Three-Body Systems with Short-Range Interactions

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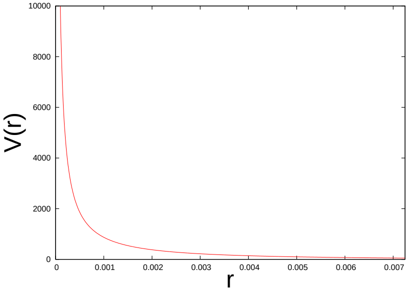
Universality at Low Energies

- If r is the typical range of a short range potential. Then for small energies $(E \leq 1/r)$ we can approximate using contact potentials.
- This is a useful description in cold atoms, halo nuclei, low energy nuclear interactions, and etc.



$$\mathcal{F}\left[\frac{1}{|\mathbf{r}|}e^{-m_\pi|\mathbf{r}|}\right] = \frac{1}{\mathbf{q}^2 + m_\pi^2}$$

$$\frac{1}{\mathbf{q}^2 + m_\pi^2} = \frac{1}{m_\pi^2} - \frac{\mathbf{q}^2}{m_\pi^4} + \frac{\mathbf{q}^4}{m_\pi^6} + \cdots$$

$$\mathcal{F}\left[\frac{1}{m_\pi^2} - \frac{\mathbf{q}^2}{m_\pi^4} + \frac{\mathbf{q}^4}{m_\pi^6} + \cdots\right] = \frac{1}{m_\pi^2}\delta^3(\mathbf{r}) - \frac{1}{m_\pi^4}\nabla^2\delta^3(\mathbf{r}) + \frac{1}{m_\pi^6}\nabla^4\delta^3(\mathbf{r}) + \cdots$$

Effective (Field) Theories

- Disparate scales can be used as an expansion parameter $\lambda_1 \gg \lambda_2, Q \sim \frac{\lambda_2}{\lambda_1}$
- Only valid in regimes where $Q < 1$.

Example 1: For objects a height h above earth the gravitational potential is given by

$$\Phi(r) = -\frac{GM_E m}{R_E}\left(1 - \frac{h}{R_E} + \left(\frac{h}{R_E}\right)^2 + \cdots\right)$$

where $Q = \frac{h}{R_E}$ is a small parameter.

Example 2: For thin sheets one can use $Q = \frac{t}{\kappa}$ where t is the thickness and κ the curvature.

- Effective (field) theories have "power counting" that organizes relative importance of terms. Gives error estimate of calculations.

Ingredients of $\mathrm{EFT}_{\not{\pi}}$

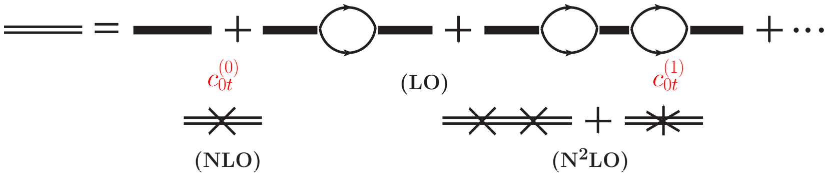
- For momenta $p < m_\pi$ pions can be integrated out as degrees of freedom and only nucleons and external currents are left.
- For any effective (field) theory write down all terms with degrees of freedom that respect symmetries.
- Develop a power counting to organize terms by their relative importance.
- Calculate respective observables up to a given order in the power counting.

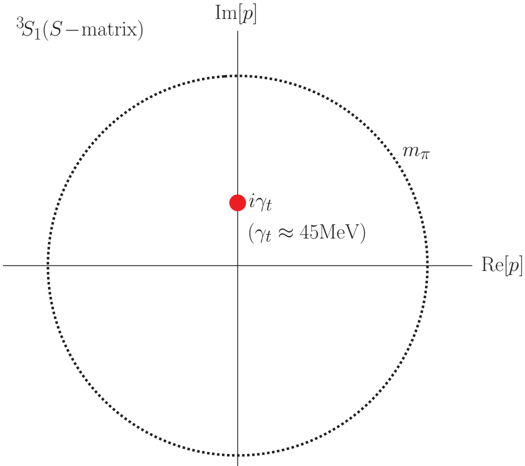
The two-body Lagrangian to $\mathrm{N}^3\mathrm{LO}$ in $\mathrm{EFT}_{\not\pi}$ is

$$\mathcal{L}_2 = \hat{N}^\dagger\left(i\partial_0+\frac{\vec{\nabla}^2}{2M_N}\right)\hat{N}+\hat{t}_i^\dagger\left(\Delta_t-\sum_{n=0}^{1}c_{nt}\left(i\partial_0+\frac{\vec{\nabla}^2}{4M_N}+\frac{\gamma_t^2}{M_N}\right)^{n+1}\right)\hat{t}_i$$
$$+\hat{s}_a^\dagger\left(\Delta_s-\sum_{n=0}^{1}c_{ns}\left(i\partial_0+\frac{\vec{\nabla}^2}{4M_N}+\frac{\gamma_s^2}{M_N}\right)^{n+1}\right)\hat{s}_a$$
$$+y_t\left[\hat{t}_i^\dagger\hat{N}^TP_i\hat{N}+\mathrm{H.c.}\right]+y_s\left[\hat{s}_a^\dagger\hat{N}^T\bar{P}_a\hat{N}+\mathrm{H.c.}\right].$$

- c_{0t}, c_{0s}-range corrections
- c_{1t}, c_{1s}-shape parameter corrections
- SD-mixing term at $\mathrm{N}^2\mathrm{LO}$
- Two-body P-wave contributions $({}^3P_J, {}^1P_1)$ at $\mathrm{N}^3\mathrm{LO}$

The LO dressed deuteron propagator is given by a bubble sum

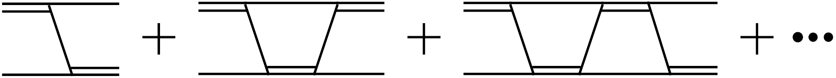




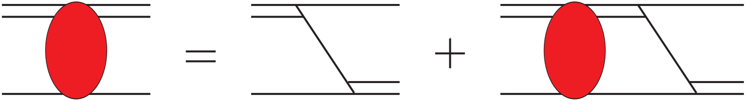
(Z-parametrization) At LO coefficients are fit to reproduce the deuteron pole and at NLO to reproduce the residue about the deuteron pole (Phillips,Rupak, and Savage (2000)).

Quartet Channel (*nd* Scattering)

At LO in the quartet channel ($1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$), *nd* scattering is given by an infinite sum of diagrams.

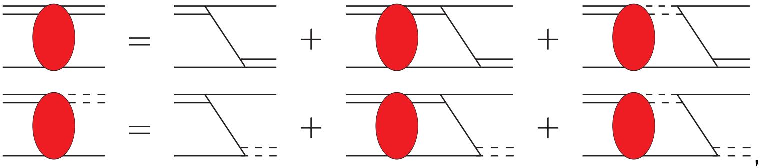


This infinite sum of diagrams can be represented by an integral equation.



Doublet Channel *nd* scattering

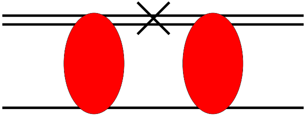
At LO in the doublet channel ($1\otimes\frac{1}{2}=\frac{1}{2}\oplus\frac{3}{2},\quad 0\otimes\frac{1}{2}=\frac{1}{2}$), *nd* scattering is given by a coupled set of integral equations



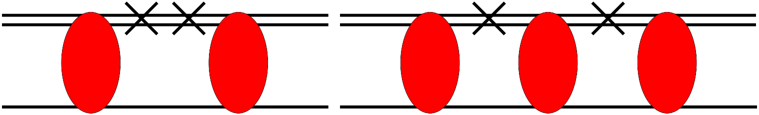
The integral equations in the doublet channel are analogous to the quartet case but are now a matrix equation in cluster configuration space (Grießhammer (2004)).

Higher Orders

NLO correction is



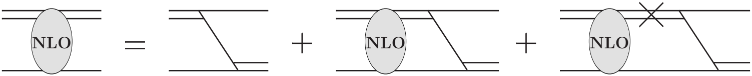
NNLO corrections are



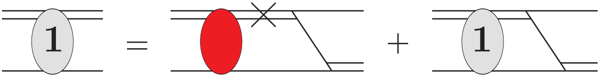
Note the second diagram contains full off-shell scattering amplitude.

Higher Order Calculations

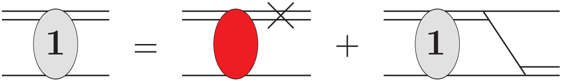
Partial resummation technique: not strictly perturbative (Bedaque, Griesshammer, and Hammer (2003))



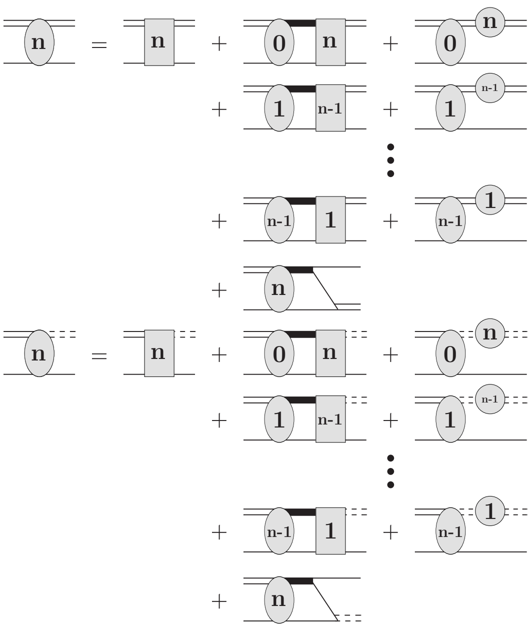
Strictly perturbative technique (Vanasse (2013)): $Z_{LO}t_{NLO}$



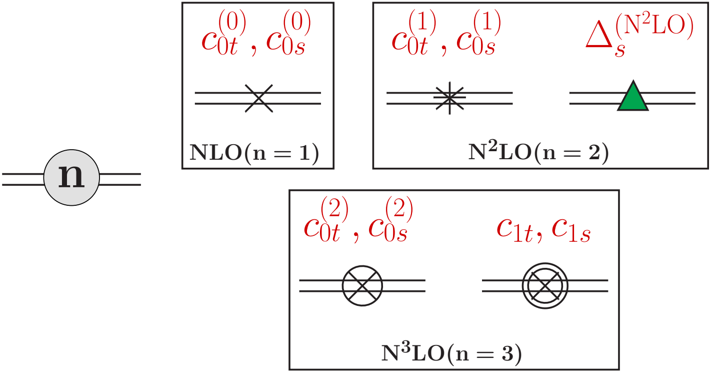
Improved strictly perturbative technique (Vanasse (2015)): $Z_{LO}t_{NLO} + Z_{NLO}t_{LO}$



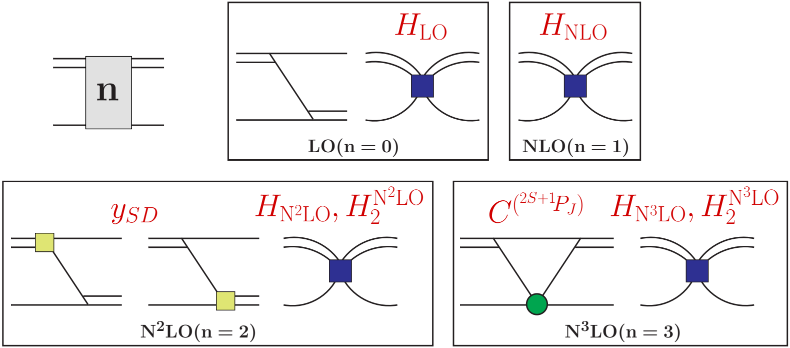
The general $\mathrm{N}^n\mathrm{LO}$ correction is given by



Two-body terms:



Three-body terms:

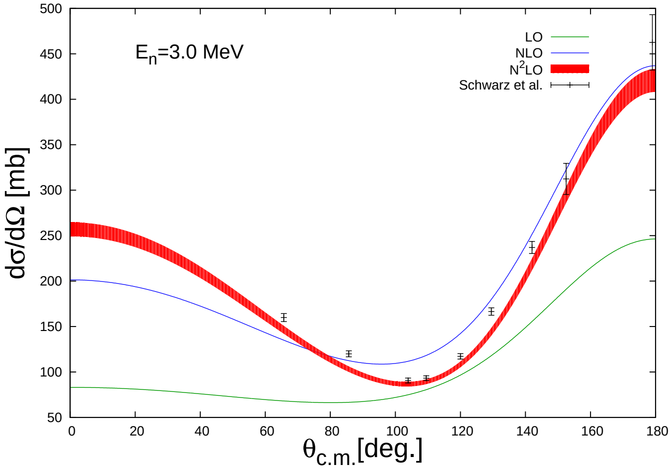


Steps for Calculation

1. Identify all two and three-body contributions to given order in power counting.
2. Calculate all diagrams to given order in power counting.
3. Project diagrams into angular momentum basis via Racah algebra.
4. Plug results into integral equations and solve scattering amplitudes numerically.
5. Use scattering amplitudes to calculate any scattering observable in *nd*

nd Scattering Results

(Margaryan,Springer, and Vanasse (2015)) arXiv:1510.07598



Nd polarization Observables

Polarizing the nucleon gives one parity-conserving polarization observable, the vector-analyzing power $A_y(\theta)$. Polarizing the nucleon transversely to the beam gives the differential scattering cross-section

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = \frac{d\bar{\sigma}}{d\Omega}(\theta)(1 - A_y(\theta)\sin(\phi)).$$

$$A_y = \frac{\frac{d\sigma}{d\Omega}_{\uparrow} - \frac{d\sigma}{d\Omega}_{\downarrow}}{\frac{d\sigma}{d\Omega}_{\uparrow} + \frac{d\sigma}{d\Omega}_{\downarrow}}$$

Polarizing the deuteron gives four parity-conserving polarization observable, the vector-analyzing power $iT_{11}(\theta)$, $T_{20}(\theta)$, $T_{21}(\theta)$, and $T_{22}(\theta)$. Polarizing the deuteron beam gives the differential scattering cross-section

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = \frac{d\bar{\sigma}}{d\Omega}(\theta)\left[1 + 2\mathrm{Re}(it_{11})iT_{11}(\theta)\sin(\phi) + t_{20}T_{20}(\theta) \right.$$
$$\left. + 2\mathrm{Re}(t_{21})T_{21}(\theta)cos(\phi) + 2Re(t_{22})T_{22}(\theta)\cos(2\phi)\right]$$

Calculation of *Nd* Polarization Observables

Write scattering amplitudes in spin-basis in terms of scattering amplitudes in spin-angular momentum basis ($\vec{\mathbf{J}} = \vec{\mathbf{L}} + \vec{\mathbf{S}}$) (Clebsch-Gordanry)

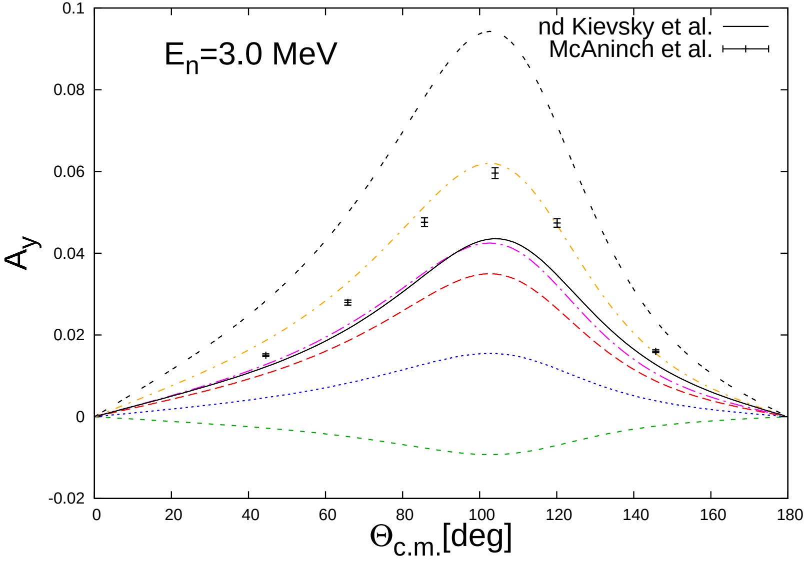
↓

Gives 36 amplitudes in spin-basis due to $(2 \times 3) \times (2 \times 3)$ (2 number of nucleon spin states) (3 number of deuteron spin states)

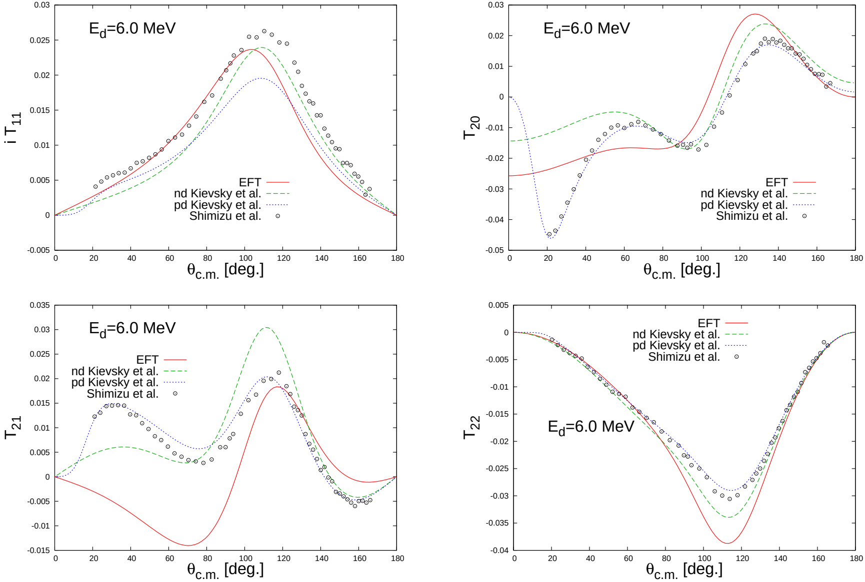
↓

Using density matrix techniques calculate polarization observables in terms of 36 amplitudes in spin-basis

nd Polarization Observables



“nd” Polarization Observables

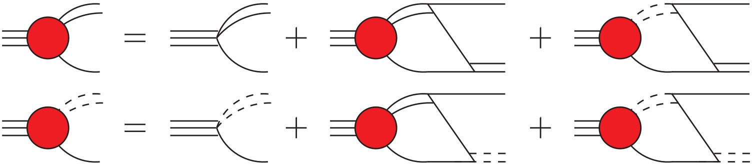


Doublet S-wave and Bound state

The three-body Lagrangian is

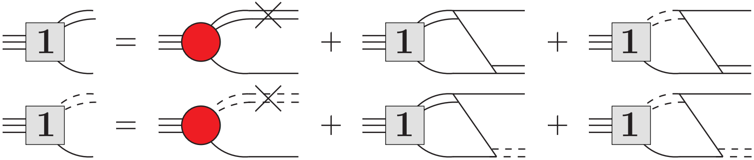
$$\mathcal{L}_3 = \hat{\psi}^\dagger \left[\Omega - h_2(\Lambda) \left(i\partial_0 + \frac{\vec{\nabla}^2}{6M_N} + \frac{\gamma_t^2}{M_N} \right) \right] \hat{\psi} + \sum_{n=0}^{\infty} \left[\omega_{t0}^{(n)} \hat{\psi}^\dagger \sigma_i \hat{N} \hat{t}_i - \omega_{s0}^{(n)} \hat{\psi}^\dagger \tau_a \hat{N} \hat{s}_a \right] + \mathrm{H.c.}.$$

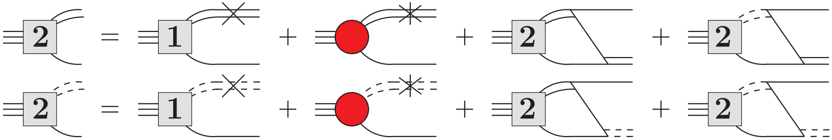
where ψ is an auxiliary triton field. The LO triton vertex function $\mathcal{G}_0(E, p)$ is given by following coupled integral equations (Hagen, Hammer, and Platter (2013))



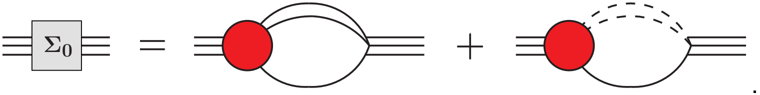
Inhomogeneous term is set to **1** to factor three-body forces out of vertex functions.

The NLO ($\mathcal{G}_1(E,p)$) and NNLO ($\mathcal{G}_2(E,p)$) triton vertex functions are





Defining



The dressed triton propagator is given by the sum of diagrams

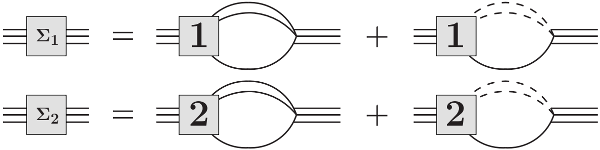
which yields

$$\begin{aligned} i\Delta_3(E) &= \frac{i}{\Omega} - \frac{i}{\Omega} H_{\rm LO}\Sigma_0(E)\frac{i}{\Omega} + \cdots \\ &= \frac{i}{\Omega}\frac{1}{1 - H_{\rm LO}\Sigma_0(E)}, \end{aligned}$$

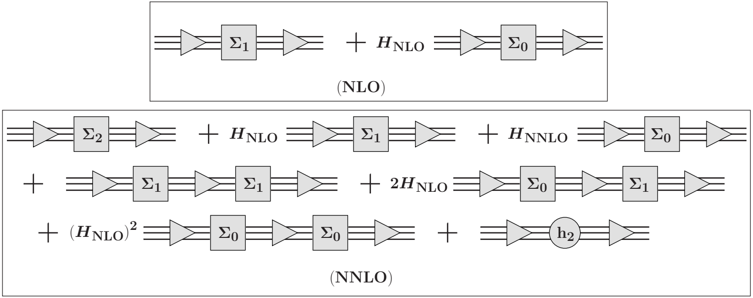
where

$$H_{\rm LO} = -\frac{3\omega_t^2}{\pi\Omega} = -\frac{3\omega_s^2}{\pi\Omega} = \frac{3\omega_t\omega_s}{\pi\Omega}.$$

Defining the functions



The NNLO triton propagator is



Properly Renormalized Vertex Function

Ensuring that triton propagator has pole at triton binding energy gives conditions

$$H_{\rm LO}=\frac{1}{\Sigma_0(B)}\quad,\quad H_{\rm LO}\Sigma_1(B)+H_{\rm NLO}\Sigma_0(B)=0,$$

$$H_{\rm LO}\Sigma_2(B)+H_{\rm NLO}\Sigma_1(B)+\left(H_{\rm NNLO}+\frac{4}{3}(M_NB+\gamma_t^2)\widehat{H}_2\right)\Sigma_0(B)=0.$$

Triton wavefunction renormalization is residue about pole leads to triton vertex functions

$$\mathbf{\Gamma}_0(p)=\sqrt{Z_\psi^{\rm LO}}\mathcal{G}_0(B,p)\quad,\quad\sqrt{Z_\psi^{\rm LO}}=\sqrt{\frac{\pi}{\Sigma_0'(B)}}$$

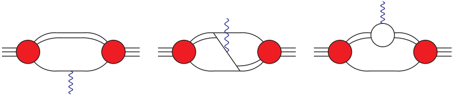
$$\mathbf{\Gamma}_1(p)=\sqrt{Z_\psi^{\rm LO}}\left[\mathcal{G}_1(B,p)-\frac{1}{2}\frac{\Sigma_1'}{\Sigma_0'}\mathcal{G}_0(B,p)\right].$$

$$\mathbf{\Gamma}_2(p)=\sqrt{Z_\psi^{\rm LO}}\left[\mathcal{G}_2(B,p)-\frac{1}{2}\frac{\Sigma_1'}{\Sigma_0'}\mathcal{G}_1(B,p)\right.$$
$$\left.+\left\{\frac{3}{8}\left(\frac{\Sigma_1'}{\Sigma_0'}\right)^2-\frac{1}{2}\frac{\Sigma_2'}{\Sigma_0'}-\frac{2}{3}M_N\widehat{H}_2\frac{\Sigma_0^2}{\Sigma_0'}\right\}\mathcal{G}_0(B,p)\right].$$

Triton Charge Form Factor

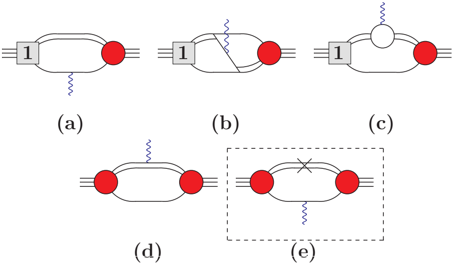
Charge form factor of triton at LO given by three diagrams

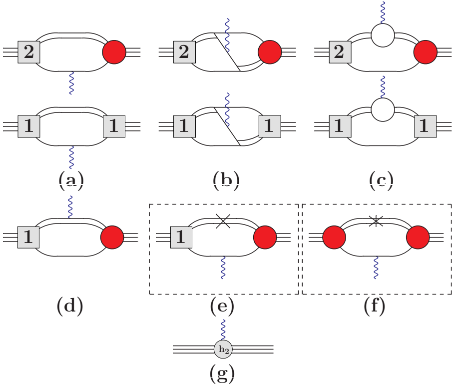
$$\hat{N}^\dagger \left[i\partial_0 + ie \left(\frac{1+\tau_3}{2} \right) \hat{A}_0 \right] \hat{N}$$



(a) (b) (c)

NLO and NNLO triton charge form factor





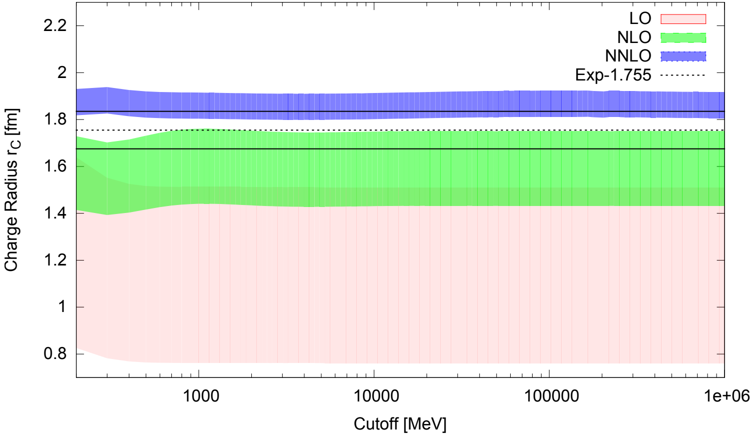
How to get charge radius

The triton charge form factor expanded in powers of Q^2 yields

$$F(Q^2) = 1 - \frac{\left\langle r^2_{^3H} \right\rangle}{6} Q^2 + \cdots$$

- **Method 1:** Calculate charge form factor for various low values of Q^2. Fit a line as function of Q^2 to the resulting data. The slope of this line is related to the charge radius.
- **Method 2:** Expand all diagrams as functions of Q^2 and take only Q^2 pieces. Then calculate this and obtain the charge radius. Has advantage of allowing more integrals to be done analytically. Therefore is more numerically stable and allows higher cutoffs to be calculated.

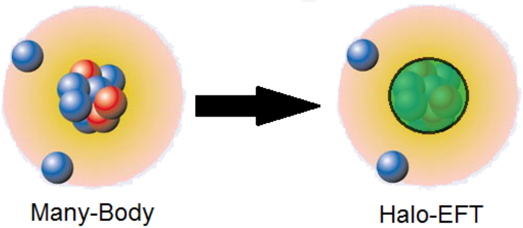
LO $\mathrm{EFT}_{\not\pi}$ prediction via wavefuntions $r_C = 2.1 \pm .6fm$ (Platter and Hammer (2005))



and now for something completely different…?

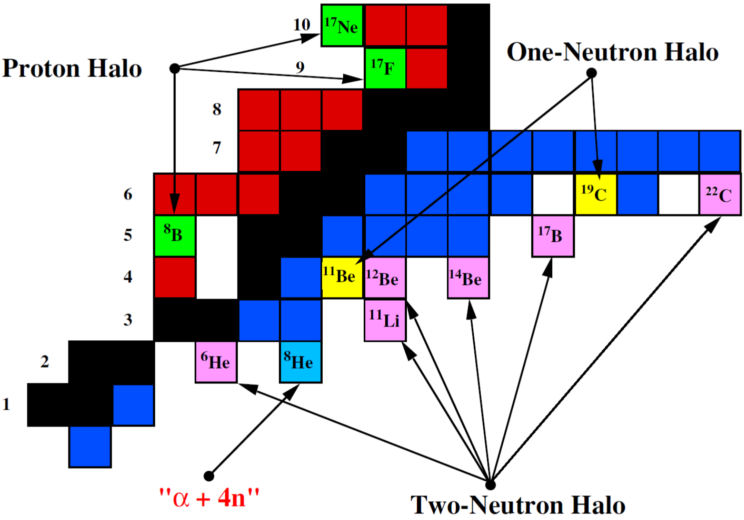
Halo-Nuclei

- For halo-nuclei $R_{halo} > R_{core}$, can expand in powers of R_{core}/R_{halo}.



- If a probe has De Broglie wavelength λ, and $\lambda > R_{core}$ the structure of the core cannot be resolved and it can be treated as a fundamental degree of freedom.
- Breakdown scales of halo-EFT set by $E_{\star}$ (first excited state energy of core) and m_{π}

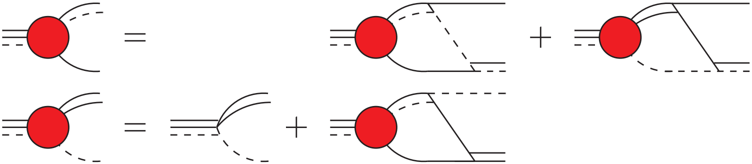
Halo-Nuclei



http://www.nupecc.org/report97/report97.pdf

Halo-Nuclei

- LO halo-nuclei vertex function given by (Hagen, Hammer, and Platter (2013))



- S-wave interactions in both two and three-body sector
- Nearly identical to pionless EFT
- Differences from pionless EFT: core is spin-0, three-body force chosen differently, and parameters will have different values

Unitary equal mass limit

Calculation of LO halo-nuclei charge radius nearly identical to triton charge radius calculation. In Unitary limit and equal mass limit it is found

Authors	$mE_{3B}\left\langle r_c^2\right\rangle$
Vanasse	.224
Hagen et al.	.265

Using analytical techniques in (Braaten and Hammer (2006)) it can be shown that $mE_{3B}\left\langle r_c^2\right\rangle = (1+s_0^2)/9 \approx .224$ in the unitary and equal mass limit. Changing a single factor in the code of Hagen et al. they would also obtain .224.

What about the lattice?

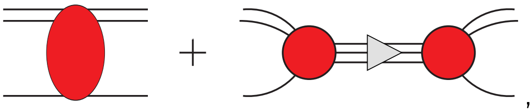
- **What I would like to see:** Low energy NN parity-violation is described by five low energy constants. These should be calculated on the lattice, and then $\mathrm{EFT}_{\not\pi}$ calculations can make predictions.
- **What we can do for you:** We can extrapolate to other physical observables using your data as input.
- Can use $\mathrm{EFT}_{\not\pi}$ to extrapolate lattice calculation to infinite volume limit.

Conclusions and Future directions

- Calculating the *nd* scattering amplitude to higher orders in $\mathrm{EFT}_{\not{\pi}}$ strictly perturbatively is made easier by new techniques.
- Calculating *nd* scattering to higher orders will allow investigation of polarization observables, in particular A_y.
- *nd* scattering to $\mathrm{N}^4\mathrm{LO}$ will require insertion of three-body *SD*-mixing terms, three-body *P*-wave corrections, and etc...
- Now that bound states can also be calculated perturbatively, one can consider calculations including external currents such as $\gamma + {}^3\mathrm{He} \rightarrow p + d$, $\gamma + {}^3\mathrm{H} \rightarrow n + d$,$\gamma + {}^3\mathrm{He} \rightarrow \gamma + {}^3\mathrm{He}$, $\gamma + {}^3\mathrm{H} \rightarrow \gamma + {}^3\mathrm{H}$, and ${}^3\mathrm{H} \rightarrow e^- + \bar{\nu}_e + {}^3\mathrm{He}$.
- Further work needs to be done on disagreement in halo-nuclei

LO three-body force

The LO doublet S-wave amplitude for nd scattering is given by the sum of diagrams



which gives

$$T_{LO} = t_{LO} + H_{LO} \frac{1}{1 - H_{LO}\Sigma_0(E)} \pi Z_{LO} \left(G_{0,Nt\to Nt}(E,k)\right)^2 ,$$

where

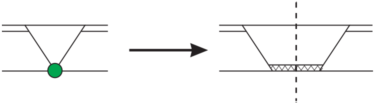
$$t_{LO} = Z_{LO} t^{\ell=0}_{0,Nt\to Nt}(k,k)$$

Fitting to the doublet S-wave nd scattering length a_{nd}, H_{LO} is given by

$$H_{LO} = \frac{x}{1 + x\Sigma_0(E)} \quad , \quad x = \frac{-\left(\frac{3\pi a_{nd}}{M_N} + T_{LO}\right)}{\pi Z_{LO}\left(G_{0,Nt\to Nt}(E,k)\right)^2}$$

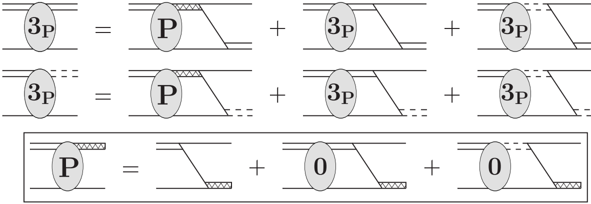
Two-body P-wave contact interactions

To calculate three-body contributions from two-body P-wave contact interactions introduce P-wave auxliary field



$$\sim \sum_{n,m} C_{n,m} \int_0^\Lambda dq q^2 Q_n(a(q,k)) Q_m(a(q,p))$$

Equivalent to projecting out loop integral in total angular momentum $\vec{\mathbf{J}}$ and then carrying out angular integration. Can also be done by solving integral equations.



Triton charge form factor

LO charge form-factor contribution from diagram (a) is

$$F_0^{(a)}(Q^2) = Z_\psi^{\mathrm{LO}} \left\{ \widetilde{\boldsymbol{\mathcal{G}}}_0^T(p) \otimes \boldsymbol{\mathcal{A}}_0(p,k,Q) \otimes \widetilde{\boldsymbol{\mathcal{G}}}_0(k) \right.$$
$$\left. +2\widetilde{\boldsymbol{\mathcal{G}}}_0^T(p) \otimes \boldsymbol{\mathcal{A}}_0(p,Q) + \mathcal{A}_0(Q) \right\}.$$

NLO contribution is

$$F_1^{(a)}(Q^2) = Z_\psi^{\mathrm{LO}} \left\{ \widetilde{\boldsymbol{\mathcal{G}}}_0^T(p) \otimes \boldsymbol{\mathcal{A}}_1(p,k,Q) \otimes \widetilde{\boldsymbol{\mathcal{G}}}_0(k) \right.$$
$$+ 2\widetilde{\boldsymbol{\mathcal{G}}}_1^T(p) \otimes \boldsymbol{\mathcal{A}}_0(p,k,Q) \otimes \widetilde{\boldsymbol{\mathcal{G}}}_0(k)$$
$$\left. +2\widetilde{\boldsymbol{\mathcal{G}}}_0^T(p) \otimes \boldsymbol{\mathcal{A}}_1(p,Q) + 2\widetilde{\boldsymbol{\mathcal{G}}}_1^T(p) \otimes \boldsymbol{\mathcal{A}}_0(p,Q) + \mathcal{A}_1(Q) \right\}.$$

where

$$\widetilde{\boldsymbol{\mathcal{G}}}_n(p) = \mathbf{D}^{(0)}\left(B_0 - \frac{\vec{\mathbf{p}}^2}{2M_N}, \vec{\mathbf{p}}\right) \boldsymbol{\mathcal{G}}_n(B_0, p).$$

Triton charge form factor

The vector term is

$$\boldsymbol{\mathcal{A}}_n(p,Q) = -\frac{M_N}{2\pi}\Bigg|_0^1 \int_0^\Lambda dq q^2 \int_{-1}^1 dx \frac{1}{qQx}\frac{1}{p\sqrt{q^2-\frac{2}{3}qQx+\frac{1}{9}Q^2}}$$

$$\times\ Q_0\left(\frac{p^2+q^2+\frac{1}{9}Q^2+(y-\frac{2}{3})qQx-M_N B_0}{p\sqrt{q^2-\frac{2}{3}qQx+\frac{1}{9}Q^2}}\right)$$

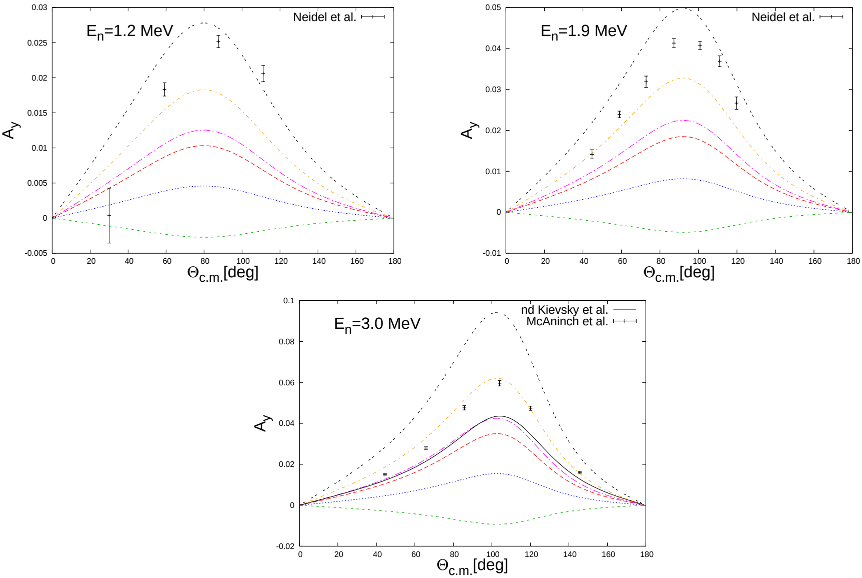
$$\times\ D_s^{(n)}\!\left(B_0-\frac{q^2}{2M_N}-\frac{Q^2}{12M_N}+\left(\frac{1}{2}-y\right)\frac{qQx}{M_N},\vec{\mathbf{q}}\right)\left(\begin{array}{c}2\\ -^2/_3\end{array}\right),$$

and scalar term is

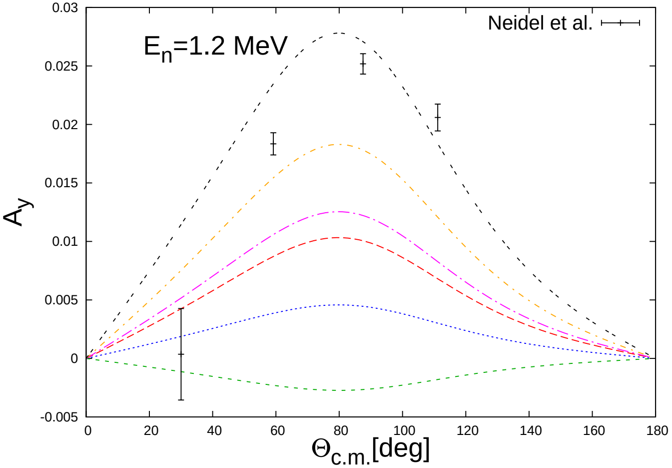
$$\mathcal{A}_n(Q) = \frac{M_N}{4\pi^2}\Bigg|_0^1 \int_0^\Lambda dq q^2 \int_{-1}^1 dx \frac{1}{qQx}\frac{2}{3}$$

$$\times\ D_s^{(n)}\!\left(B_0-\frac{q^2}{2M_N}-\frac{Q^2}{12M_N}+\left(\frac{1}{2}-y\right)\frac{qQx}{M_N},\vec{\mathbf{q}}\right).$$

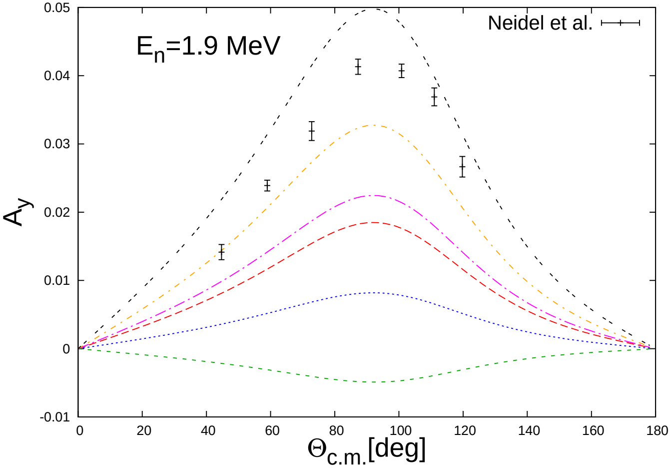
nd Polarization Observables



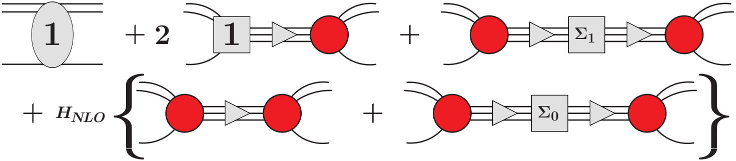
nd Polarization Observables



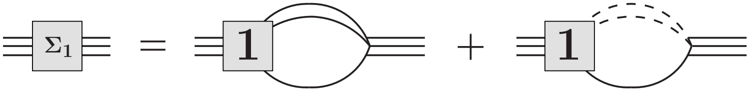
nd Polarization Observables



The NLO doublet S-wave amplitude for *nd* scattering is given by the sum of diagrams



where



The $\mathrm{N}^2\mathrm{LO}$ contribution can be calculated similarly but has many more contributions.

Integral equations for *nd* scattering amplitude

Projecting out in total angular momentum $\vec{\mathbf{J}} = \vec{\mathbf{L}} + \vec{\mathbf{S}}$ we obtain the set of coupled inegral equations in cluster configuration space

$$\begin{aligned}
\mathbf{t}^{(n)}{}_{\beta,\alpha}(k,p) &= \mathbf{K}^{(n)}{}_{\beta,\alpha}(k,p,E) \\
&+ \sum_{\gamma}\sum_{i=1}^{n-1} \mathbf{K}^{(n-i)}{}_{\beta,\gamma}(q,p,E) \otimes \left(\mathbf{R}^{(0)}\left(E - \frac{q^2}{2M_N}, q\right) \circ \mathbf{t}^{(i)}_{\gamma,\alpha}(k,q)\right) \\
&+ \sum_{i=1}^{n-1} \mathbf{R}^{(n-i)}\left(E - \frac{p^2}{2M_N}, p\right) \circ \mathbf{t}^{(i)}_{\beta,\alpha}(k,p) \\
&+ \mathbf{K}^{(0)}_{\beta,\beta}(q,p,E) \otimes \left(\mathbf{R}^{(0)}\left(E - \frac{q^2}{2M_N}, q\right) \circ \mathbf{t}^{(n)}_{\beta,\alpha}(k,q)\right)
\end{aligned}$$

where $\alpha = J, L, S, \beta = J, L', S'$, and $\gamma = J, L'', S''$ and

$$A(q) \otimes B(q) = \frac{1}{2\pi^2}\int_0^{\Lambda} dq q^2 A(q) B(q),$$

$\circ$: Schur product in cluster-configuration space

Three-body breakup cross-section