Transport Properties in Magnetic Field

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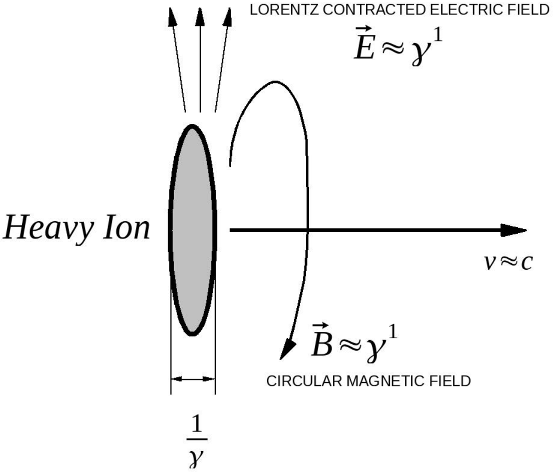
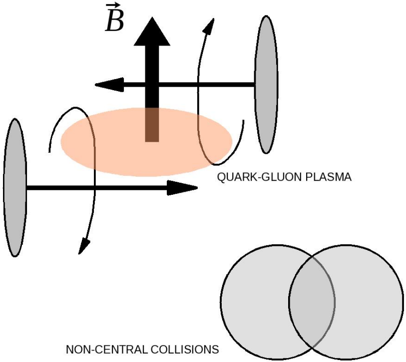
The magnetic field in heavy-ion collisions



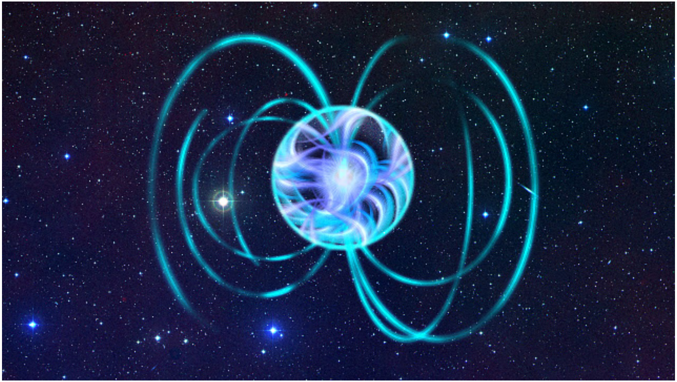
In heavy-ion collisions, two magnetic fields from the projectiles overlap along the same direction out of reaction plane



$$eB \sim (300\,\mathrm{MeV})^2 \sim T^2,$$

but the life time may be short $\tau \lesssim 1$ fm/c

Astrophysical eB of magnetars is about 10^{15-16} $G \sim (1\,\mathrm{MeV})^2$, but $T^2 \ll eB$



Transport properties in strong magnetic field

We will select the two topics:

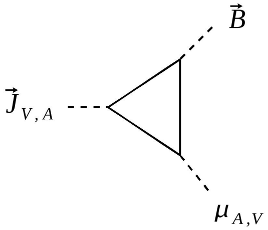
- **Chiral magnetic wave**
  (Kharzeev-HUY, arXiv:1012.6026)
- **Longitudinal electric conductivity in perturbative QCD in leading log**
  (Work in progress with Koichi Hattori and Shiyong Li)

Chiral Magnetic Wave

Chiral Magnetic Effect (CME) and Chiral Separation Effect (CSE)

(Fukushima-Kharzeev-Warringa, Son-Zhitnitsky, Vilenkin)

$$\vec{J}_V = \frac{eN_c}{2\pi^2}\mu_A \vec{B}\,, \quad \vec{J}_A = \frac{eN_c}{2\pi^2}\mu_V \vec{B}$$

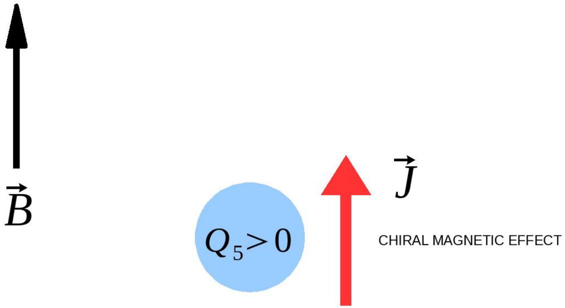


Note the $< AVV >$ structure

CME+CSE+Hydrodynamics = CMW

CMW is the inclusive universal language of CME/CSE in hydrodynamics

Why do we have waves ?



$$\vec{J}_V = \frac{N_c e \vec{B}}{2\pi^2} \mu_A \quad , \quad \vec{J}_A = \frac{N_c e \vec{B}}{2\pi^2} \mu_V$$

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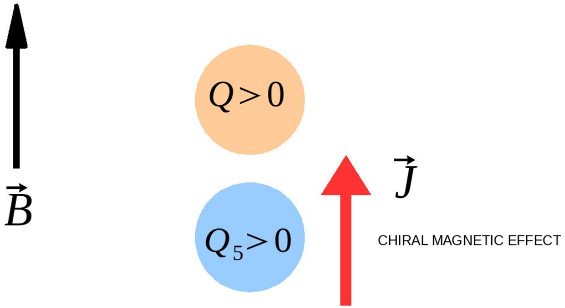

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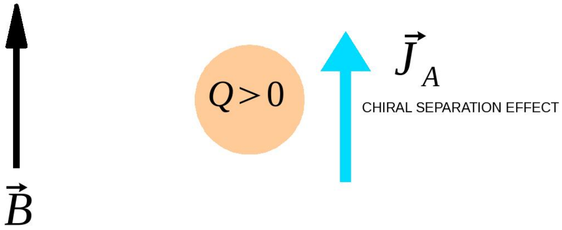

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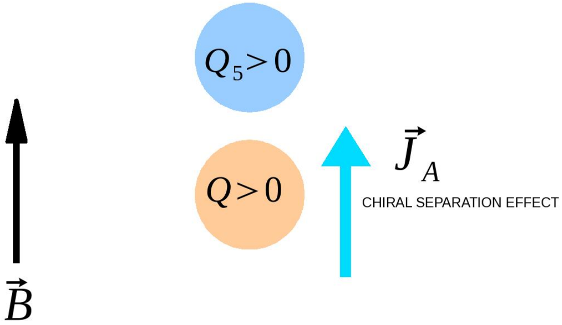

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CMW: Sound Waves of Chiral Charges

Add and subtract CME and CSE to go to
Left/Right-handed chiralities

$$\vec{J}_V = \frac{eN_c}{2\pi^2}\mu_A\vec{B}\,, \quad \vec{J}_A = \frac{eN_c}{2\pi^2}\mu_V\vec{B}$$

$$J_{R/L} \equiv \frac{1}{2}(J_V \pm J_A)$$

Then, we have a "diagonalization" of the CME/CSE

$$\vec{J}_R = \frac{eN_c}{4\pi^2}\mu_R\vec{B}\,, \quad \vec{J}_L = -\frac{eN_c}{4\pi^2}\mu_L\vec{B}\,, \quad \mu_{L/R} \approx \frac{1}{\alpha}n_{L/R}$$

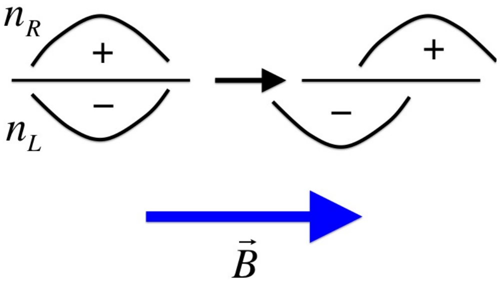
Hydro charge conservation $\partial_\mu J^\mu_{L/R} = 0$ gives

$$\left(\partial_t + \vec{v}_\chi \cdot \vec{\nabla}\right) n_R = 0\,, \left(\partial_t - \vec{v}_\chi \cdot \vec{\nabla}\right) n_L = 0\,, \vec{v}_\chi = \frac{eN_c}{4\pi^2\alpha}\vec{B}$$

Two Independent Uni-directional Propagating Waves !

Development of a charge dipole from initial axial charge

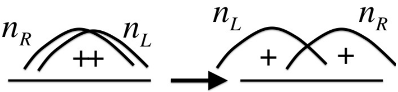
$$n_A > 0\,, \quad n_V = 0 \longrightarrow n_R = \frac{1}{2} n_A\,, \quad n_L = -\frac{1}{2} n_A = -n_R$$

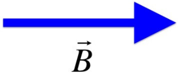


Development of a charge quadrupole from initial electric charge

(Burnier-Kharzeev-Liao-HUY, Gorba-Miransky-Shovkovy)

$$n_A = 0\,, \quad n_V > 0 \longrightarrow n_R = \frac{1}{2}n_V\,, \quad n_L = \frac{1}{2}n_V = n_R$$





Q: Axial charge is not conserved

Conservation law is modified to $\partial_\mu J_A^\mu = -\frac{1}{\tau_R} n_A$ where the relaxation rate is given by

$$\frac{1}{\tau_R} \sim \alpha_s^5 \log(1/\alpha_s) T + \alpha_s m_q^2 / T$$

(the second term is from the quark mass (Kaplan-Reddy))

The CMW dispersion relation becomes (Stephanov-HUY-Yin)

$$\omega = -\frac{i}{2\tau_s} \pm \sqrt{-\frac{1}{4\tau_s^2} + v_\chi^2 k^2}$$

A transition from CMW ($k \gg 1/\tau_R$) to pure diffusion ($k \ll 1/\tau_R$)

τ_R is numerically larger than 10 fm with $\alpha_s = 0.2$ and $m_q = 10$ MeV

It is okay to neglect this in heavy-ion collisions

Q: Electromagnetic field is dynamical

Longitudinal Mode:

The CMW dispersion becomes (Kharzeev-HUY)

$$\omega^2 = v_\chi^2 k^2 + m_B^2$$

where

$$v_\chi = \frac{eN_c}{2\pi^2\alpha} B\,, \quad m_B \equiv \frac{e^2 N_c}{2\pi^2\sqrt{\alpha}} B$$

CMW becomes "massive"

Numerically, $m_B^{-1} \approx 20$ fm with $eB \sim m_\pi^2$ and $T \approx 200$ MeV, the basic reason is the smallness of α_{EM}

It is okay to neglect this in heavy-ion collisions

Q: Electromagnetic field is dynamical

Transverse Mode:

Transverse modes become helicity-polarized via the transition from the fermion helicity (axial charge) into the magnetic helicity. For low enough k, the resulting chiral magneto-hydrodynamics features inverse cascades with turbulence (eg. a recent work by Hirono-Kharzeev-Yin and Yamamoto).

Due to the smallness of α_{EM} and μ_A/T, the space-time scale $k \sim \alpha_{EM}\mu_A$ is numerically much larger than 10 fm

It is okay to neglect this in heavy-ion collisions, unless $\mu_A/T \sim 1/\alpha_{EM} \sim 100$

Longitudinal Electric Conductivity in Strong Magnetic Field in Perturbative QCD

Work in progress with Koichi Hattori and Shiyong Li

We will compute the electric conductivity along the direction of magnetic field

$$J = \sigma_{zz} E$$

in the limit of

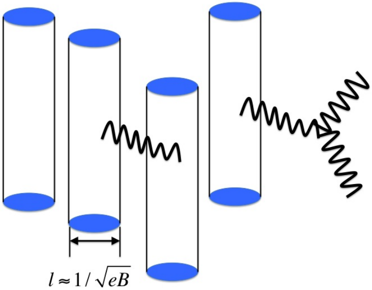
- $eB \gg T^2$, **which means we can focus on the lowest Landau levels (LLL)**
- **pertubative QCD in the deconfined phase, and will further assume** $\alpha_s eB \ll T^2$
- **small enough electric field** $eE \ll m_q^2$ **to avoid quantum anomaly (Schwinger mechanism)**
- **dynamical degrees of freedom are thermal quarks and gluons that are nearly massless**
- **neutral quark-gluon plasma**

The final result at leading log (preliminary) is

$$\sigma_{zz} = e^2 \left(\sum_F q_F^3 \right) \frac{9}{\pi^3} \frac{eB}{T} \frac{1}{\alpha_s^2 \log(T^2/\alpha_s eB)}$$

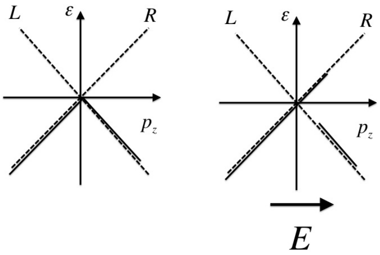
We will show some technical details in the rest of the talk

Lowest Landau Level (LLL) thermal quarks-anti quarks are the charge carriers, which are interacting with themselves and thermal gluons



These states move only in 1+1 dimensions and their transverse size is $l_B \sim 1/\sqrt{eB}$. The latter will introduce a "form factor" $R(\boldsymbol{q}_\perp) \sim e^{-\frac{\boldsymbol{q}_\perp^2}{4eB}}$ along the transverse direction in the interaction vertex. We will also see a "Schwinger phase" in the vertex

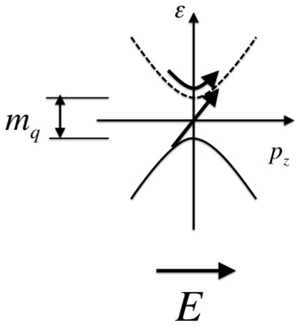
We introduce a small, but finite quark mass $T^2 \gg m_q^2 \gg |e\boldsymbol{E}|$ to avoid Schwinger mechanism in the massless limit



Since the 1+1 dim density of states is $p_z/(2\pi)$, the transverse density of states is $eB/(2\pi)$, and $\dot{p}_z = eE$

$$\dot{n}_A = (eB/2\pi)\dot{p}_z/(2\pi) = \frac{e^2}{4\pi^2} E \cdot B$$

which is precisely the chiral anomaly (Gribov)



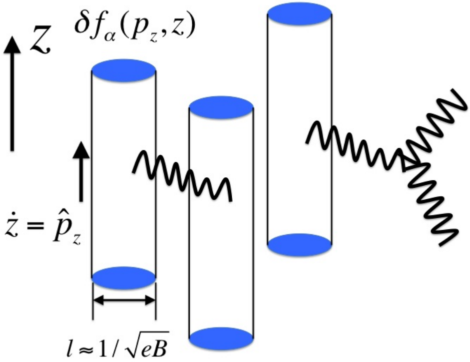
Semi-classical motion vs. quantum tunneling

quantum tunneling amplitude for pair creation $\sim e^{-\frac{m_q^2}{|e\boldsymbol{E}|}}$

However, m_q^2 will be neglected in the dispersion relation of the hard modes ($\boldsymbol{p} \sim T$) which dominate the charge transports.

A hybrid kinetic theory description

Introduce a distribution function for each of the LLL states: $f_{\alpha,\pm}(p_z, z)$, where α labels the LLL states and (p_z, z) is the longitudinal momentum and position, and $\pm$ refers to quark-antiquark.



Since the applied electric field is homogeneous, the solution will be independent of α and z. Also by charge conjugation and z-reflection,

$$\delta f_-(p_z) = -\delta f_+(p_z) = +\delta f_+(-p_z).$$

The longitudinal current will be

$$J = 2e\left(\frac{eB}{2\pi}\right)\int_{-\infty}^{+\infty}\frac{dp_z}{(2\pi)}\hat{p}_z\delta f_+(p_z) = 4e\left(\frac{eB}{2\pi}\right)\int_0^{\infty}\frac{dp_z}{(2\pi)}\delta f_+(p_z)$$

where $\hat{p}_z = \mathrm{sgn}(p_z)$ is the longitudinal velocity

Boltzmann equation for $f_{\alpha,+}(p_z, z)$

$$\partial_t f_{\alpha,+}(p_z, z) + \dot{z}\partial_z f_{\alpha,+}(p_z, z) + \dot{p}_z \partial_{p_z} f_{\alpha,+}(p_z, z) = C[f]$$

Homogeneous electric field in static limit gives $\dot{p}_z = eE$, $\partial_t = \partial_z = 0$, and to linear order in E, $f_{\alpha,+}(p_z, z) = f^{eq}(|p_z|) + \delta f(p_z)$ and the Boltzmann equation becomes

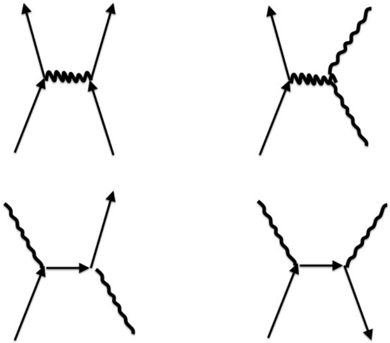
$$-\beta(eE)\hat{p}_z f^{eq}(|p_z|)(1 - f^{eq}(|p_z|)) = C[\delta f]$$

where $f^{eq}(\epsilon) \equiv 1/(e^{\beta\epsilon} + 1)$

The key is the collision term $C[\delta f]$

We will see a leading log of $\sim \alpha_s^2 \log\left(\frac{T^2}{\alpha_s eB}\right)$

Following the previous studies on the case without magnetic fields (Arnold-Moore-Yaffe), we look at the potentially important t-channel scattering diagrams involving LLL quarks-antiquarks.



Feynman Rules of LLL States

We choose to work in the Landau gauge $\boldsymbol{A} = (0, Bx^1, 0)$. Then (p_2, p_z) are well-defined quantum numbers. $p_2 = \alpha$ is the label for LLL states.

Note that there is no concept of p_1 for LLL fermions

The wave functions are localized in x^1 as

$$\phi_{p_2} = e^{ip_2 x^2} H_0\left(x_1 - \frac{p_2}{eB}\right)$$

Quantization:

$$\psi(\boldsymbol{x}) = \frac{1}{\sqrt{L_2 L_z}} \sum_{p_z, p_2} \frac{1}{\sqrt{2|p_z|}} e^{ip_2 x^2 + ip_z z} H_0\left(x^1 - \frac{p_2}{eB}\right) u_{1+1}(p_z) \times a_{p_2, p_z} + \mathrm{h.c.}$$

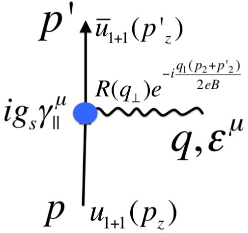
with $\{a_{p_2,p_z}, a^\dagger_{p_2',p_z'}\} = \delta_{p_2,p_2'}\delta_{p_z,p_z'}$

Form Factor and Schwinger Phase

Working out the interaction vertex

$$H_I = g_s \int d^3x\, A_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)$$

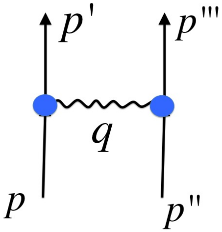
- **Only the longitudinal current exists:** $\mu = t$ **or** $\mu = z$
- **The vertex has a form factor and a Schwinger phase (Hattori, Kojo, Su)**



$$ig_s\epsilon_\mu[\bar{u}_{1+1}(p'_z)\gamma^\mu_{||}u_{1+1}(p_z)]R(\boldsymbol{q}_\perp)e^{-i\frac{q_1}{2eB}(p_2+p'_2)}$$

where $R(\boldsymbol{q}_\perp) = e^{-q_\perp^2/(4eB)}$, $\boldsymbol{q} = (q_1, p'_2 - p_2, p'_z - p_z)$

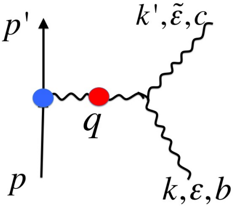
Examples of Collision Terms



$$C_{q-q}[f_{p_2}(p_z)] = \int_{p'}\int_{p''}\int_{p'''}\left|\int \frac{dq^1}{(2\pi)} e^{-i\frac{q_1}{eB}(p_2'-p_2'')}[R(\boldsymbol{q}_\perp)]^2\mathcal{M}\right|^2$$
$$\times\quad (2\pi)^3\delta^{(3)}(p+p''-p'-p''')$$
$$\times\quad f(p)f(p'')(1-f(p'))(1-f(p'''))$$

where $\int_p = \int \frac{d^2p}{2|p_z|(2\pi)^2}$ **and** $\mathcal{M}$ **is the usual matrix element with 1+1 dim LLL quarks and 3+1 dim gluon exchange**

A leading log appears from the gluon scatterings



$$\mathcal{M} = -ig_s^2 f^{abc} t^a \frac{2(p_\parallel \cdot k_\parallel)(\epsilon \cdot \tilde{\epsilon}^*)}{Q^2} R(\boldsymbol{q}_\perp)$$

Small Q gives a IR logarithmic divergence, which is screened by the 1-loop self energy from LLL quark loop

$$Q^2 \to Q^2 + m_D^2\,, \quad m_D^2 = \frac{g_s^2}{\pi} T_R N_F (eB/2\pi) e^{-\frac{q_\perp^2}{2eB}}$$

Writing $\delta f = \beta f^{eq}(1 - f^{eq})\delta\chi$

$$C_{qg}[\delta\chi(p_z)] = \frac{\delta I}{\delta\chi(p_z)}$$

where

$$\begin{aligned} I &= -\frac{1}{4}\int \frac{dp_z}{(2E_p)^2}\int \frac{d^4Q}{(2\pi)^4}(\delta\chi(p_z+q_z)-\delta\chi(p_z))^2 \\ &\times (2\pi)\delta(q^0 - \hat{p}_z q_z) f^{eq}(p)(1 - f^{eq}(p+q)) \\ &\times \int \frac{d^3k}{(2\pi)^3(2E_k)^2}|\mathcal{M}|^2(2\pi)\delta(q^0 - \hat{k}\cdot\boldsymbol{q}) \\ &\times \beta f^{eq}(k)(1 + f^{eq}(k-q)) \end{aligned}$$

We do small IR expansion for leading IR log

$$\delta\chi(p_z+q_z) - \delta\chi(p_z) \approx q_z(\delta\chi(p_z))'$$

The result is

$$C_{qg}[\delta\chi(p_z)] = \frac{\pi}{12}\alpha_s^2 N_c C_R T^2 \log\left(\frac{T^2}{\alpha_s eB}\right) \times [f^{eq}(|p_z|)(1-f^{eq}(|p_z|))\delta\chi'(p_z)]'$$

and the Boltzmann equation becomes

$$(eE)\hat{p}_z f^{eq}(|p_z|)(1-f^{eq}(|p_z|)) = -\frac{\pi}{12}\alpha_s^2 N_c C_R T^3 \log\left(\frac{T^2}{\alpha_s eB}\right) \times [f^{eq}(|p_z|)(1-f^{eq}(|p_z|))\delta\chi'(p_z)]'$$

We have a nice solution

$$\delta\chi(p_z) = \frac{12}{\pi N_c C_R T^2 \alpha_s^2 \log\left(\frac{T^2}{\alpha_s eB}\right)} (eE)\left(p_z + T\,\hat{p}_z(1 - e^{-\beta|p_z|})\right)$$

and the conductivity is

$$\begin{aligned} \sigma_{zz} &= e^2 \frac{12}{\pi^3} \frac{d_R N_F}{N_c C_R} \left(\frac{eB}{T\alpha_s^2 \log(T^2/\alpha_s eB)} \right) \\ &= e^2 \left(\sum_F q_F^3 \right) \frac{9}{\pi^3} \left(\frac{eB}{T\alpha_s^2 \log(T^2/\alpha_s eB)} \right) \end{aligned}$$

for fundamental quarks for $N_c = 3$

Further Directions:

Dense system/Other phases (e.g. Alford-Nishimura-Sedrakian), Viscosities, Smaller magnetic fields

Thank you!