Collective Neutrino Oscillations as a many-body problem

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INT
August 2016

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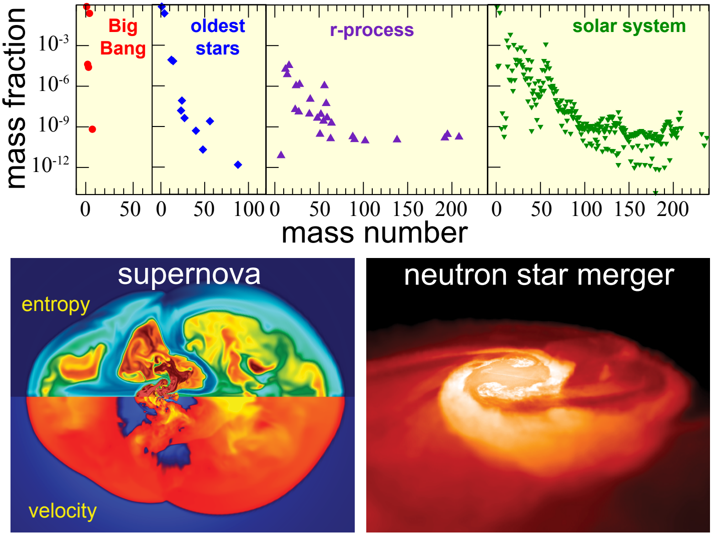
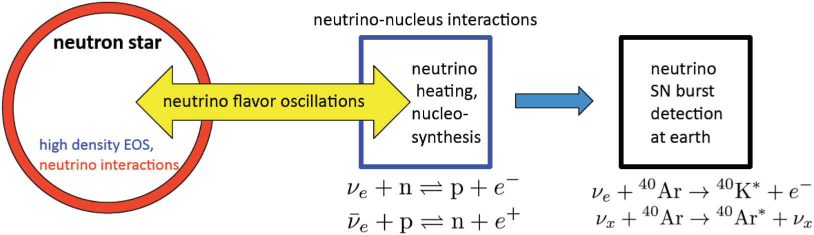
Motivation

Supernova neutrinos

- $M_{\rm progenitor} \geq 8 M_{\odot} \Rightarrow \Delta E \sim 10^{59}$ MeV
- 99 % of this energy is carried away by neutrinos and antineutrinos with $10 \leq E_{\nu} \leq 30$ MeV $\Rightarrow 10^{58}$ neutrinos!
  A neutrino many-body system!



Balantekin, *et al.*, arXiv:1401.6435 [nucl-th]



Balantekin and Fuller, arXiv:1303.3874 [nucl-th]

Neutrino many-body system

Many neutrino system

This is the only many-body system driven by the weak interactions:

Table: Many-body systems

Nuclei	Strong	at most $\sim$250 particles
Condensed matter	E&M	at most N_A particles
Neutron Star	Gravity + Strong	?
ν**'s in SN**	Weak	$\sim 10^{58}$particles

Astrophysical extremes allow us to test physics that cannot be tested elsewhere!

Neutrino Mixing

Mass and Flavor States

$$a_1(\mathbf{p}) = \cos\theta\, a_e(\mathbf{p}) - \sin\theta\, a_x(\mathbf{p})$$

$$a_2(\mathbf{p}) = \sin\theta\, a_e(\mathbf{p}) + \cos\theta\, a_x(\mathbf{p})$$

Neutrino Flavor Isospin Operators

$$\hat{J}^+_{\mathbf{p}} = a_e^\dagger(\mathbf{p}) a_x(\mathbf{p}) \ , \qquad \hat{J}^-_{\mathbf{p}} = a_x^\dagger(\mathbf{p}) a_e(\mathbf{p}) \ ,$$

$$\hat{J}^0_{\mathbf{p}} = \frac{1}{2}\left(a_e^\dagger(\mathbf{p}) a_e(\mathbf{p}) - a_x^\dagger(\mathbf{p}) a_x(\mathbf{p})\right)$$

$$[\hat{J}^+_{\mathbf{p}}, \hat{J}^-_{\mathbf{q}}] = 2\delta_{\mathbf{pq}} \hat{J}^0_{\mathbf{p}} \ , \qquad [\hat{J}^0_{\mathbf{p}}, \hat{J}^\pm_{\mathbf{q}}] = \pm\delta_{\mathbf{pq}} \hat{J}^\pm_{\mathbf{p}} \ ,$$

Neutrino Hamiltonian

Vacuum Oscillation Term

$$\hat{H}_\nu^{(1)} = \sum_{\mathbf{p}} \left(\frac{m_1^2}{2p} a_1^\dagger(\mathbf{p}) a_1(\mathbf{p}) + \frac{m_2^2}{2p} a_2^\dagger(\mathbf{p}) a_2(\mathbf{p}) \right) + \hat{I}(\ldots).$$

$$\hat{H}_\nu^{(1)} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p, \qquad \hat{B} = (\sin 2\theta, 0, -\cos 2\theta)$$

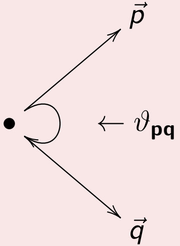
One-Body Hamiltonian including interactions with the electron background

$$\hat{H}_\nu = \sum_p \left(\frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p - \sqrt{2} G_F N_e J_p^0 \right)$$

Neutrino Hamiltonian

Neutrino-Neutrino Interactions

$$\hat{H}_{\nu\nu} = \frac{\sqrt{2}G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos\vartheta_{\mathbf{pq}})\, \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$



Note: due to the $(1 - cos\vartheta)$ term there is no interaction between neutrinos moving in the same direction.

Neutrino Hamiltonian

The total neutrino Hamiltonian

$$\hat{H}_{\text{total}} = H_\nu + H_{\nu\nu} = \left(\sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p - \sqrt{2} G_F N_e J_p^0 \right) + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} (1 - \cos \vartheta_{\mathbf{pq}}) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

Dasgupta, Duan, Fogli, Friedland, Fuller, Lisi, Lunardini, McKellar, Mirizzi, Qian, Pantaleone, Pastor, Pehlivan, Raffelt, Sawyer, Sigl, Smirnov, Balantekin, $\cdots$

Antineutrinos and three flavors

Including antineutrinos

$$H = H_\nu + H_{\bar{\nu}} + H_{\nu\nu} + H_{\bar{\nu}\bar{\nu}} + H_{\nu\bar{\nu}}$$

Requires introduction of a second set of SU(2) algebras!

Including three flavors

Requires introduction of SU(3) algebras.

Both extensions are straightforward, but tedious!
Balantekin and Pehlivan, J. Phys. G **34**, 1783 (2007).

Neutrino Hamiltonian

Neutrino Hamiltonian with $\nu - \nu$ interactions

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \sum_{\mathbf{p},\mathbf{q}} \left(1 - \cos \vartheta_{\mathbf{pq}}\right) \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

Single-angle approximation $\Rightarrow$

$$\hat{H}_{\text{total}} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \langle (1 - \cos \vartheta_{\mathbf{pq}}) \rangle \sum_{\mathbf{p} \neq \mathbf{q}} \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

Defining $\mu = \frac{\sqrt{2} G_F}{V} \langle (1 - \cos \vartheta_{\mathbf{pq}}) \rangle$, and $\omega_p = \frac{\delta m^2}{2p}$ one can write

$$\hat{H}_{\text{total}} = \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \mu \sum_{\mathbf{p} \neq \mathbf{q}} \vec{J}_{\mathbf{p}} \cdot \vec{J}_{\mathbf{q}}$$

BCS Hamiltonian

Hamiltonian in Quasi-spin basis

$$\hat{H}_{\text{BCS}} = \sum_k 2\epsilon_k \hat{t}_k^0 - |G| \hat{T}^+ \hat{T}.$$

Quasi-spin operators:

$$\hat{t}_k^+ = c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger, \qquad \hat{t}_k^- = c_{k\downarrow} c_{k\uparrow}, \qquad \hat{t}_k^0 = \frac{1}{2}\left(c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow} - 1\right)$$

$$[\hat{t}_k^+, \hat{t}_l^-] = 2\delta_{kl}\hat{t}_k^0 \ , \qquad [\hat{t}_k^0, \hat{t}_l^\pm] = \pm\delta_{kl}\hat{t}_k^\pm \ .$$

Richardson gave a solution of this problem. Hence there exist invariants of motion.

Gaudin method in neutrino physics

Gaudin algebra

$$[S^+(\lambda), S^-(\mu)] = 2\frac{S^0(\lambda) - S^0(\mu)}{\lambda - \mu}$$

$$[S^0(\lambda), S^\pm(\mu)] = \pm\frac{S^\pm(\lambda) - S^\pm(\mu)}{\lambda - \mu}$$

$$[S^0(\lambda), S^0(\mu)] = [S^\pm(\lambda), S^\pm(\mu)] = 0$$

λ is an arbitrary complex parameter. The operators

$$X(\lambda) = S^0(\lambda)S^0(\lambda) + \frac{1}{2}S^+(\lambda)S^-(\lambda) + \frac{1}{2}S^-(\lambda)S^+(\lambda)$$

satisfy $[X(\lambda), X(\mu)] = 0, \quad \lambda \neq \mu.$

Lowest weight vector is chosen to satisfy

$$S^-(\lambda)|0\rangle = 0, \quad \text{and} \quad S^0(\lambda)|0\rangle = W(\lambda)|0\rangle,$$

$$\Rightarrow X(\lambda)|0\rangle = \left[W(\lambda)^2 - \frac{\partial W(\lambda)}{\partial \lambda}\right]|0\rangle.$$

Excited states are given by

$$|\xi > \equiv |\xi_1, \xi_2, \ldots, \xi_n > \equiv S^+(\xi_1) S^+(\xi_2) \ldots S^+(\xi_n)|0 > .$$

The complex numbers $\xi_1, \xi_2, \ldots, \xi_n$ satisfy the Bethe Ansatz equations:

$$W(\xi_\alpha) = \sum_{\substack{\beta=1 \\ (\beta \neq \alpha)}}^{n} \frac{1}{\xi_\alpha - \xi_\beta} \quad \text{for} \quad \alpha = 1, 2, \ldots, n.$$

Corresponding eigenvalue of $X(\lambda)$ is

$$E_n(\lambda) = \left[W(\lambda)^2 - W'(\lambda)\right] - 2 \sum_{\alpha=1}^{n} \frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha}.$$

Neutrino representation of the Gaudin algebra

$$S^0(\lambda) = A + \sum_k \frac{\hat{J}_k^0}{\omega_k - \lambda} \qquad S^\pm(\lambda) = \sum_k \frac{\hat{J}_k^\pm}{\omega_k - \lambda}$$

ω_k and A are arbitrary constants. We choose the mass basis for the operators, $\omega_k = \delta m^2/2k$ and $A = -1/2\mu$.

$$X(\lambda) = \sum_p \frac{\mathbf{J}_p^2}{(\omega_p - \lambda)^2} + A^2 + Y(\lambda)$$

$$Y(\lambda) = \sum_{p,q,p\neq q} \frac{\mathbf{J}_p \cdot \mathbf{J}_q}{(\omega_p - \lambda)(\omega_q - \lambda)} + 2A \sum_p \frac{J_p^0}{(\omega_p - \lambda)}.$$

$$[X(\lambda), Y(\nu)] = 0 = [Y(\lambda), Y(\nu)]$$

$$\frac{h_p}{\mu} \equiv \lim_{\lambda \to \omega_p} (\lambda - \omega_p) Y(\lambda) = 2 \sum_{q, q \neq p} \frac{\mathbf{J}_p \cdot \mathbf{J}_q}{\omega_p - \omega_q} + \frac{1}{\mu} J_p^0$$

$$\frac{H}{\mu} = \sum_p \omega_p \frac{h_p}{\mu} = 2 \sum_{q,p,q \neq p} \mathbf{J}_p \cdot \mathbf{J}_q + \frac{1}{\mu} \sum_p \omega_p J_p^0.$$

$$[H, h_p] = 0$$

Above equations are written in the mass basis, but the transformation to flavor basis is easy

$$H_{\rm flavor} = T \, H_{\rm mass} T^{-1} \qquad T = e^{\theta (a_1^\dagger a_2 - a_2^\dagger a_1)}$$

The duality between $\nu - \nu$ and BCS Hamiltonians

The ν-ν Hamiltonian

$$\hat{H} = \sum_p \frac{\delta m^2}{2p} \hat{B} \cdot \vec{J}_p + \frac{\sqrt{2} G_F}{V} \vec{J} \cdot \vec{J}$$

$\Longleftrightarrow$

The BCS Hamiltonian

$$\hat{H}_{\mathrm{BCS}} = \sum_k 2\epsilon_k \hat{t}_k^0 - |G| \hat{T}^+ \hat{T}$$

Same symmetries leading to Analogous (dual) dynamics!

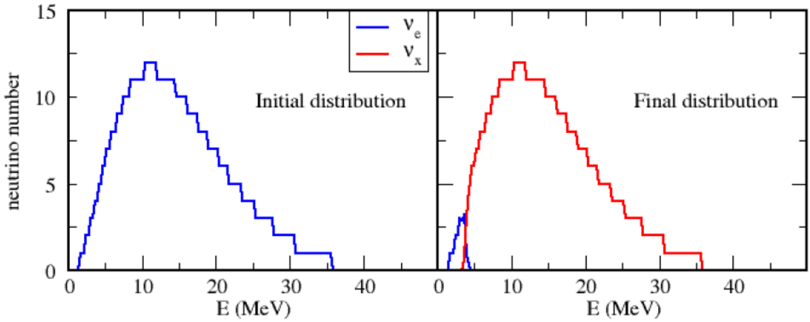
Pehlivan, Balantekin, Kajino, and Yoshida, Phys.Rev. D **84**, 065008 (2011).

Bethe ansatz equations

$$\underbrace{\sum_p \frac{-j_p}{\omega_p - \xi_\alpha} = \frac{1}{2\mu} + \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{N} \frac{1}{\xi_\alpha - \xi_\beta}\,.}_{\text{Bethe ansatz equations}}$$

Adiabatic solution of the exact many-body problem

Pehlivan et al., AIP Conf. Proc. **1743**, 040007 (2016)



Mean Field Approximations

When $[\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2] \sim 0$. Approximate the operator product as

$$\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2 \sim \hat{\mathcal{O}}_1\langle\hat{\mathcal{O}}_2\rangle + \langle\hat{\mathcal{O}}_1\rangle\hat{\mathcal{O}}_2 - \langle\hat{\mathcal{O}}_1\rangle\langle\hat{\mathcal{O}}_2\rangle \ ,$$

where the expectation values should be calculated with respect to a state $|\Psi\rangle$ which satisfies the condition $\langle\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2\rangle = \langle\hat{\mathcal{O}}_1\rangle\langle\hat{\mathcal{O}}_2\rangle$. This reduces the two-body problem to a one-body problem:

$$a^\dagger a^\dagger aa \Rightarrow \langle a^\dagger a\rangle a^\dagger a + \langle a^\dagger a^\dagger\rangle aa + \mathrm{h.c.}$$

Mean Field Approximations

$$\hat{H} \sim \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{P} \cdot \vec{J}$$

Polarization vector: $\vec{P}_{\mathbf{p},s} = 2\mu \langle \vec{J}_{\mathbf{p},s} \rangle$. Use SU(2) coherent states for the expectation value.

Mean-neutrino field

Polarization vectors

$$\hat{H} \sim \sum_p \omega_p \hat{B} \cdot \vec{J}_p + \vec{P} \cdot \vec{J}$$

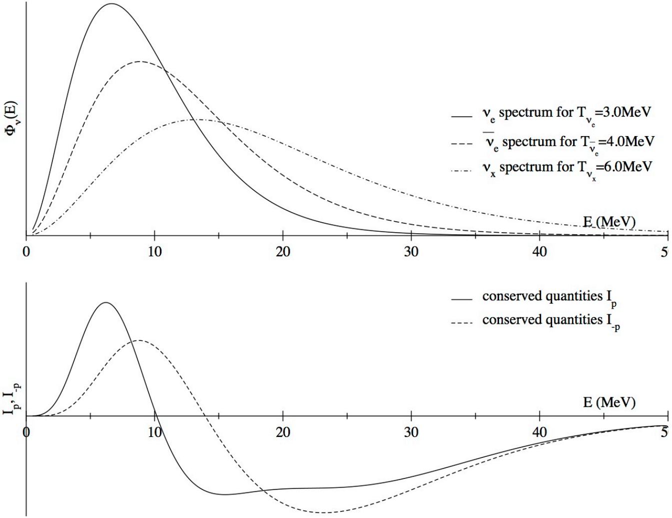
$$\vec{P}_{\mathbf{p},s} = 2\langle \vec{J}_{\mathbf{p},s} \rangle$$

Eqs. of motion: $$\frac{d}{d\tau}\vec{J}_p = -i[\vec{J}_p, \hat{H}^{\mathrm{RPA}}] = (\omega_p \hat{B} + \vec{P}) \times \vec{J}_p$$

Mean Field Consistency requirement $\Rightarrow \frac{d}{d\tau}\vec{P}_p = (\omega_p \hat{B} + \vec{P}) \times \vec{P}_p$

Invariants $$I_p = 2\langle \hat{h}_p \rangle = \hat{B} \cdot \vec{P}_p + \sum_{q(\neq p)} \frac{\vec{P}_p \cdot \vec{P}_q}{\omega_p - \omega_q} \Rightarrow \frac{d}{d\tau} I_p = 0$$

Raffelt; Pehlivan, Balantekin, Kajino, Yoshida


Φν(E)
νe spectrum for Tνe=3.0MeV
ν̄e spectrum for Tν̄e=4.0MeV
νx spectrum for Tνx=6.0MeV
E (MeV)
0
10
20
30
40
5
conserved quantities Ip
conserved quantities I-p
Ip, I-p
E (MeV)

Mean-neutrino field

Possible mean fields

Neutrino-Neutrino Interaction:

$$\overline{\Psi}_{\nu L}\gamma^{\mu}\Psi_{\nu L}\overline{\Psi}_{\nu L}\gamma_{\mu}\Psi_{\nu L} \Rightarrow \overline{\Psi}_{\nu L}\gamma^{\mu}\Psi_{\nu L}\langle\overline{\Psi}_{\nu L}\gamma_{\mu}\Psi_{\nu L}\rangle + \cdots$$

Antineutrino-Antineutrino Interaction:

$$\overline{\Psi}_{\overline{\nu} R}\gamma^{\mu}\Psi_{\overline{\nu} R}\overline{\Psi}_{\overline{\nu} R}\gamma_{\mu}\Psi_{\overline{\nu} R} \Rightarrow \overline{\Psi}_{\overline{\nu} R}\gamma^{\mu}\Psi_{\overline{\nu} R}\langle\overline{\Psi}_{\overline{\nu} R}\gamma_{\mu}\Psi_{\overline{\nu} R}\rangle + \cdots$$

Neutrino-Antineutrino Interaction:

$$\overline{\Psi}_{\nu L}\gamma^{\mu}\Psi_{\nu L}\overline{\Psi}_{\overline{\nu} R}\gamma_{\mu}\Psi_{\overline{\nu} R} \Rightarrow \overline{\Psi}_{\nu L}\gamma^{\mu}\Psi_{\nu L}\langle\overline{\Psi}_{\overline{\nu} R}\gamma_{\mu}\Psi_{\overline{\nu} R}\rangle + \cdots$$

Balantekin and Pehlivan, JPG **34**, 1783 (2007)

Mean-neutrino field

Possible mean fields

Neutrino-Antineutrino can also have an additional mean field:

$$\overline{\Psi}_{\nu L}\gamma^{\mu}\Psi_{\nu L}\overline{\Psi}_{\overline{\nu}R}\gamma_{\mu}\Psi_{\overline{\nu}R} \Rightarrow \overline{\Psi}_{\nu L}\gamma^{\mu}\langle\Psi_{\nu L}\overline{\Psi}_{\overline{\nu}R}\gamma_{\mu}\rangle\Psi_{\overline{\nu}R} + \cdots$$

However note that

$$\langle\Psi_{\nu L}\overline{\Psi}_{\overline{\nu}R}\gamma_{\mu}\rangle \propto m_{\nu}$$

(negligible is the medium isotropic)

Fuller et al., Volpe

CP-violating phases in collective oscillations

$$\begin{aligned} \text{Neutrinos}: \quad T_{ij}(p,\vec{p}) &= a_i^\dagger(\vec{p})a_j(\vec{p}) \\ \text{Antineutrinos}: \quad T_{ij}(-p,\vec{p}) &= -b_j^\dagger(\vec{p})b_i(\vec{p}) \end{aligned}$$

$$H_{\nu\nu} = \frac{G_F}{\sqrt{2}V} \sum_{i,j=1}^{3} \sum_{E,\vec{p}} \sum_{E',\vec{p}\,'} (1-\cos\theta_{\vec{p}\vec{p}\,'})\, T_{\alpha_i\alpha_j}(E,\vec{p})\, T_{\alpha_j\alpha_i}(E',\vec{p}\,')$$

$$\underbrace{H_\nu + H_{\nu\nu}}_{\text{with } \delta\neq 0} = S_\tau^\dagger \underbrace{H_\nu + H_{\nu\nu}}_{\text{with } \delta=0} S_\tau$$

$$S_\tau = e^{-i\delta(T_{\tau\tau}+\bar{T}_{\tau\tau})}$$

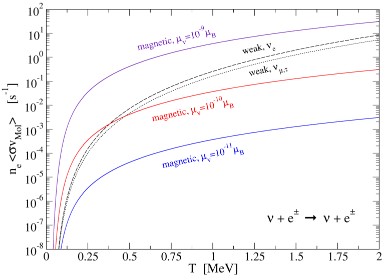
Including Spin-Flavor Precession

$$\underbrace{H_\nu + H_{\nu\nu} + H_{\mathrm{SFP}}(\mu)}_{\text{with } \delta\neq 0} = S_\tau^\dagger \underbrace{H_\nu + H_{\nu\nu} + H_{\mathrm{SFP}}(\mu^{\mathrm{eff}})}_{\text{with } \delta = 0} S_\tau$$

$$\mu^{\mathrm{eff}} = S\mu S = \begin{pmatrix} 0 & \mu_{12} & \mu_{13} e^{i\delta} \\ -\mu_{12} & 0 & \mu_{23} e^{i\delta} \\ -\mu_{13} e^{i\delta} & -\mu_{23} e^{i\delta} & 0 \end{pmatrix}.$$

Pehlivan *et al.*, Phys. Rev. D **90**, 065011 (2014)

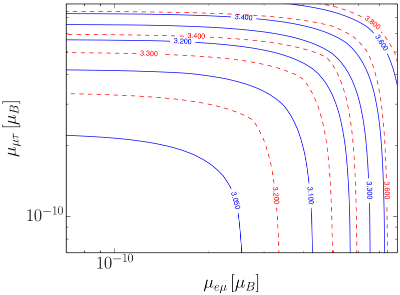
In the Early Universe weak and magnetic cross sections have a very different energy dependence. This has potentially many interesting ramifications for decoupling in the BBN epoch.



Vassh *et al.*. Phys. Rev. D **92**, 125020 (2015)

Majorana magnetic moment in the Early Universe

$$\rho_{\mathrm{relativistic}} = \frac{\pi^2}{15} T_\gamma^4 \left[1 + \frac{7}{8} N_{\mathrm{eff}} \left(\frac{4}{11} \right)^{4/3} \right]$$



Planck:

$$N_{\mathrm{eff}} = 3.30 \pm 0.27 \quad \Rightarrow$$

$$\mu_{\mathrm{Majorana}} \le 6 \times 10^{-10} \mu_B$$

Conclusions

How can we make further progress?

- We examined the many-neutrino gas both from the exact many-body perspective and an effective one-body description following introduction of a mean field. In the limit of the single angle approximation, both pictures possess constants of motion.
- At least in the single angle approximation, we can solve the full many-body problem in the adiabatic limit for a few simple cases. To go beyond those special cases we need to solve the Bethe ansatz equations.
- The condensed-matter community has developed many advanced tools to solve those equations. Bringing those two communities together to exchange ideas would significantly help SN neutrino physics community.