ESTIMATING AND CHECKING TRUNCATION ERRORS IN EFFECTIVE FIELD THEORY

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For the BUQEYE collaboration

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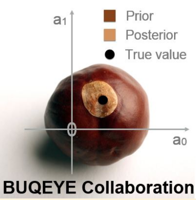
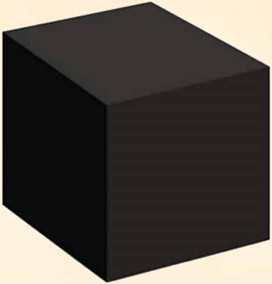


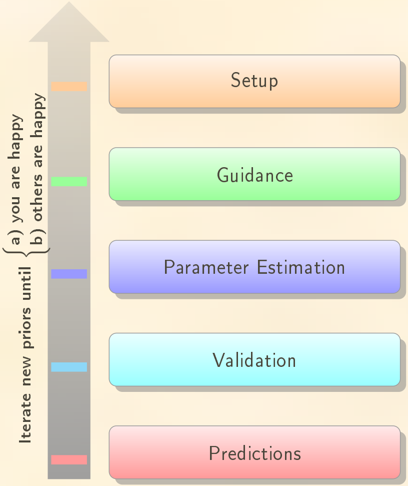
Research Supported by the US DOE and NSF

How many would like to see Bayesian analysis:



How Bayesian analysis actually is:

Bayesian Flow



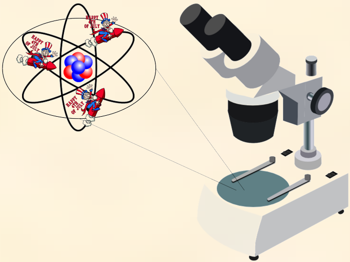
Wesolowski, NK, Furnstahl, DP, Thapaliya, J.Phys.G (2016)

Diagnostics:

- Setup
  - specify priors
- Guidance
  - Evidence Ratios
  - Hyperparameter posteriors
- Parameter Estimation
  - Stability (x_{max}, $\bar{a}$)
- Validation
  - Cross-validation
  - Lepage Plots

Lepage Plots

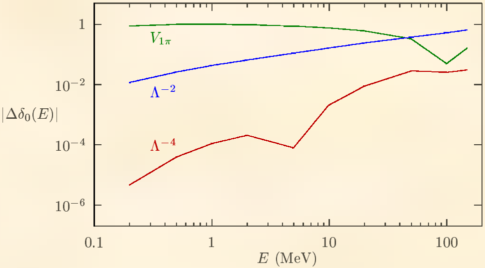
- 1997 - How to Renormalize the Schrödinger equation
- "..mimic the *real* short-distance structure of the target and probe by *simple* short-distance structure..."
- Low-energy data will **never** contain sufficient information to tell the difference between this mimicry and reality



- structure as an expansion in a small parameter
- Lepage plot = Error Plot
- Goal to diagnose whether the expansion is "working"

Lepage Plots

- Approximate unknown high-energy potentials with smeared delta functions
- Impose an ultraviolet cutoff to remove less-understood physics, **Λ**
- Add correction terms which imitate short range physics – each one will bring an additional parameter

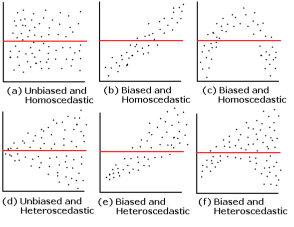


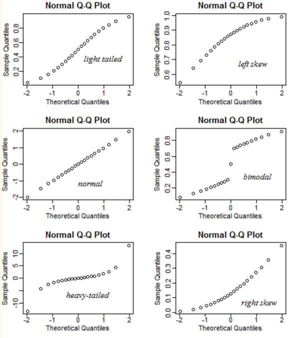
G.P.Lepage (1997)

- Look at $^1S_0$ phase shifts as corrections are added
- should see power law scaling

Lepage Plot as Error Plot

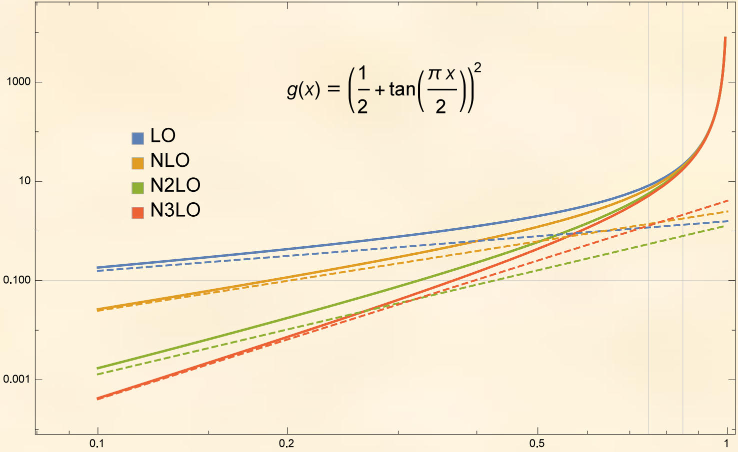
- Validation Stage
- Q: Can we convince ourselves that the next term indeed behaves as x^{k+1}?
  - Translated as expectation on residuals
  - not a new concept





- Is it distributed around zero?
- Is it normal?

Idealized Lepage Plot – Polynomial Residuals



Is there a way to account for errors from data and lower-order coefficients? Yes!

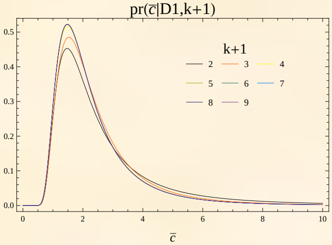
Statistical Errors

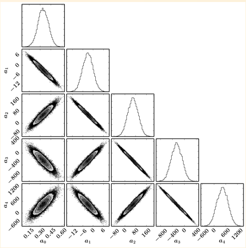
- Both statistical and truncation errors are expressed through marginalization integrals
  - Truncation: Retains information only of the posterior of $\bar{c}$

$$\mathrm{pr}(\Delta_k(x)|D,k) = \frac{1}{x^{k+1}} \int \mathrm{d}\bar{c} \int \cdots \int \mathrm{d}c_{k+2} ... \mathrm{d}c_\infty$$

$$\mathrm{pr}\left(c_{k+1} = \frac{1}{x^{k+1}} \left(\Delta_k(x) - \sum_{n=k+2}^{\infty} c_n x^n\right), c_{k+2}, ... c_\infty | \bar{c}\right) \mathrm{pr}(\bar{c}|D,k)$$

  - Statistical: Benefits from coefficient posteriors and correlation matrices





Statistical Errors

- Account for errors from data and lower-order coefficients

$$c_0 = \mu_{c_0} \pm \sigma_{c_0}$$

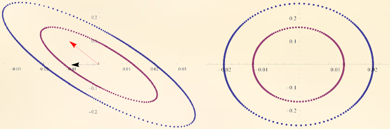
$$\vdots$$

$$c_k = \mu_{c_k} \pm \sigma_{c_k}$$

- Anti-correlations expected from polynomial structure

$$\mathbf{C} = \left(\mathbf{G}^{\mathsf{T}}\mathbf{E}^{-1}\mathbf{G} + \frac{1}{\bar{c}^2}\mathcal{I}\right)^{-1}$$

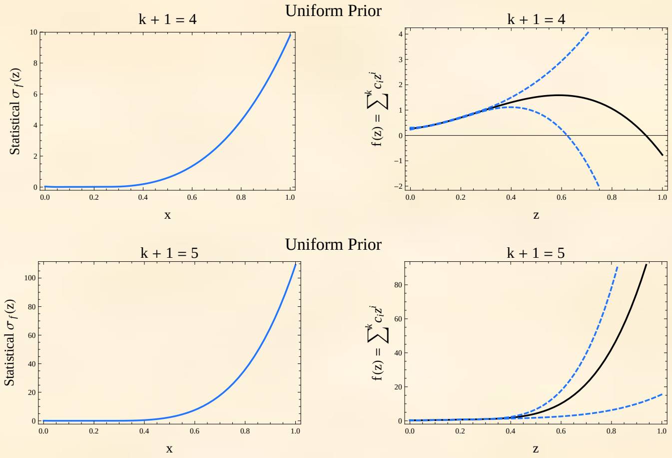
$$= \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{\mathsf{T}}$$



- $\mathbf{R}$ - orthogonal matrix of eigenvectors
- $\mathbf{\Lambda}$ - Diagonal matrix of inverse variances in SVD frame.
- Transfer between spaces:
  $\mathbf{c}_{\mathrm{EFT}} = \mathbf{R}^{-1}\mathbf{c}_{\mathrm{SVD}}$
- Errors now independent:
  $\sigma_{f(x)} = \sqrt{((\mathbf{RG})^{\mathsf{T}})^2 \cdot \mathit{Diagonal}\,[\mathbf{\Lambda}]}$

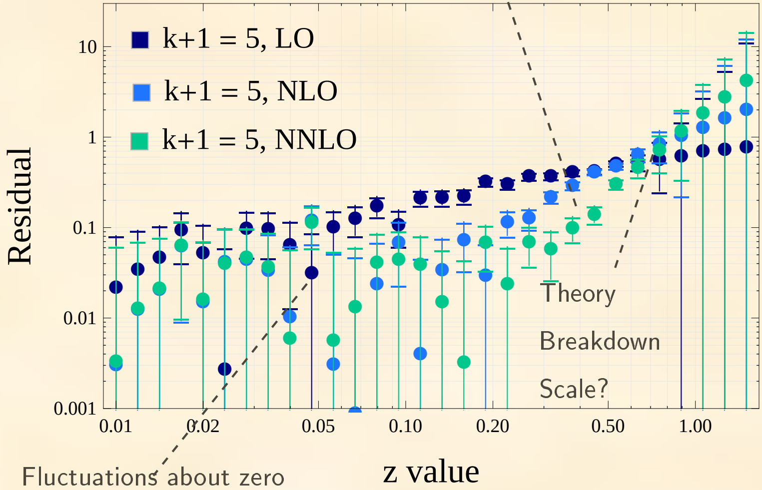
Statistical Errors

- check analytics with MCMC



Lepage Plots in Practice – Polynomial Residuals

linear behaviour, dominance of first-order truncation?



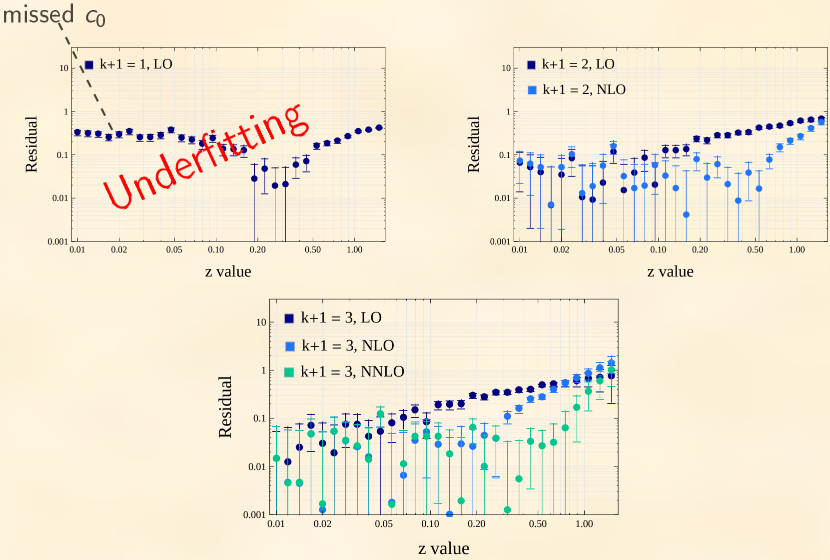
Theory

Breakdown

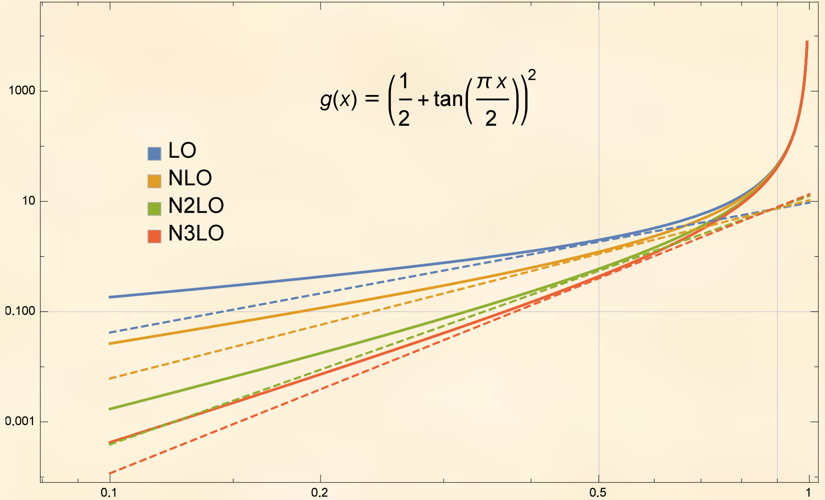
Scale?

Fluctuations about zero

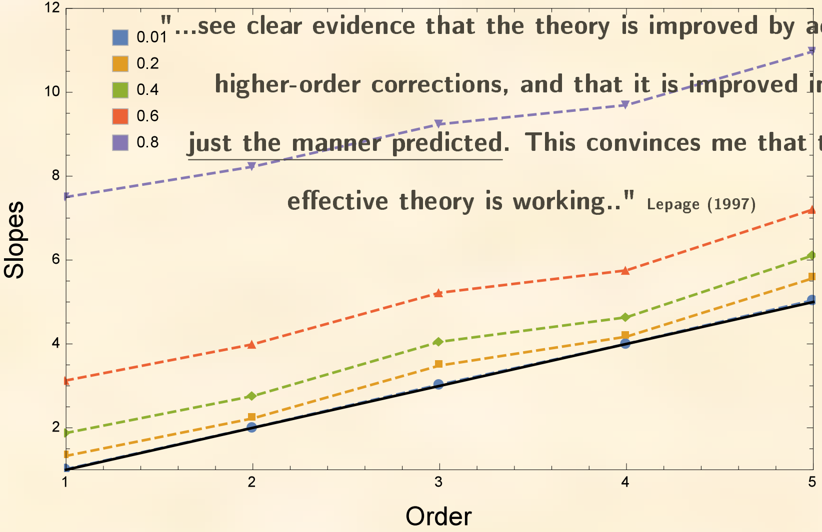
Evolution of Lepage Plots in k



Slopes vs Noise



Scaling > Value

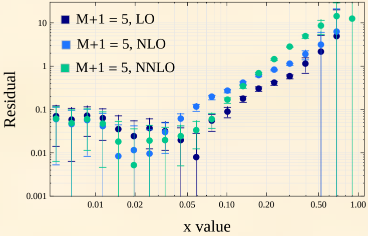


"...see clear evidence that the theory is improved by adding higher-order corrections, and that it is improved in just the manner predicted. This convinces me that the effective theory is working.." Lepage (1997)

LP as Diagnostic

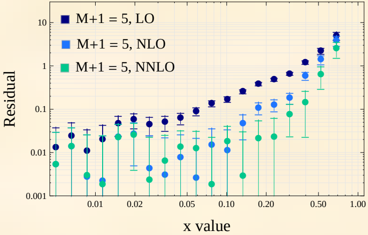
- Uniform Prior

$$\sigma_{M=4}(x) = (0.33 \pm 0.07) - (1.88 \pm 2.69)x + (44.65 \pm 32.6)x^2 - (181.9 \pm 149.79)x^3 + (263.61 \pm 228.5)x^4$$



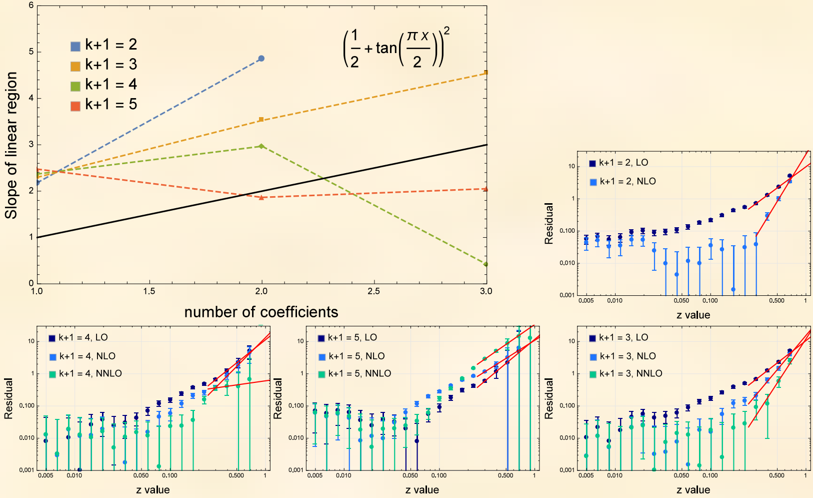
- Gaussian Prior ($\bar{c} = 5$)

$$\sigma_{M=4}(x) = (0.247 \pm 0.024) + (1.65 \pm 0.46)x + (2.98 \pm 2.38)x^2 + (0.38 \pm 4.4)x^3 - (0.02 \pm 4.9)x^4$$



Can we quantify this objection to the results of the Uniform Prior?

Residual Scaling - Uniform Prior



Residual Scaling - Uniform Prior

M	$\chi^2/d.o.f.$	a_0	a_1	a_2
1	2.24	0.203 ± 0.014	2.55 ± 0.11	
2	1.64	0.250 ± 0.023	1.57 ± 0.40	3.33 ± 1.31
3	1.85	0.269 ± 0.039	0.954 ± 1.094	8.16 ± 8.05
4	1.96	0.333 ± 0.067	-1.88 ± 2.69	44.7 ± 32.6
5	1.39	0.566 ± 0.132	-14.8 ± 6.85	276 ± 117
6	1.85	0.590 ± 0.291	-16.4 ± 18.1	311 ± 395
7	2.67	0.242 ± 0.788	8.97 ± 56.3	-373 ± 1494

TABLE I: Fit results for standard χ^2 approach with $x_{max} = 1/\pi$ and $c = 0.05$.

stolen from Schindler,DP (2009)

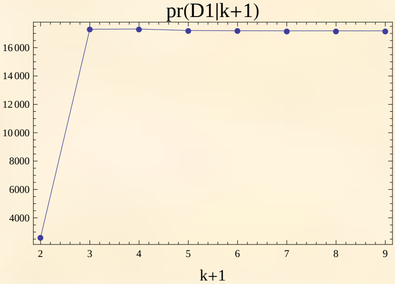
- "if one did not know the underlying values of a0 and a1 one might be hard put to explain the extent to which the fit at order 2 is superior to that at order 3, or indeed, that at order 5."
- Coincidentally (?), the fit at order 2 is the only order where we see the correct scaling.
- Lepage plots as model selection?
- Remember: this is still a toy-model-sample-size of 1

Model Selection

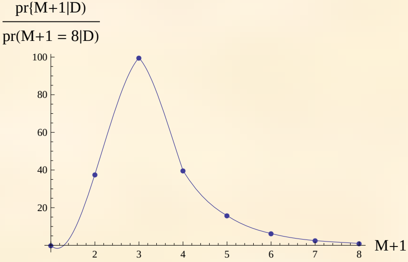
$$\mathrm{pr}(\bar{c}|k, D) = \frac{\mathrm{pr}(D|\bar{c},k)\mathrm{pr}(\bar{c}|k)}{\mathrm{pr}(D|k)}$$

- $\mathrm{pr}(\mathrm{Pregnant}|\mathrm{Woman}) \neq \mathrm{pr}(\mathrm{Woman}|\mathrm{Pregnant})$
- Take uniform prior on k
- $\mathrm{pr}(D|k) \propto \mathrm{pr}(k|D)$

$$\frac{\mathrm{pr}(M_i|D)}{\mathrm{pr}(M_8|D)} = \frac{\int_0^{100} \mathrm{pr}(D|M_i,R)\mathrm{pr}(M_i|R)\mathrm{pr}(R)}{\int_0^{100} \mathrm{pr}(D|M_8,R)\mathrm{pr}(M_8|R)\mathrm{pr}(R)}$$

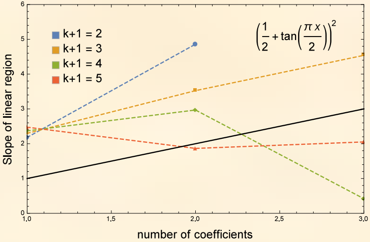


D1 {R,0,100}

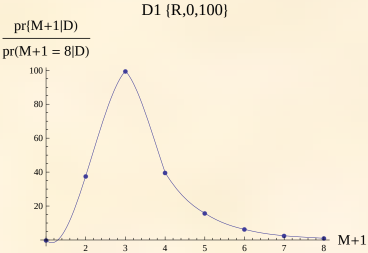


Model Selection

Is

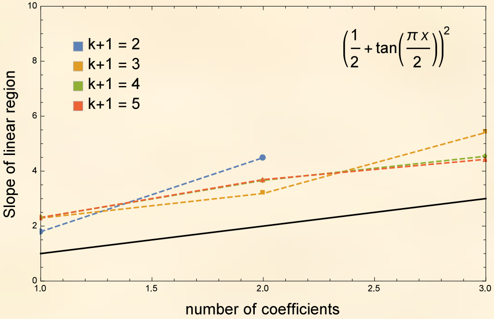


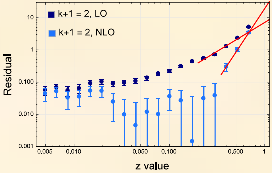
different information than

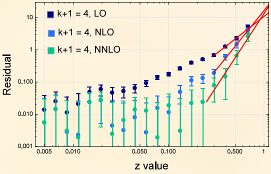


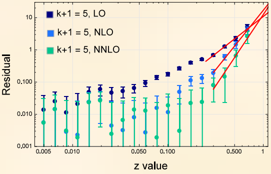
- do Lepage plots offer a new window to the Bayesian analysis or a different perspective on an old one?
- Different Questions?
  - How big of a model is justified by the data?
  - Which model scales correctly?
- May be too soon to tell..

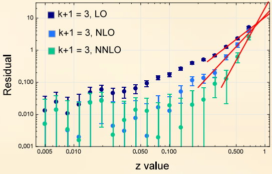
Residual Scaling - Gaussian Prior











The Diagnostic

- Slopes of first-order approximation obscured by statistical fluctuations.
  - Seeing **statistically significant** changes in slope at values of x near the breakdown scale may be sufficient?
- To what extent may this inform model selection?
- When parameter estimation fails, slopes will be defined by residuals as e.g.

$$\delta c_0 + \delta c_1 x + \sum_{n=2}^{\infty} c_n x^n$$

  - Could this discrepancy be turned into a parameter estimation diagnostic?
- This has been a **quick** glance at a **single** toy problem...more for the future.

THEORISTS ANONYMOUS

- Admit that you have a problem: your theory has uncertainties
- Acknowledge the existence of a higher power
- Seek to understand its impact on out theory
- Make a searching and fearless inventory of errors
- Acknowledge your mistakes
- Make amends for those mistakes
- Help others who must deal with the same issues

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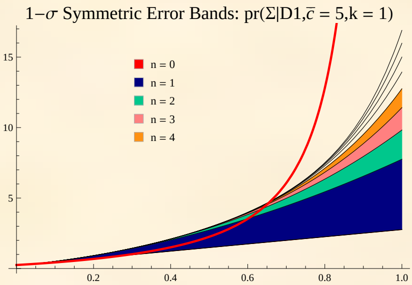
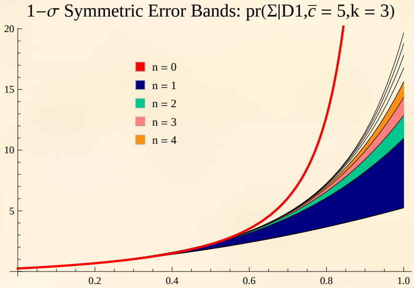
  - **Attend INT Bayesian Program**

Thank You!

Concerns...continued

- Is the first term expansion good enough?
- Can we extrapolate the correlation matrix from the fit $c_0, \ldots c_k$ to the marginalization for truncated terms?





$$\sigma_\Sigma^2 = \sum_{j=0}^{k} \sigma^2_{c_j,SVD} x^{2j} + \sum_{j=k+1}^{k+1+n} \bar{c}^2 x^{2j}$$