 X_5 + 2\sqrt{\rho}\nu H X_6 + 2\nu^2 H X_7\right\} \\
W^{T}_{semi} &= -2X_1 + X_5 \eta_T^2 \\
W^{TT}_{semi} &= -X_5 \eta_T^2 \cos 2\phi_N \\
W^{TC}_{semi} &= \frac{2\sqrt{2}}{\rho}\eta_T\left\{H X_5 + \sqrt{\rho}\nu X_6 + X_7\right\}\cos\phi_N \\
W^{TL}_{semi} &= \frac{2\sqrt{2}}{\rho}\eta_T\left\{\nu H X_5 + \sqrt{\rho} X_6 + \nu X_7\right\}\cos\phi_N \\
W^{T'}_{semi} &= \frac{1}{\sqrt{\rho}}\left\{Z_1 + H Z_2\right\} \\
W^{TC'}_{semi} &= \frac{2\sqrt{2}}{\rho}\eta_T\left\{-\left(\sqrt{\rho}\nu Y_2 + Y_3\right)\sin\phi_N + \left(\sqrt{\rho} Z_2 + \nu Z_3\right)\cos\phi_N\right\} \\
W^{TL'}_{semi} &= \frac{2\sqrt{2}}{\rho}\eta_T\left\{-\left(\sqrt{\rho} Y_2 + \nu Y_3\right)\sin\phi_N + \left(\sqrt{\rho}\nu Z_2 + Z_3\right)\cos\phi_N\right\},
\end{aligned}
$$

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\end{aligned}
$$

where

$$
\begin{aligned}
\eta_T &\equiv \frac{p_N}{m_N}\sin\theta_N \\
H &\equiv \frac{1}{m_N}\left[E_N - \nu p_N \cos\theta_N\right],
\end{aligned}
$$

and where the Xs, Ys and Zs are functions of $(k; q, \omega; p, \mathcal{E})$, that is, are responses in 5-dimensional space.

All of this is fine; however, it does not capture where the nuclear response is large or small. To do this, it is better to change variables yet again to the missing energy and missing momentum, E_m and p_m, as is well known from studies of (e,e'p) reactions.

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See the Appendix for a summary of all of these coordinate transformations and definitions of all variables

Example: $^{16}O(\nu_\mu,\mu^- p)$ with

$k' = 1$ GeV/c for the muon

$\theta = 10$ deg.

$p_N = 50$ MeV/c for the proton

$q_N^L = 10$ deg.

$\phi_N^L = 180$ deg.

all of which will be kept fixed for this example

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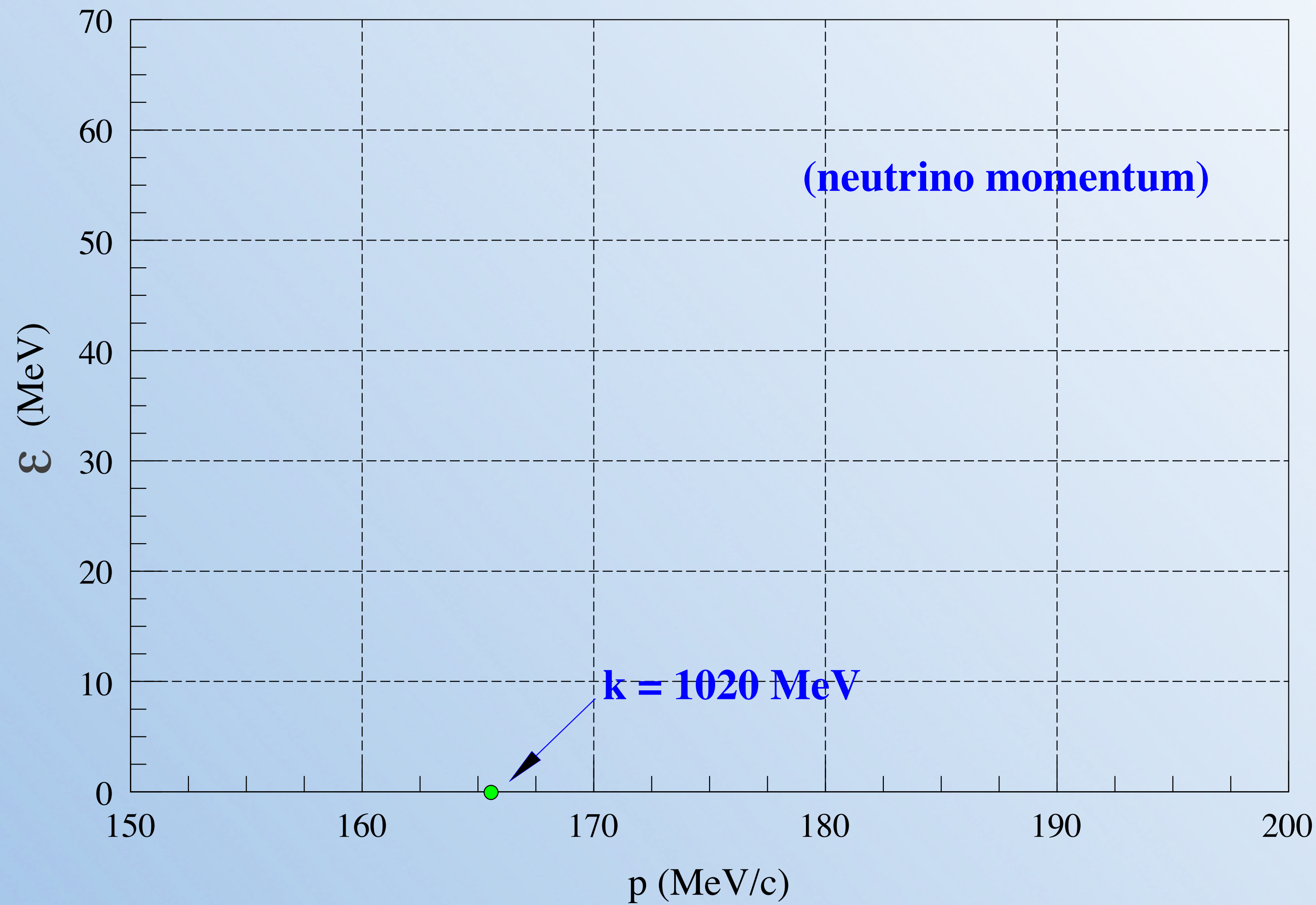
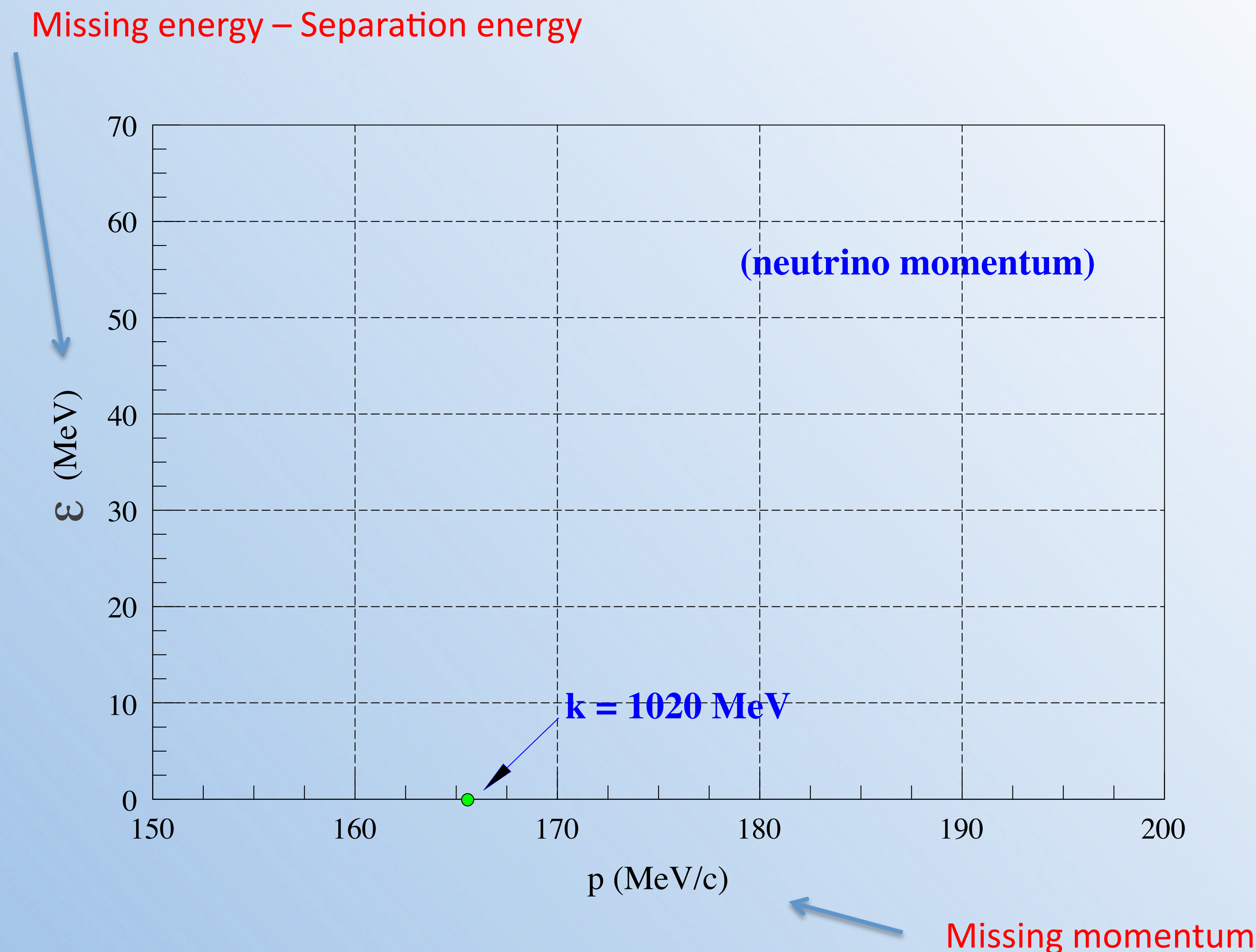
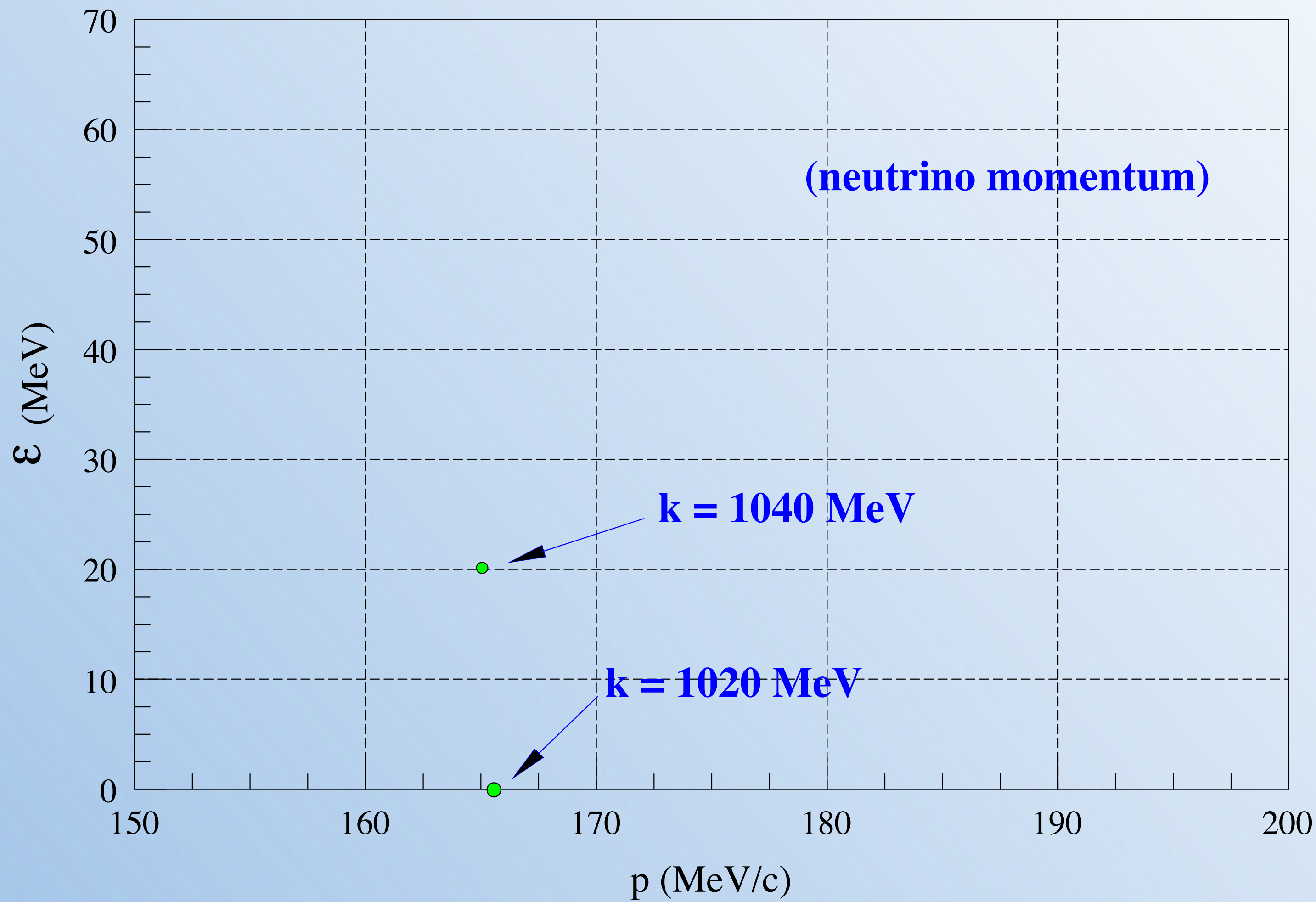
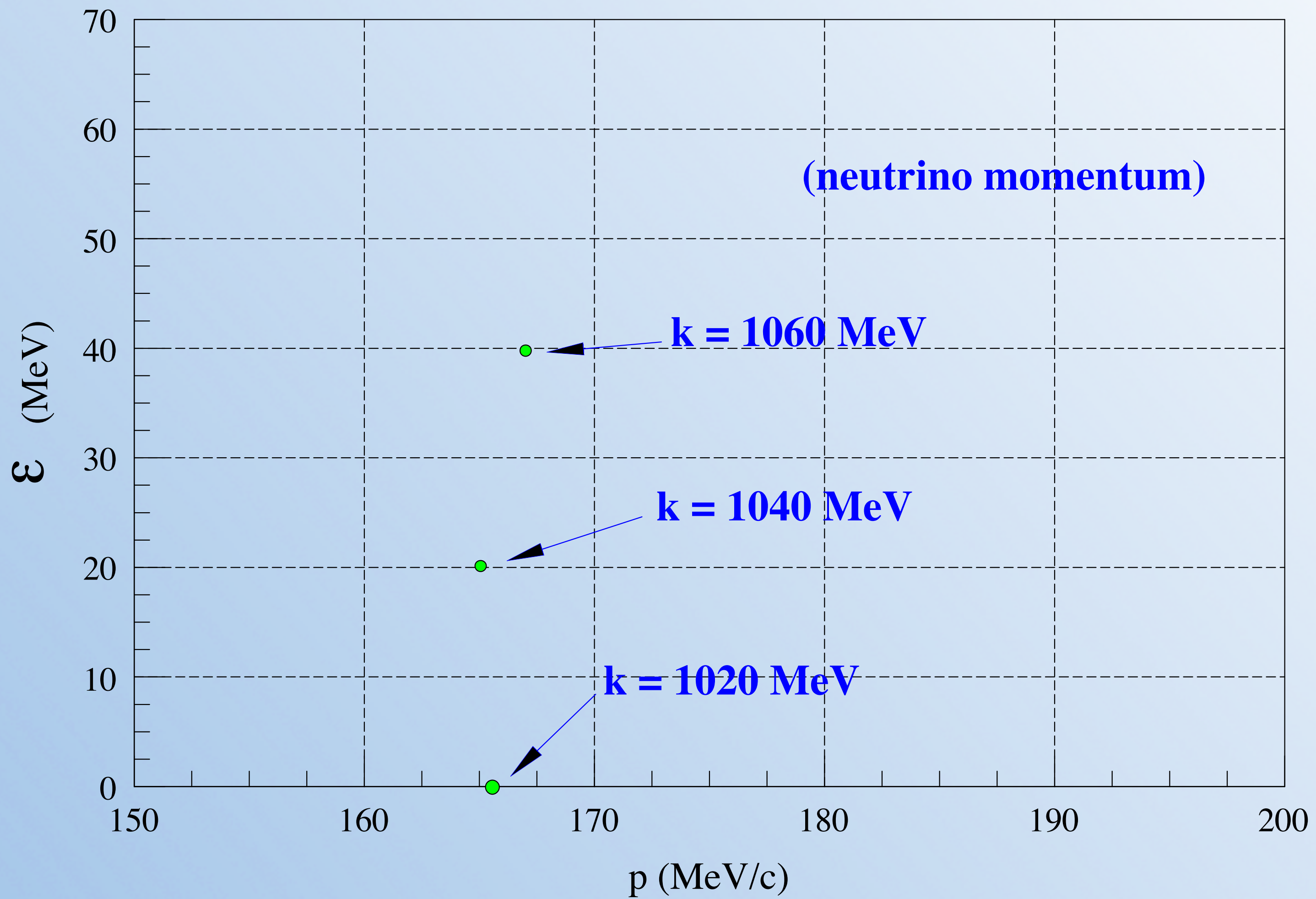
… starting with neutrino momentum k = 1020 MeV/c


70
60
50
40
30
20
10
0
150
160
170
180
190
200
ε (MeV)
p (MeV/c)
(neutrino momentum)
k = 1020 MeV

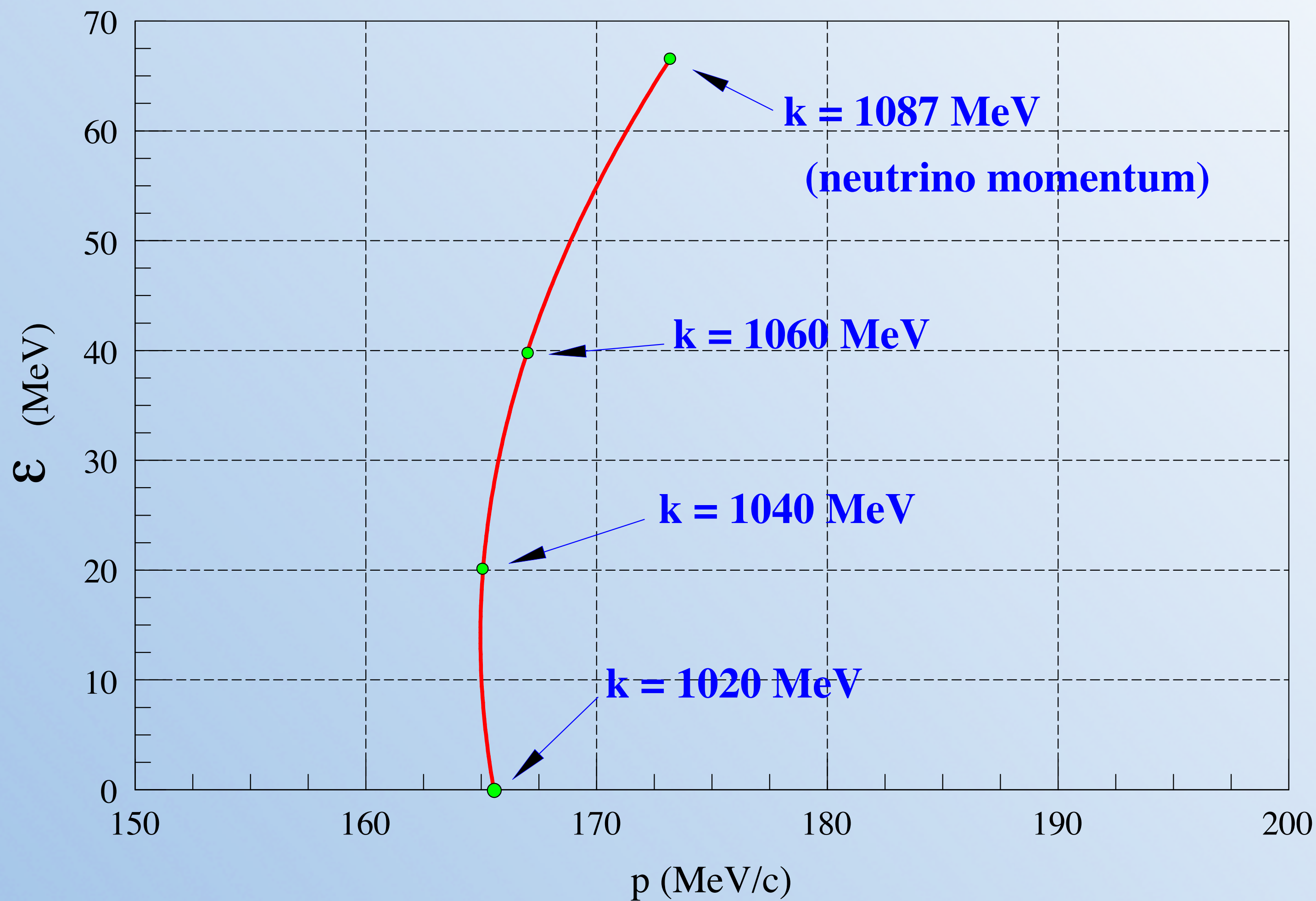

Missing energy – Separation energy
70
60
50
40
30
20
10
0
ε (MeV)
(neutrino momentum)
k = 1020 MeV
150
160
170
180
190
200
p (MeV/c)
Missing momentum



(neutrino momentum)
k = 1040 MeV
k = 1020 MeV
ε (MeV)
p (MeV/c)
0
10
20
30
40
50
60
70
150
160
170
180
190
200



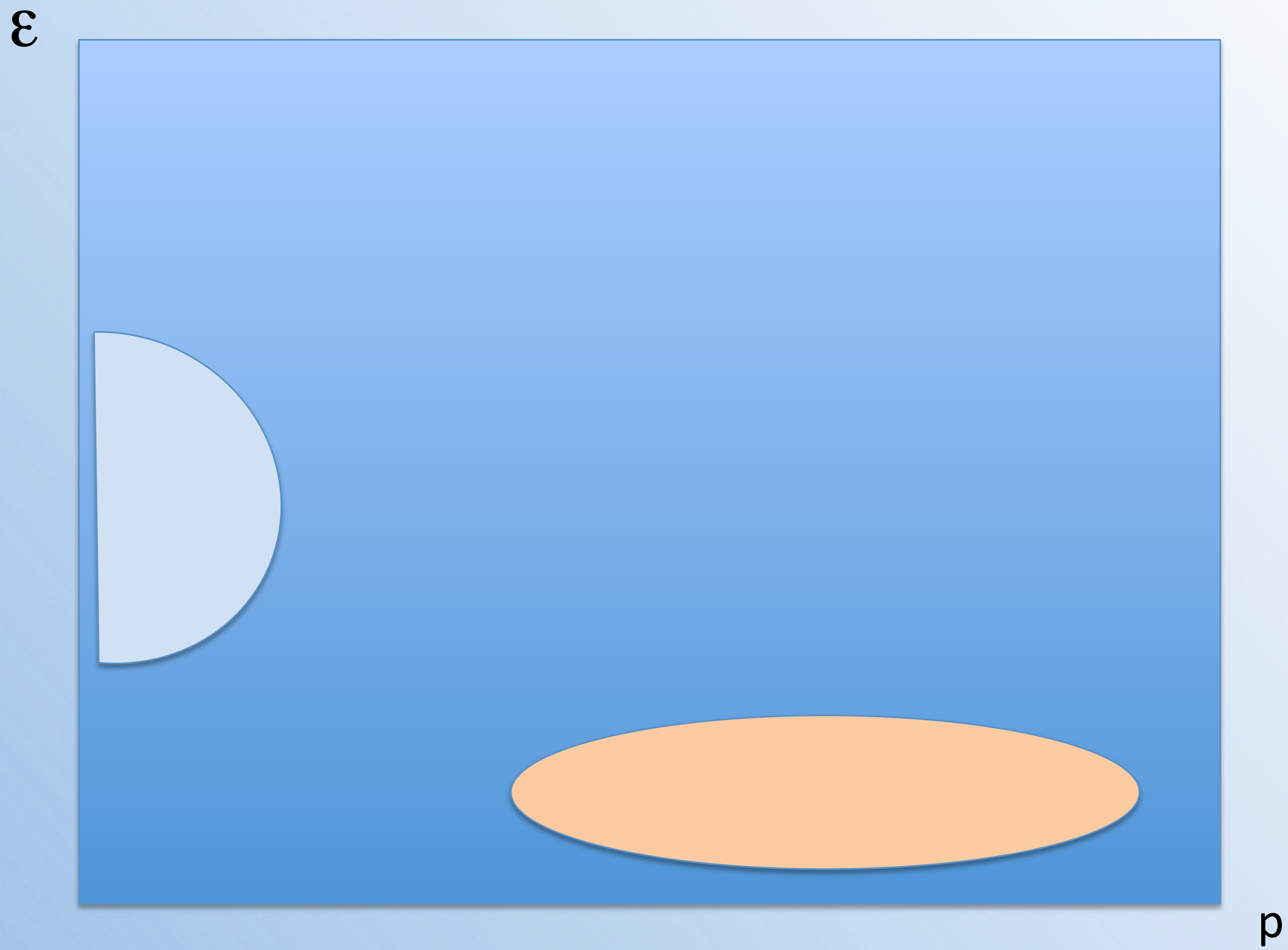
70
60
50
40
30
20
10
0
150
160
170
180
190
200
ε (MeV)
p (MeV/c)
(neutrino momentum)
k = 1060 MeV
k = 1040 MeV
k = 1020 MeV



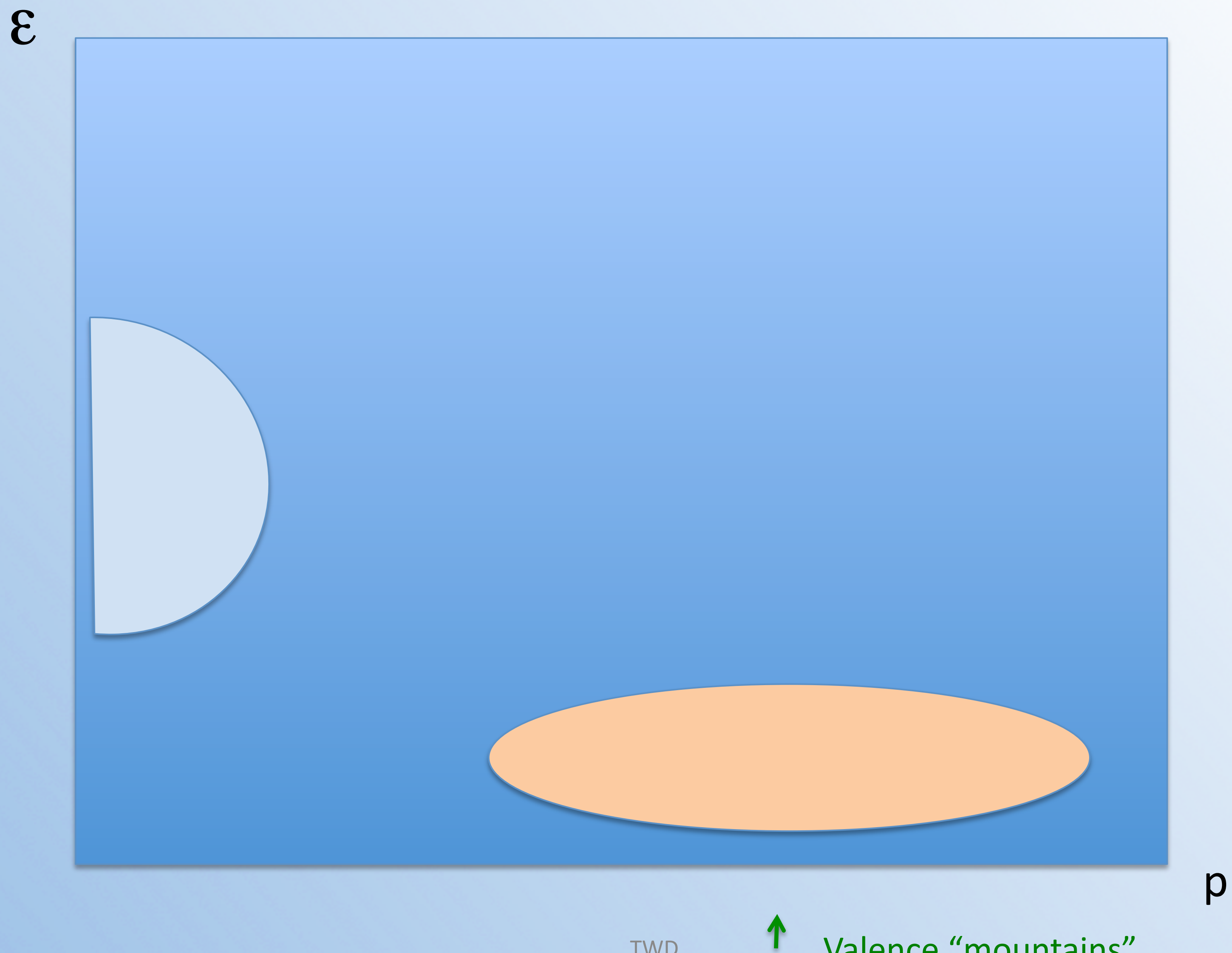
70
60
50
40
30
20
10
0
150
160
170
180
190
200
ε (MeV)
p (MeV/c)
k = 1087 MeV
(neutrino momentum)
k = 1060 MeV
k = 1040 MeV
k = 1020 MeV

Discussion of the **landscape** and **trajectories** in the missing energy – missing momentum plane:
this is where modeling comes into play

Generic landscape



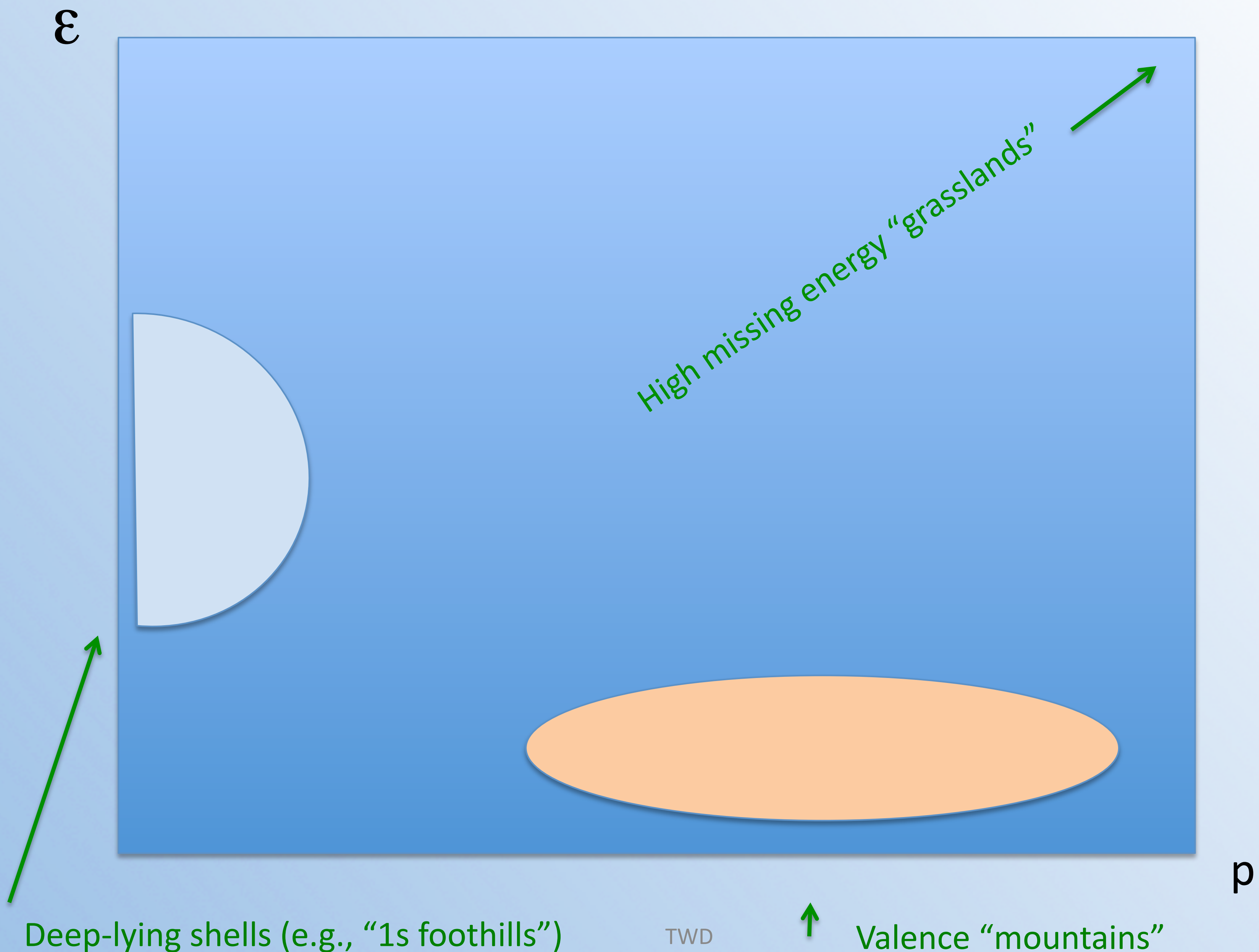
Generic landscape

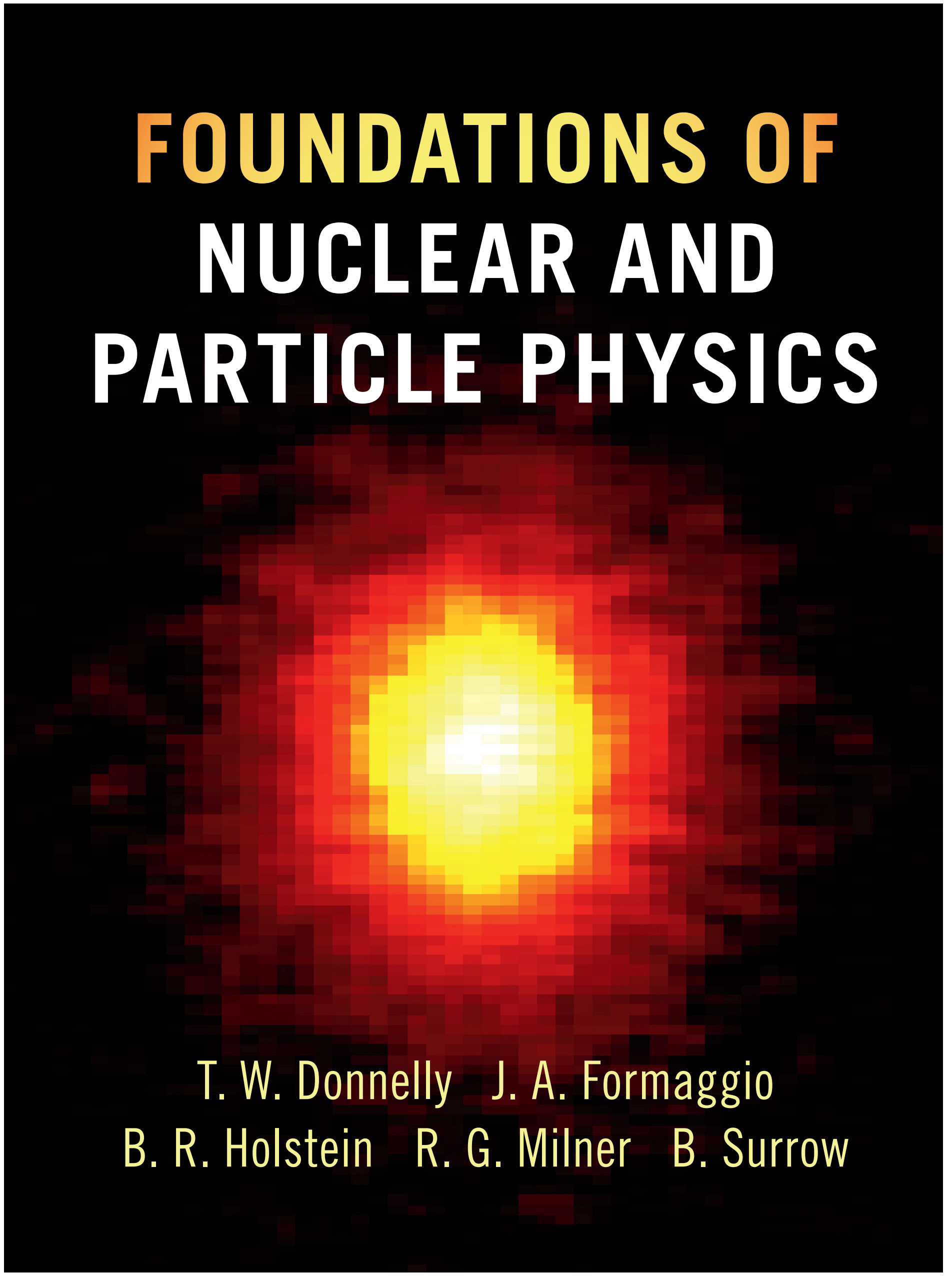


Valence "mountains"

Generic landscape

Generic landscape





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APPENDIX: NOTES ON SEMI-INCLUSIVE TRAJECTORIES

The following is a summary of the material in O. Moreno, TWD, J. W. Van Orden and W. P. Ford, *Phys. Rev.* **D90** (2014) 013014 [MDVF]. For these developments we begin in the "laboratory system" defined by the incident neutrino beam and the final-state charged lepton momentum. We assume a given neutrino momentum k and hence energy $\epsilon = \sqrt{k^2 + m^2}$, where m is the neutrino mass (usually taken to be zero). In the laboratory system we take the $3'$ axis to be along the incident neutrino momentum, the final-state charged lepton to lie in the $1' - 3'$ plane, and the normal to that plane to be in the $2'$ direction (see Fig. 1). The final-state charged lepton then has 3-momentum

$$\mathbf{k}' = k'(\sin\theta \mathbf{u}_{1'} + \cos\theta \mathbf{u}_{3'}), \quad (1)$$

where θ is the neutrino-charged lepton scattering angle. Defining m' to be the final-state charged lepton's mass we have for its energy $\epsilon' = \sqrt{k'^2 + m'^2}$. Then, with $M^2 \equiv (m^2 + m'^2)/2$, we have for the energy transfer, the 4-momentum transfer squared, and the 3-momentum transfer, respectively, the following:

$$\omega = \epsilon - \epsilon' \quad (2)$$

$$-Q^2 = q^2 - \omega^2 = 2(\epsilon\epsilon' - kk'\cos\theta - M^2) \quad (3)$$

$$q = \sqrt{-Q^2 + \omega^2}. \quad (4)$$

Of course, one must have

$$-Q^2 > 0 \text{ (spacelike)}. \quad (5)$$

If we define

$$E_f \equiv M_A^0 + \omega \qquad (6)$$

the final hadronic energy, where M_A^0 is the target rest mass, and note that the final total hadronic 3-momentum is

$$\mathbf{p}_f = \mathbf{q} \qquad (7)$$

in the laboratory system where the target is at rest, then the invariant mass of the final hadronic state is given by

$$W_A = \sqrt{E_f^2 - p_f^2} = \sqrt{(M_A^0 + \omega)^2 - q^2}. \qquad (8)$$

Since this must at least be greater than or equal to the breakup threshold, we have

$$W_A \geq W_{A-1}^0 + m_N = M_A^0 + E_s, \qquad (9)$$

where W_{A-1}^0 is the ground state invariant mass of the A-1 system, m_N is the nucleon mass and E_s is the separation energy. This leads to the constraint

$$\omega \geq \omega_T \equiv E_s + \frac{(-Q^2) + E_s^2}{2M_A^0}. \qquad (10)$$

Next let us rotate to the coordinate system oriented along the direction of the momentum transfer; this system is denoted the "q-system". The q-axis is oriented in the $1' - 3'$ plane at an angle θ_q, where

$$\cos\theta_q = \frac{1}{q}\left(k - k'\cos\theta\right) \tag{11}$$

$$\sin\theta_q = \frac{1}{q}k'\sin\theta. \tag{12}$$

Letting the rotated system have axes 3 (along $\mathbf{q}$), 2 (normal to the lepton scattering plane and equal to the $2'$ direction), together with 1 (forming a right-handed coordinate system 123; see Fig. 2), we have the following relating the unit vectors:

$$\mathbf{u}_{1'} = \cos\theta_q \mathbf{u}_1 - \sin\theta_q \mathbf{u}_3 \tag{13}$$

$$\mathbf{u}_{2'} = \mathbf{u}_2 \tag{14}$$

$$\mathbf{u}_{3'} = \sin\theta_q \mathbf{u}_1 + \cos\theta_q \mathbf{u}_3. \tag{15}$$

We now assume that a nucleon is detected in the laboratory system with 4-momentum $p_N^\mu = (E_N, p_N)$, where

$$E_N = \sqrt{p_N^2 + m_N^2}; \tag{16}$$

that is, in the laboratory system

$$\mathbf{p}_N = p_N(\sin\theta_N^L \cos\phi_N^L \mathbf{u}_{1'} + \sin\theta_N^L \sin\phi_N^L \mathbf{u}_{2'} + \cos\theta_N^L \mathbf{u}_{3'}), \tag{17}$$

where the magnitude of the nucleon's 3-momentum is denoted $p_N \equiv |\mathbf{p}_N|$ and the direction in the laboratory system (the neutrino-charged lepton system) is specified by the angles (θ_N^L, ϕ_N^L). Of course the 3-momentum may also be written in terms of the q-system unit vectors

$$\mathbf{p}_N = p_N(\sin\theta_N \cos\phi_N \mathbf{u}_1 + \sin\theta_N \sin\phi_N \mathbf{u}_2 + \cos\theta_N \mathbf{u}_3), \tag{18}$$

where now the angles (θ_N, ϕ_N) are employed (see Fig. 2). Upon equating these two expressions and using Eqs. (13-15) that relate the unit vectors we have the following relationships:

$$\sin\theta_N \cos\phi_N = \sin\theta_N^L \cos\phi_N^L \cos\theta_q + \cos\theta_N^L \sin\theta_q \tag{19}$$

$$\sin\theta_N \sin\phi_N = \sin\theta_N^L \sin\phi_N^L \tag{20}$$

$$\cos\theta_N = -\sin\theta_N \cos\phi_N \sin\theta_q + \cos\theta_N^L \cos\theta_q. \tag{21}$$

Since the polar angle θ_N is assumed to be in the range $[0^0, 180^0]$, the sine is non-negative and so

$$\sin\theta_N = \sqrt{1-\cos^2\theta_N}, \tag{22}$$

where $\cos\theta_N$ is given by Eq. (21), and hence the azimuthal angle in the q-systen is determined by

$$\cos\phi_N = \frac{1}{\sin\theta_N}\left(\sin\theta_N^L\cos\phi_N^L\cos\theta_q + \cos\theta_N^L\sin\theta_q\right) \tag{23}$$

$$\sin\phi_N = \frac{1}{\sin\theta_N}\sin\theta_N^L\sin\phi_N^L \tag{24}$$

using Eqs. (19,20,22). Knowing when the sines and cosines here are positive or negative we can determine the quadrant in which the angle ϕ_N lies. This fixes all of the variables in the q-system in terms of the laboratory variables.

Next it is useful to transform to variables that are well-suited to modeling and where the nuclear dynamics come into play. One can define the missing 3-momentum by

$$\mathbf{p}_m \equiv -\mathbf{p} = \mathbf{q} - \mathbf{p}_N \tag{25}$$

and immediately have

$$p = \sqrt{p_N^2 + q^2 - 2p_N q \cos\theta_N}, \tag{26}$$

namely, given in terms of the variables above. The energy of the recoiling (in general excited) A-1 system is given by

$$E_{A-1} = E_f - E_N \equiv \sqrt{W_{A-1}^2 + p^2}, \tag{27}$$

that is, in terms of the results given in Eqs. (6,16). Here W_{A-1} is the invariant mass of the A-1 system, again, in general in an excited state. It is convenient, as usual, to define the following effective excitation energy (the "daughter energy difference")

$$\mathcal{E} \equiv E_{A-1} - E_{A-1}^0 \geq 0, \tag{28}$$

where

$$E_{A-1}^0 = \sqrt{\left(W_{A-1}^0\right)^2 + p^2}. \tag{29}$$

This definition is such that $\mathcal{E}$ =0 when the final hadronic state is the ground state of the A-1 system. Typically one has the approximation

$$\mathcal{E} = (E_m - E_s)\left[1 - \frac{p^2}{2W_{A-1}W_{A-1}^0} + \cdots\right] \simeq E_m - E_s, \tag{30}$$

where E_m is the traditional definition of the missing energy, $E_m \equiv W_{A-1} + m_N - M_A^0$.

Finally, we have the variables that characterize the allowed region in the $(\mathcal{E}, p)$-plane:

$$y = \frac{1}{W_A^2}[Z_1 - Z_2] \qquad (31)$$

$$Y = \frac{1}{W_A^2}[Z_1 + Z_2], \qquad (32)$$

where W_A is given in Eq. (8) and

$$Z_1 = (M_A^0 + \omega) \times \sqrt{\Lambda - W_{A-1}^0(W_{A-1}^0 + m_N)}\sqrt{\Lambda - W_{A-1}^0(W_{A-1}^0 - m_N)} \qquad (33)$$

$$Z_2 = q\Lambda \qquad (34)$$

with

$$\Lambda = \frac{1}{2}\left[W_A^2 + \left(W_{A-1}^0\right)^2 - m_N^2\right]. \qquad (35)$$

Two curves in the $(\mathcal{E}, p)$-plane may be defined:

$$\mathcal{E}^{\pm}(q, y; p) \equiv \sqrt{m_N^2 + (q+y)^2} - \sqrt{m_N^2 + (q \pm p)^2} + \sqrt{\left(W_{A-1}^0\right)^2 + y^2} - \sqrt{\left(W_{A-1}^0\right)^2 + p^2} \tag{36}$$

and, since

$$\mathcal{E}^{-}(q, y; p) - \mathcal{E}^{+}(q, y; p) = \frac{4qp}{\sqrt{m_N^2 + (q+p)^2} + \sqrt{m_N^2 + (q-p)^2}} \geq 0, \tag{37}$$

one has $\mathcal{E}^{-}(q, y; p) \geq \mathcal{E}^{+}(q, y; p)$ (and equal to zero only when $p = 0$). From the original derivation of these results we have that the following constraints must be satisfied, defining the physical region:

$$\max(0, \mathcal{E}^{+}(q, y; p)) \leq \mathcal{E} \leq \mathcal{E}^{-}(q, y; p). \tag{38}$$

Thus, as long as the constraints in Eqs. (5,10,38) are all satisfied, one has a physical point in the $(\mathcal{E}, p)$-plane for fixed final-state charged lepton kinematics together with fixed laboratory system nucleon kinematics, all for a given neutrino energy. As the neutrino energy changes (for fixed final-state charged lepton kinematics and fixed laboratory system nucleon kinematics) the other variables defined above may also change (some must) and accordingly a different point will be found in the $(\mathcal{E}, p)$-plane. When the entire range of neutrino energies is spanned from threshold to very large values, a trajectory will the emerge as the line of solutions for given charged lepton and nucleon kinematics.