Inelastic scattering on the lattice

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Plan

- Introduction: scattering processes on the lattice and non-relativistic EFT
- Definition of the optical potential
- The limit $L \to \infty$ and smoothing
- Energy scan: twisted boundary conditions
- Conclusions, outlook

Scattering, resonances and the bound states

- How are the scattering processes described on the lattice?
  (Performing the limit $L \to \infty$)
- How are the inelastic channels included?
- How does one include the multi-particle bound states?
- How does one study the properties of the inelastic resonances?

$\cdots$

Lüscher's approach

M. Lüscher, lectures given at Les Houches (1988); NPB 364 (1991) 237, $\cdots$

- Lattice simulations are always done in a finite box of size L
- The interaction range is much smaller than the box size: $R \ll L$
- The box is small enough, so that the individual energy levels can be separated
- $p \simeq 2\pi/L$: non-relativistic EFT can be applied

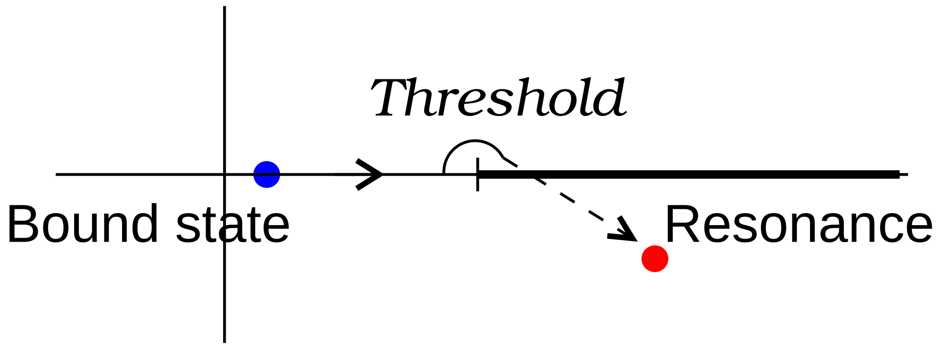
$\Rightarrow$ Single channel case: Lüscher equation relates the scattering phase to the measured energy level

$\Rightarrow$ The finite-volume corrections to the scattering phase are exponentially suppressed

$\Rightarrow$ Masses and widths of resonances are extracted from measured phases

Resonances

- Lüscher approach: spectrum $\leftrightarrow$ scattering phase, real axis
- Resonances correspond to S-matrix poles on the unphysical Riemann sheets after continuation into the complex plane



- Resonances are characterized by their mass, their lifetime, . . .
  These are the *intrinsic* properties of a resonance that should not depend neither on a particular *experiment* nor a particular *theoretical model* which is used to describe the data

How are the mass and width extracted from the measured phase?

Where are the resonance poles?

Assume, e.g., the effective range expansion (S-wave):

$$p\cot\,\delta(p) = A_0 + A_1 p^2 + \cdots\,,$$

Analytic continuation to the resonance pole:

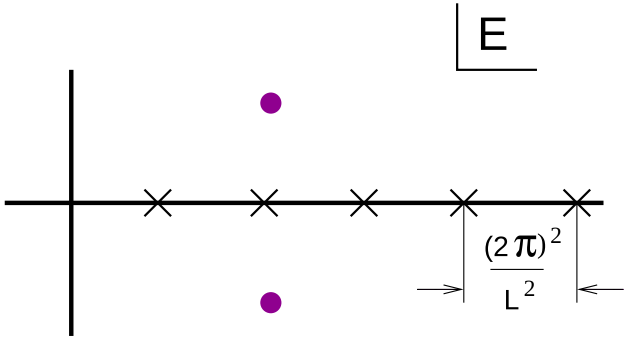
$$p_R\cot\,\delta(p_R) = -ip_R$$

$\Rightarrow$ $A_0, A_1, \cdots$ are measured on the lattice (real)

$\Rightarrow$ Resonance pole p_R in the complex momentum plane

The limit $L \to \infty$

Analytic structure of the two-point function in the complex plane



- The distance between energy levels $\sim (2\pi)^2/L^2$.
- The poles merge in the cut as $L \to \infty$

$$F_L(E) = \frac{1}{1 - E + g^2\sqrt{E}\cot(\sqrt{E}L)} \to \frac{1}{1 - E + g^2\sqrt{E}}\,, \qquad \mathrm{Im}E \neq 0$$

- Lüscher equation: $p\cot\delta(p)$ is a *meromorphic* function
- Phase can be extracted at *real* energies, $L \to \infty$ limit can be performed, only exponentially suppressed corrections

Covariant NREFT in the infinite volume

G. Colangelo, J. Gasser, B. Kubis and A. Rusetsky, PLB 638 (2006) 187
J. Gasser, B. Kubis and A. Rusetsky, NPB 850 (2011) 96

The Lagrangian:

- Contains non-relativistic field operators
- Particle number is conserved
- Counting rules observed after applying “threshold expansion”
- Electromagnetic and weak interactions can be systematically included

$$\mathcal{L}_I = C_0 \Phi_1^\dagger \Phi_2^\dagger \Phi_1 \Phi_2 + \text{derivatives}$$

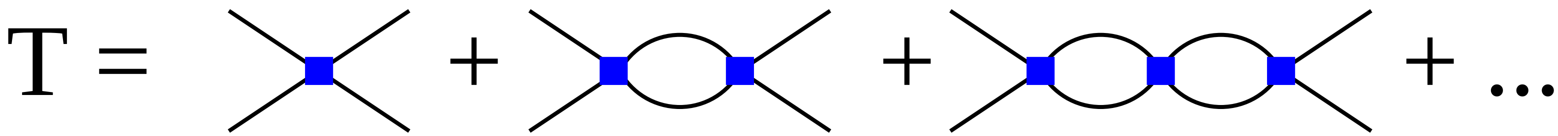
The propagator with the relativistic dispersion law:

$$D(p) = \frac{1}{2w(\mathbf{p})} \frac{1}{w(\mathbf{p}) - p_0 - i0}\,, \qquad w(\mathbf{p}) = \sqrt{M^2 + \mathbf{p}^2}$$

Lippmann-Schwinger equation

- Threshold expansion: in Feynman integrals, expand all integrands in powers of three-momenta, integrate in the dimensional regularization and sum up again

$$\hookrightarrow \text{loop} = \frac{ip}{8\pi\sqrt{s}}\,, \qquad p = \frac{\lambda^{1/2}(s, M_1^2, M_2^2)}{2\sqrt{s}}$$



$\hookrightarrow$ Scattering amplitude is Lorentz-invariant:

$$T_l = \frac{8\pi\sqrt{s}}{p\cot\delta_l(p) - ip}\,, \qquad p^{2l+1}\cot\delta_l(p) = -\frac{1}{a_l} + \frac{1}{2}r_l p^2 + \cdots$$

- Important in nonrest frames (formfactors, 3-body scattering)

Covariant NREFT in a finite volume

EFT in a finite volume with the *same* Lagrangian $\Rightarrow$ lattice QCD

Feynman loops are modified in a finite volume

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \to \frac{1}{L^3}\sum_{\mathbf{p}}$$

(finite box with periodic b.c.)

Lippmann-Schwinger equation in a finite volume

$$T_L(E) = V + VG_L(E)T_L(E)$$

- The potential V is a low-energy polynomial, only exponentially suppressed finite-volume corrections
- The finite-volume Green function $G_L(E)$ contains a tower of the real poles. The limit $L \to \infty$ does not exist for Im$E = 0$.

Lüscher equation

Poles in the amplitude = spectrum of the Hamiltionan

$\hookrightarrow$ Lüscher equation: (see S. R. Beane *et al.*, NPA 747 (2005) 55)

$$\det\left(\delta_{ll'}\delta_{mm'} - \tan\delta_l(s)\mathcal{M}_{lm,l'm'}\right) = 0$$

$\mathcal{M}_{lm,l'm'}$ is a linear combination of Lüscher zeta-functions

$$Z_{lm}(1,q^2) = \lim_{s\to 1}\sum_{\mathbf{n}\in\mathbb{Z}^3}\frac{\mathcal{Y}_{lm}(\mathbf{n})}{(\mathbf{n}^2-q^2)^s}\,, \qquad q = \frac{pL}{2\pi}$$

- In case of multi-channel scattering, Lippmann-Schwinger equation becomes a *matrix equation* in the channel space
- In order to ensure a smooth limit $L\to\infty$ for real energies E, one should determine *all elements* of the matrix V separately

Optical potential for the multi-channel scattering

H. Feshbach, '58; A.K. Kerman *et al*, '59

- Coupled channel Lüscher approach: more unknowns than measurements at a single energy, use of phenomenological parameterizations inevitable (effective-range expansion, UChPT,...)

  M. Lage, U.-G. Meißner and AR, PLB 681 (2009) 439

  V. Bernard, M. Lage, U.-G. Meißner and AR, JHEP 1101 (2011) 019

  M. Döring, U.-G. Meißner, E. Oset and AR, EPJA 47 (2011) 139

- Intermediate and final states with three and more particles: the proposed framework still very cumbersome

- Many interesting exotic states decaying into different channels:

$$Z_c(3900)^\pm \to J/\psi\pi^\pm,\ h_c\pi^\pm,\ (D\bar{D}^*)^\pm$$

$$Z_c(4025) \to D^*\bar{D}^*\ ,\ h_c\pi\ \ ,\quad \text{etc}$$

$$\boxed{\text{Change of a paradigm?}}$$

Optical potential in the infinite volume: $\pi\eta - K\bar{K}$

- Multi-channel Lippmann-Schwinger equation reduces to algebraic equations in each partial wave; Potential = K-matrix

$$T(E) = V + VG_0(E)T(E)\,.$$

- Projectors: "primary": $P = K\bar{K}$, "secondary": $Q = 1 - P = \pi\eta$

- Definition of the optical potential

$$\begin{aligned} T_P(E) &= W(E) + ip_{K\bar{K}}\, W(E) T_P(E) \\ W(E) &= V_{PP} + \frac{ip_{\pi\eta} V_{PQ}^2}{1 - ip_{\pi\eta} V_{QQ}} \end{aligned}$$

- Introduce $M = V^{-1}$: useful in the vicinity of the K-matrix poles

$$W^{-1}(E) = M_{PP} - \frac{M_{PQ}^2}{M_{QQ} - ip_{\pi\eta}}\,.$$

Optical potential: short summary

- Optical potential contains less information than the multi-channel potential V. Interactions between different channels are not resolved separately
- Optical potential contains full information *about the primary channel:* it leads to the same $T_P(E)$, as the solution of full Lippmann-Schwinger equation and can be used to extract poles in the primary channel
- The imaginary part of the optical potential is given by a sum of the cross sections into the secondary channels. Multi-channel unitarity is obeyed

Is it possible to directly extract the optical potential on the lattice without resolving scattering into individual final states?

Scattering equations in a finite volume

- Imposing periodic boundary conditions: $\mathbf{p}=(2\pi/L)\mathbf{n}\,,\;\mathbf{n}\in\mathbb{Z}^3$

$$ip_k \to \frac{2}{\sqrt{\pi}L}\,Z_{00}(1;q_k^2)\,,\qquad q_k=\frac{p_kL}{2\pi}\,,\qquad k=\pi\eta,\;K\bar{K}$$

- The secular equation determines the energy levels:

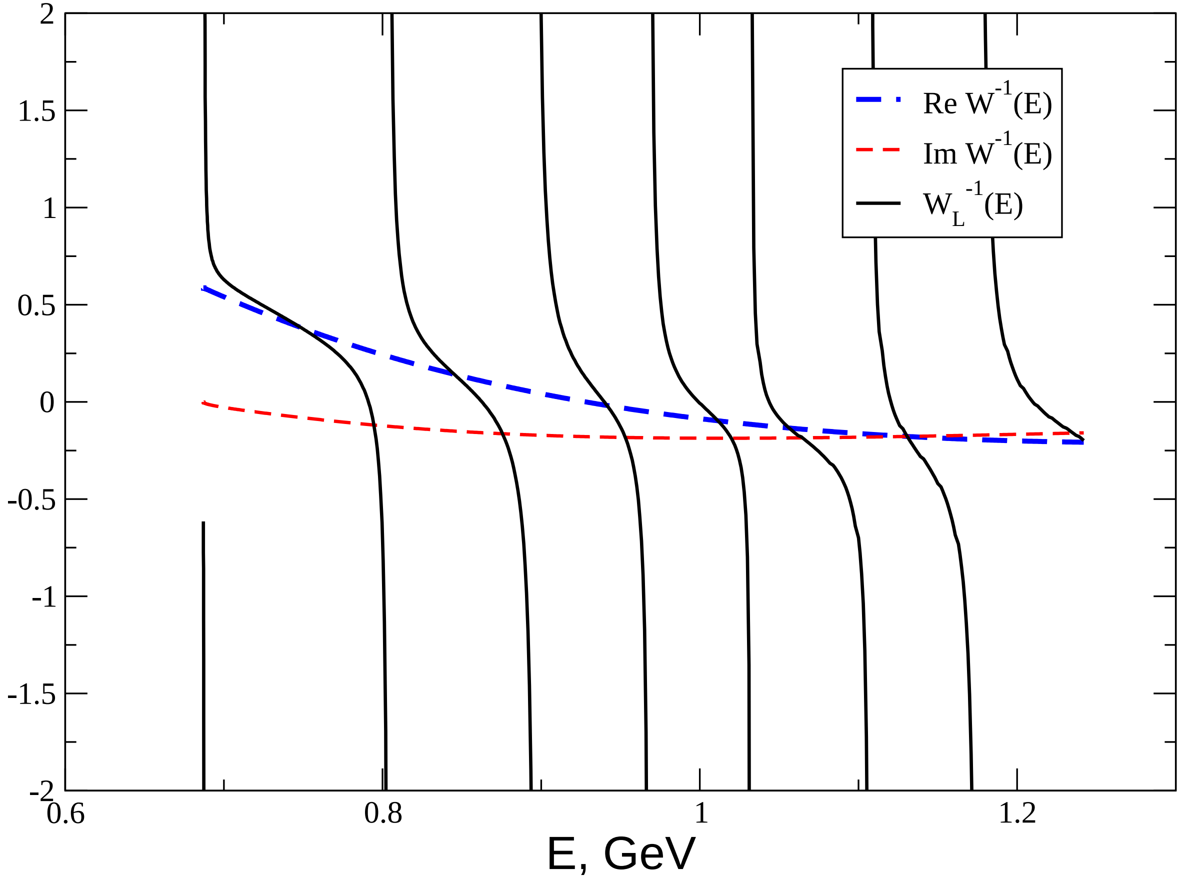
$$\biggl(M_{PP}-\frac{2}{\sqrt{\pi}L}\,Z_{00}(1;q_{K\bar{K}}^2)\biggr)\biggl(M_{QQ}-\frac{2}{\sqrt{\pi}L}\,Z_{00}(1;q_{\pi\eta}^2)\biggr)-M_{PQ}^2=0$$

- Finite-volume analog of the optical potential:

$$W_L^{-1}(E)\doteq\frac{2}{\sqrt{\pi}L}\,Z_{00}(1;q_{K\bar{K}}^2)\quad=\quad M_{PP}-\frac{M_{PQ}^2}{M_{QQ}-\frac{2}{\sqrt{\pi}L}\,Z_{00}(1;q_{\pi\eta}^2)}$$

$$\frac{2}{\sqrt{\pi}L}\,Z_{00}(1;q_{\pi\eta}^2)\to ip_{\pi\eta}\quad\Rightarrow\quad W_L^{-1}(E)\to W^{-1}(E)$$

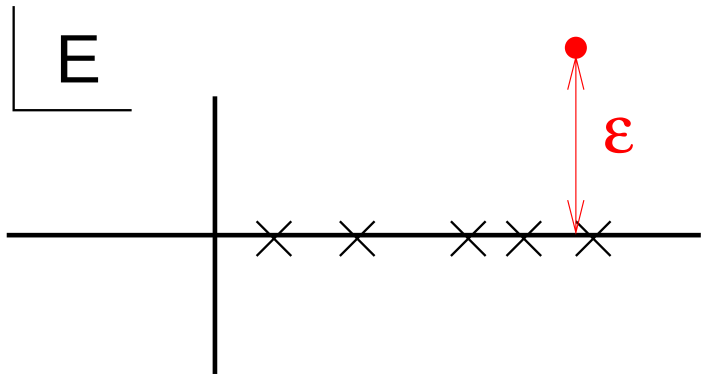
The infinite-volume limit of $W_L^{-1}(E)$



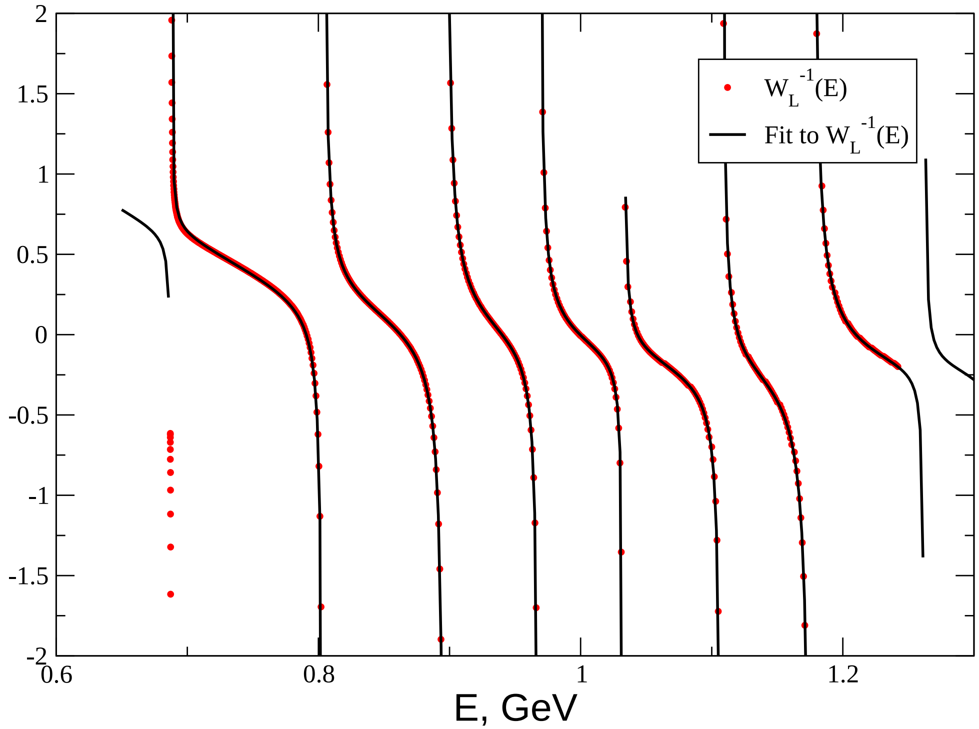
W_L^{-1}: no limit $L \to \infty$ in the presence of inelastic channels!

Causal prescription

- Introduce causal prescription $E \to E + i\varepsilon$, equivalent to adiabatic turning on/off the interaction
- The limits $L \to \infty$ and $\varepsilon \to 0$ are not interchangeable. Infinite-volume limit is obtained with first $L \to \infty$ and then $\varepsilon \to 0$ (DeWitt, 1956)
- If $\varepsilon \gg$ (distance between energy levels), the individual levels merge in a cut



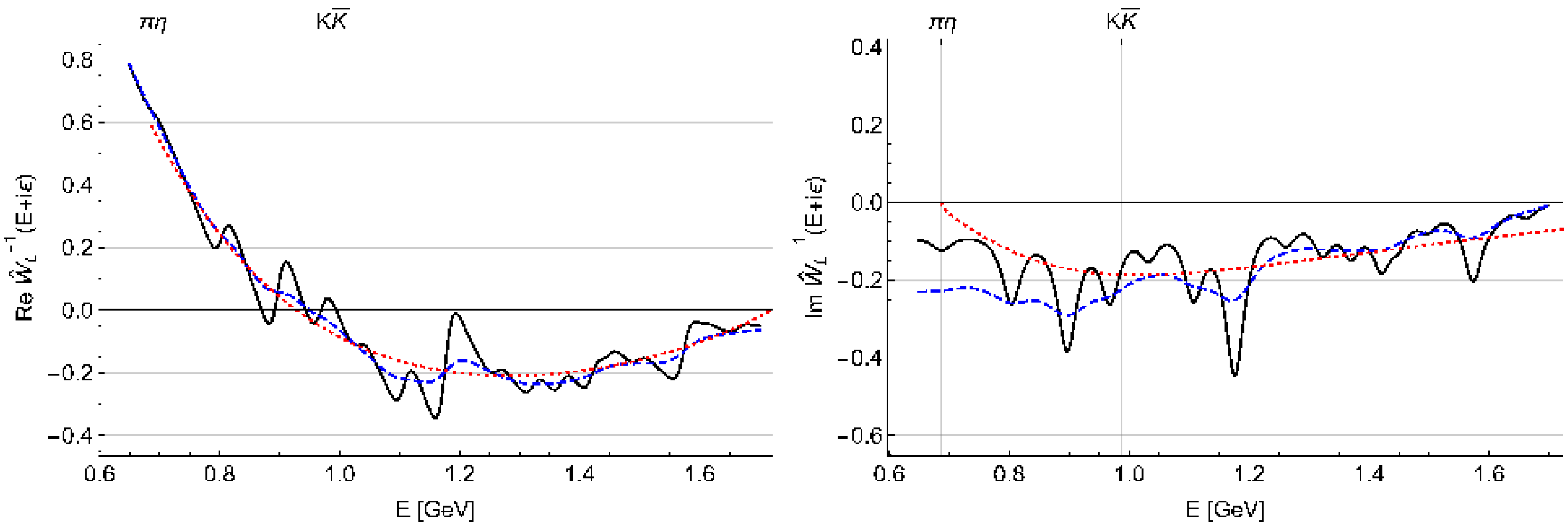
Fit to the pseudophase on the real axis



$$\hat{W}_L^{-1}(E) = \sum_i \frac{Z_i}{E - Y_i} + D_0 + D_1 E + D_2 E^2 + D_3 E^3 + \cdots$$

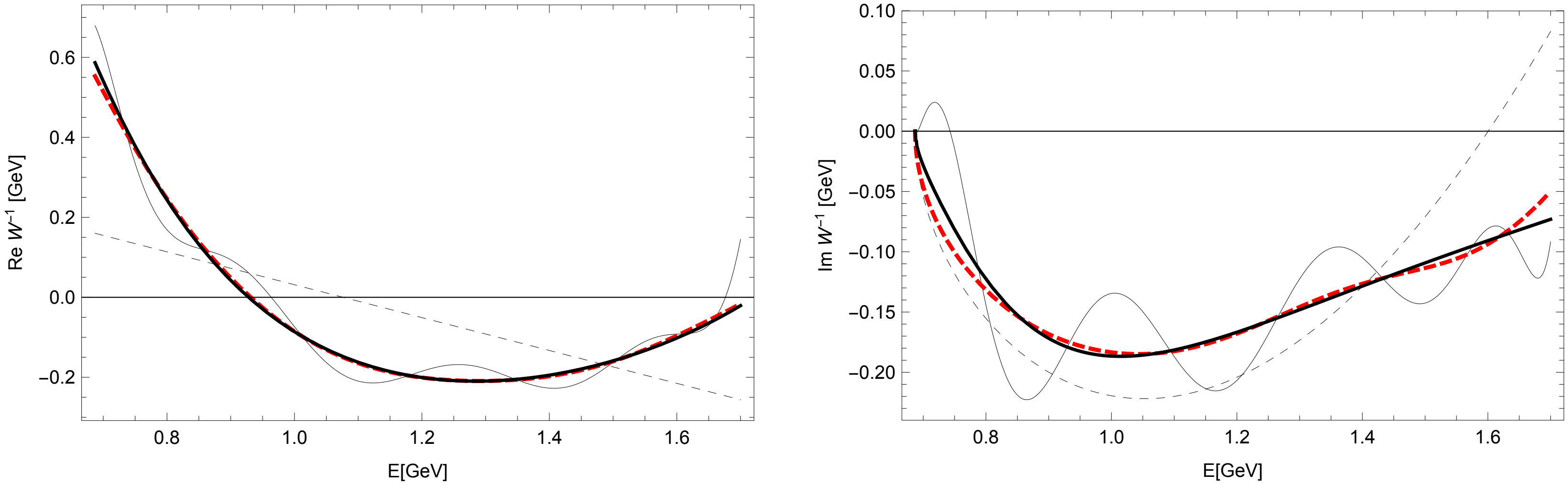
- Only simple poles + regular background, even when multiparticle states are present!

Continuation into the complex plane



- For $E \to E + i\varepsilon$, the regular summation theorem (Lüscher) can be applied to the diagrams containing *any number of intermediate particles.*
- The limit $L \to \infty$ exists, if $\varepsilon \neq 0$

Smoothing: threshold behavior built in



- Fit the oscillating functions by

$$\hat{W}^{-1}(E) = \sum_k (a_k + i p_{\pi\eta}(E + i\varepsilon) b_k)(E + i\varepsilon - E_0)^k$$

- Evaluate the result at $\varepsilon \to 0$

The penalty factor

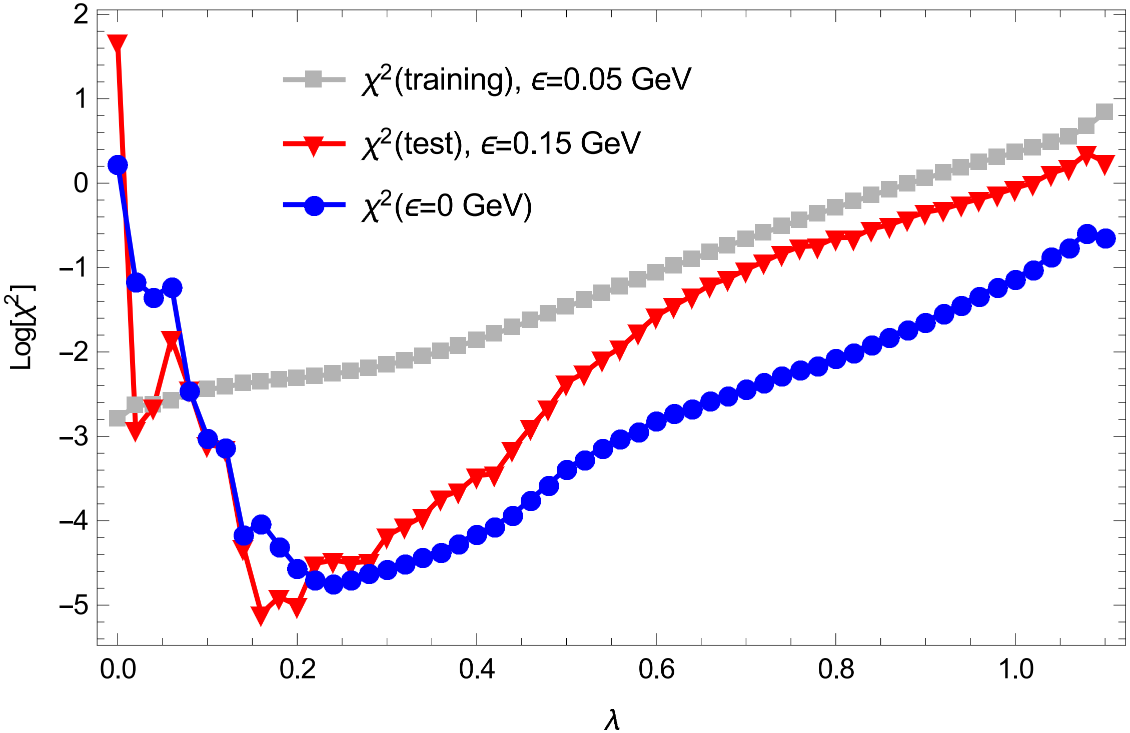
$$\chi^2 = \sum_k \frac{|\hat{W}^{-1}(E_k) - \hat{W}_L^{-1}(E_k)|^2}{\sigma_k^2} + P(a_j, b_j)$$

example: $$P(a_j, b_j) = \lambda^4 \int_{E_{\mathrm{min}}+i\varepsilon}^{E_{\mathrm{max}}+i\varepsilon} \left| \frac{\partial^2 \hat{W}^{-1}(E)}{\partial E^2} \right|$$

Looking for the *sweet spot* in λ (LASSO method):

- Choose arbitrarily the *training set,* determine χ^2 for this set
- From the rest of the data *(test-validation set)* determine χ_V^2 without altering the fit parameters a_j, b_j
- The minimum of χ_V^2 as a function of λ determines the *sweet spot*

Finding the sweet spot



- Training and test/validation sets: different ε
- Test with the “true” potential yields a very similar value of λ

Non-parametric methods: Gaussian smearing

- Each data point is replaced by a linear combination of the neighbouring data points with the weights:

$$w(x) \propto \exp(-(x - x_0)^2/2\sigma_0^2)$$

- Smearing radius = $2\sigma_0$ = 2 $\times$ distance between peaks
- Numerically extrapolate $\varepsilon \to 0$

Measuring $W_L^{-1}(E)$ by using twisted boundary conditions

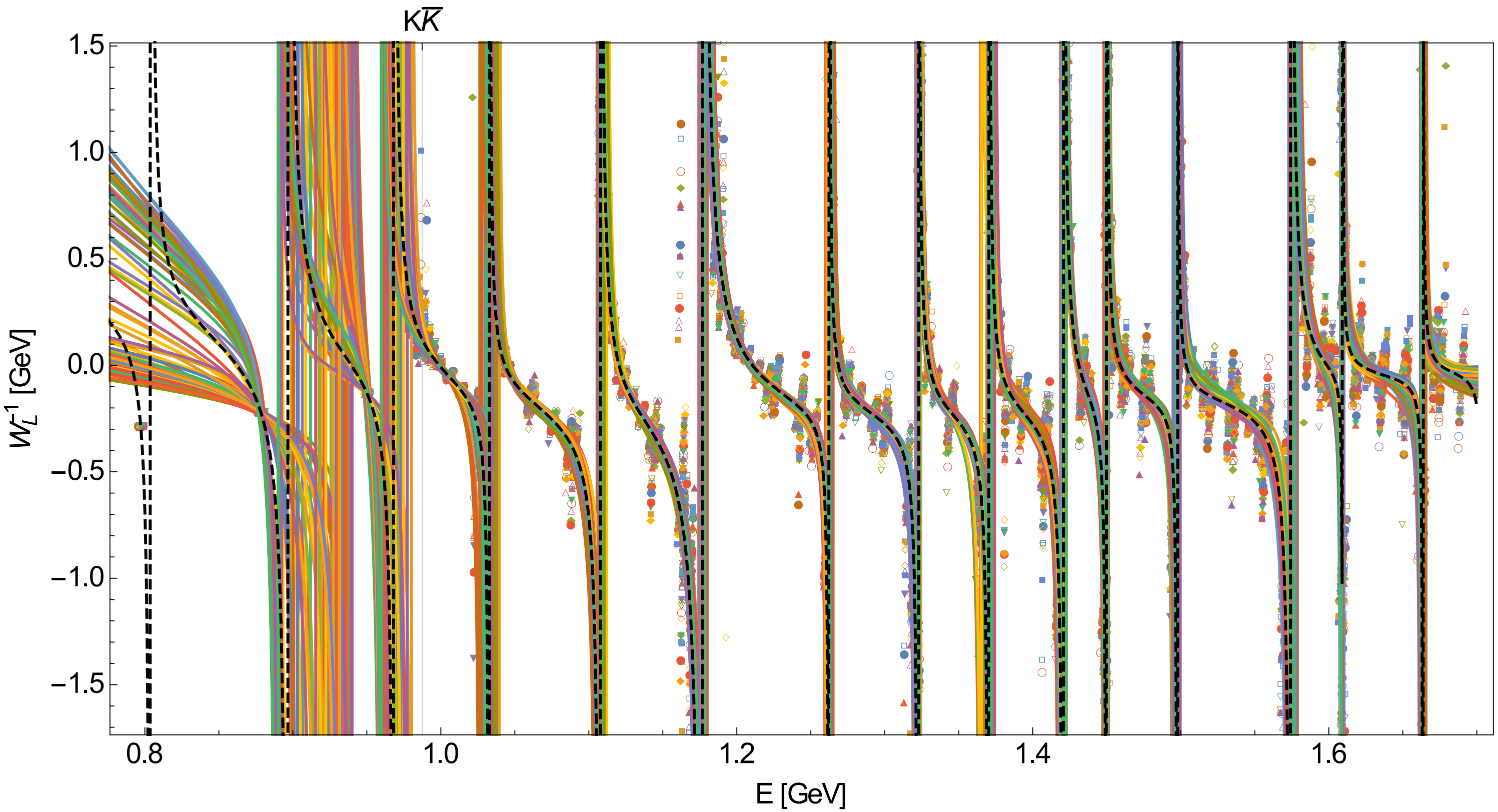
- $W_L^{-1}(E)$ depends on L, possibly on other parameters. In the fit, all these parameters should be fixed
- In some cases, tuning the energy can be achieved by using (partially) twisted boundary conditions: The elastic threshold moves and the inelastic stays put

$$u(\mathbf{x}+L\mathbf{e}) = e^{i\theta}u(\mathbf{x})\,, \quad d(\mathbf{x}+L\mathbf{e}) = e^{i\theta}d(\mathbf{x})\,, \quad s(\mathbf{x}+L\mathbf{e}) = s(\mathbf{x})$$

$$Z_{00}(1;q_{K\bar{K}}^2) \to Z_{00}^{\theta}(1;q_{K\bar{K}}^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n}\in\mathbb{Z}^3} \frac{1}{(\mathbf{n}+\theta/2\pi)^2 - q_{K\bar{K}}^2}$$

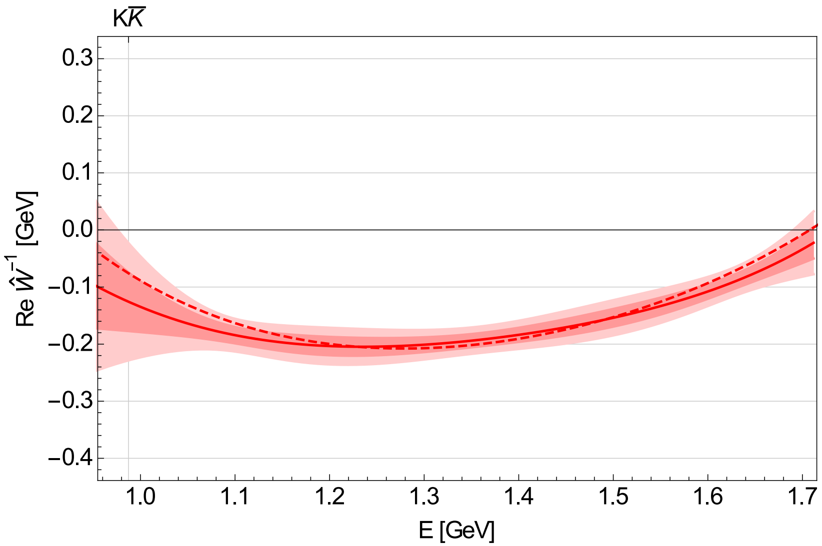
$$W_L^{-1}(E) = \frac{2}{\sqrt{\pi}L} Z_{00}^{\theta}(1;q_{K\bar{K}}^2(\theta)) :$$ No explicit dependence on θ!

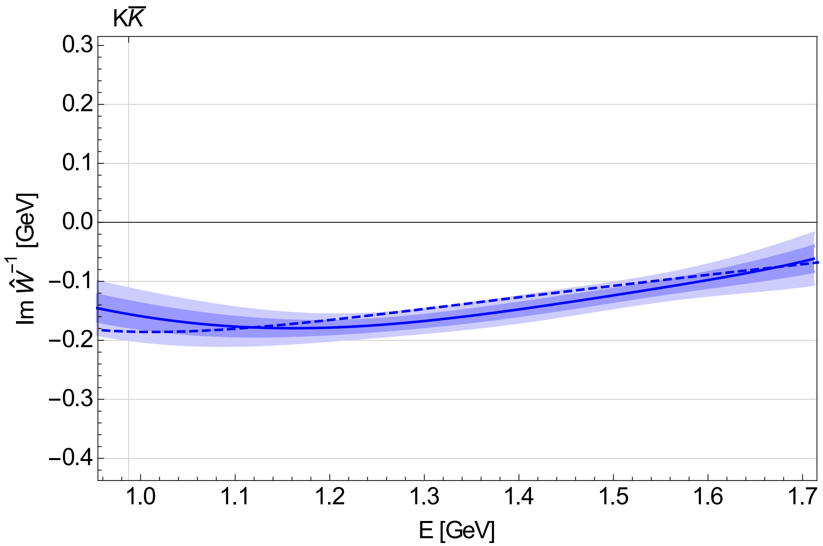
Synthetic data obtained with partial twisting



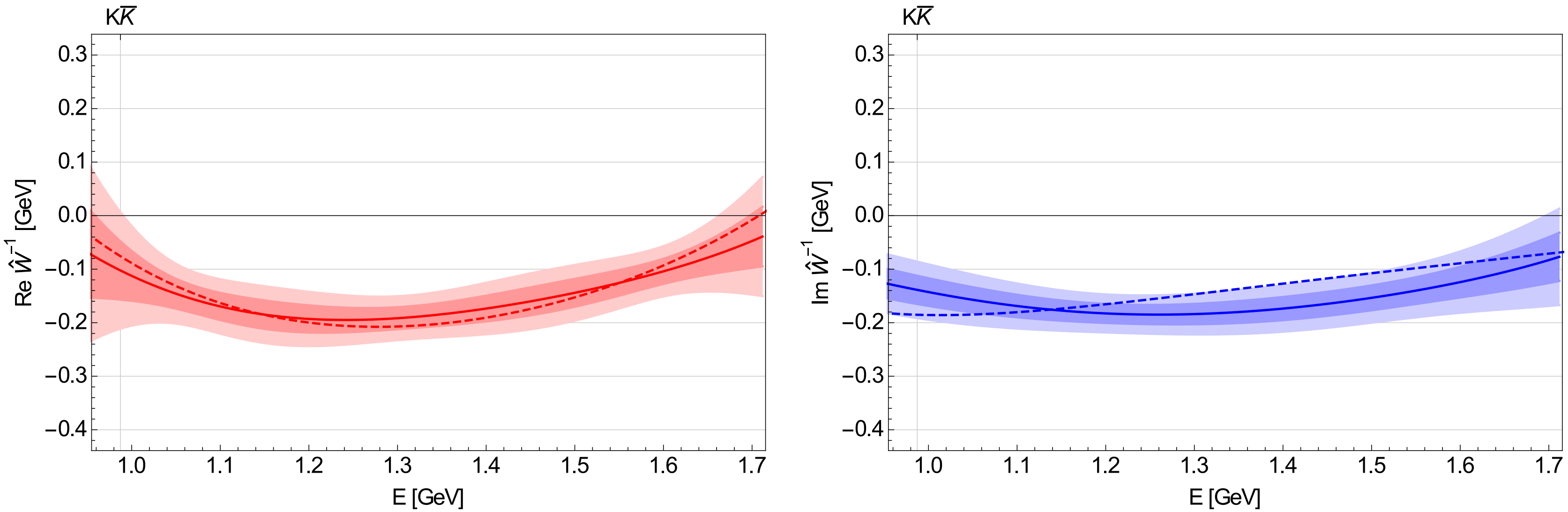
- Can not scan below primary threshold!

1 MeV error, polynomial smearing

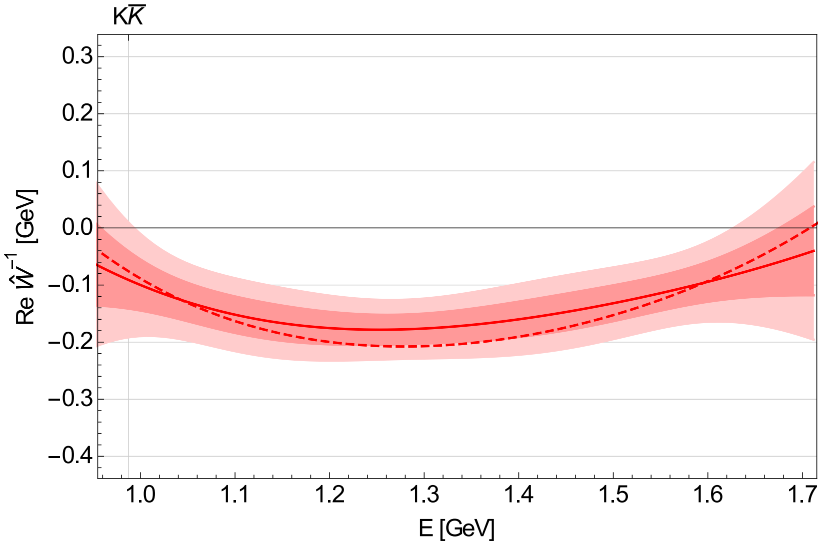


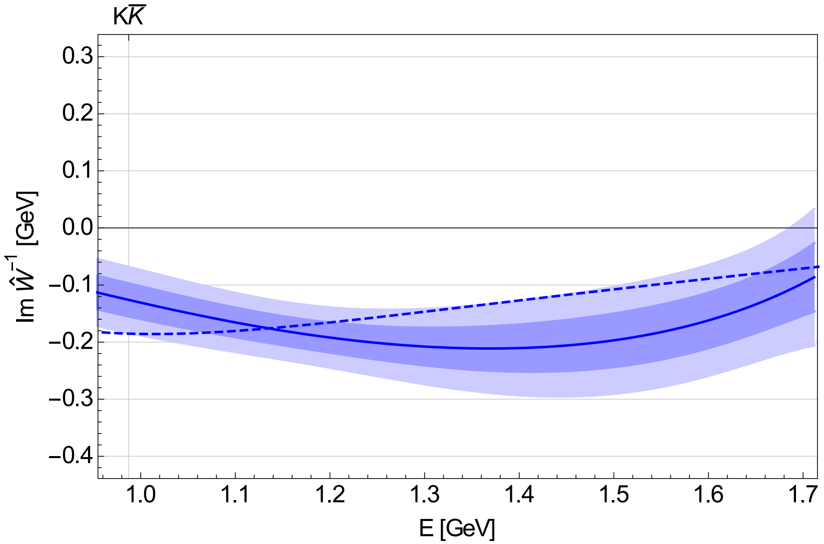


2 MeV error, polynomial smearing

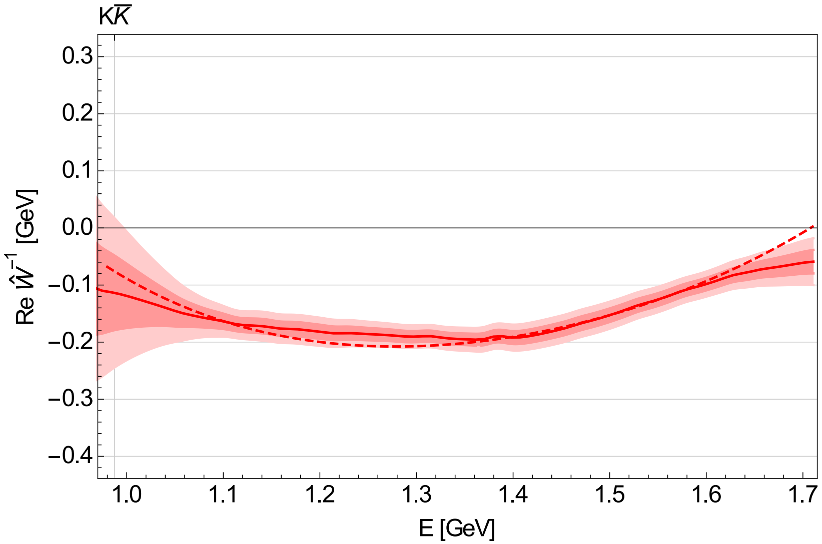


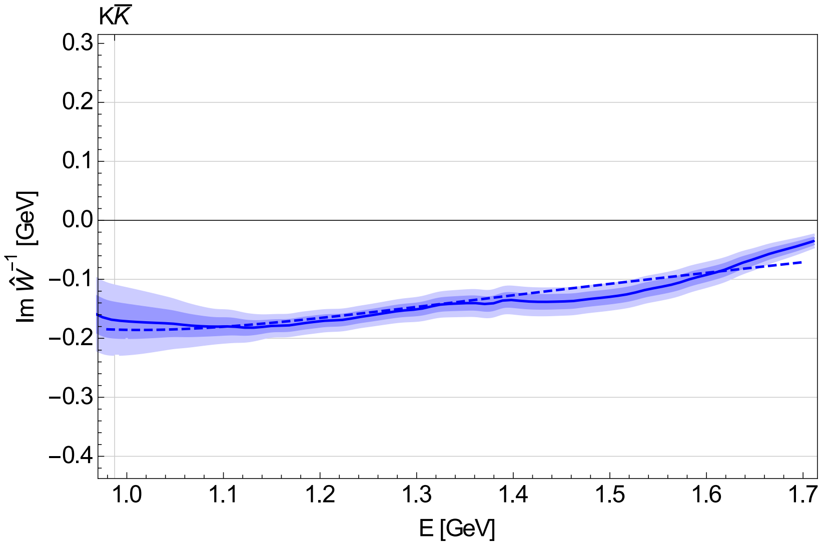
3 MeV error, polynomial smearing



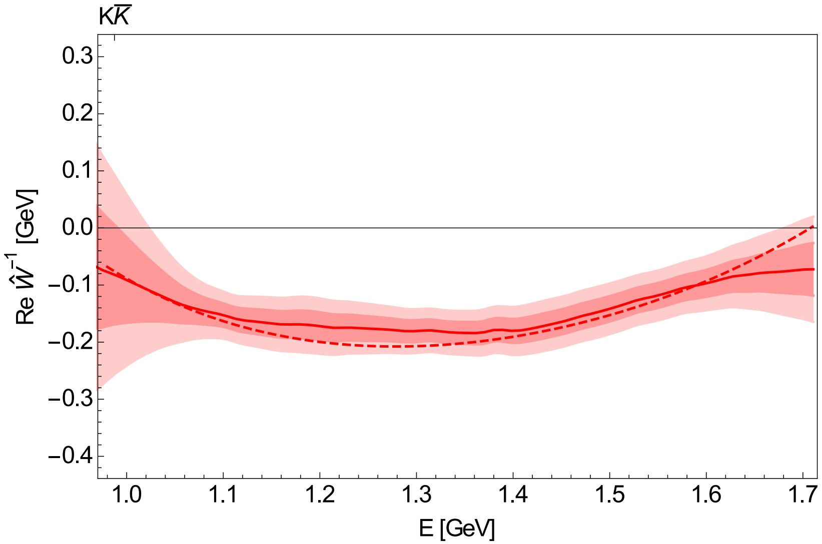


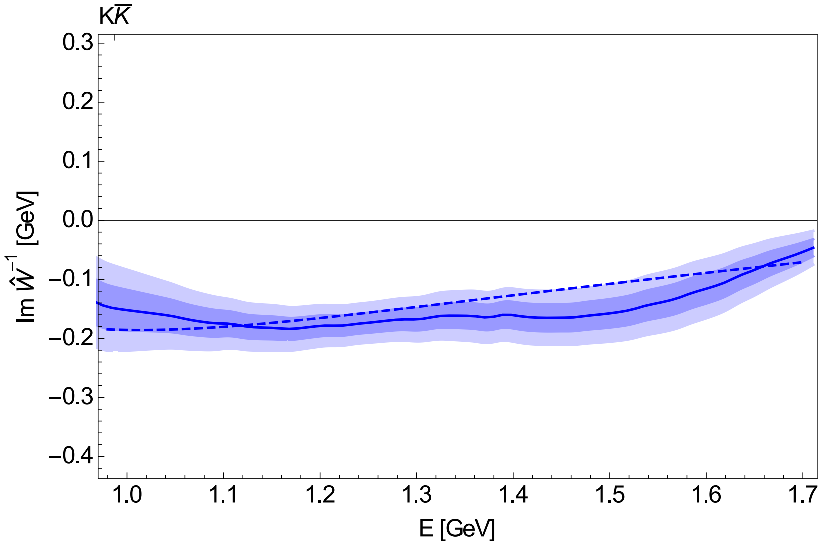
1 MeV error, Gaussian smearing



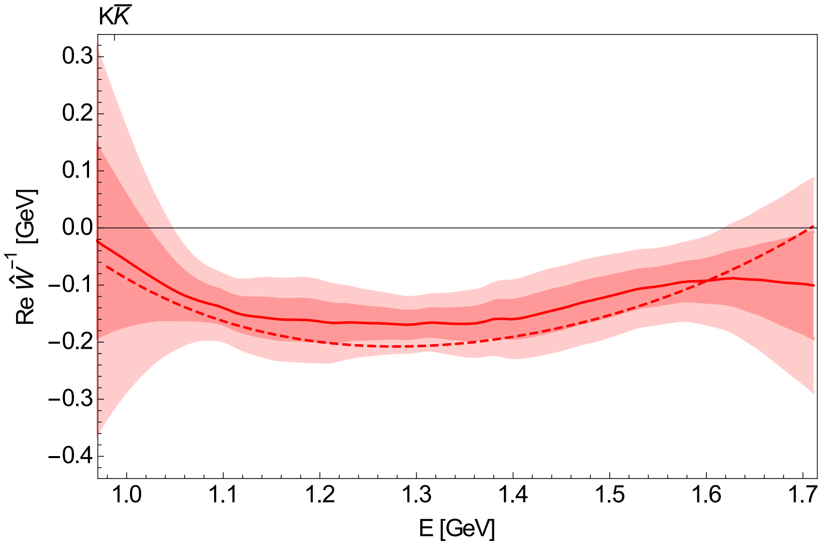


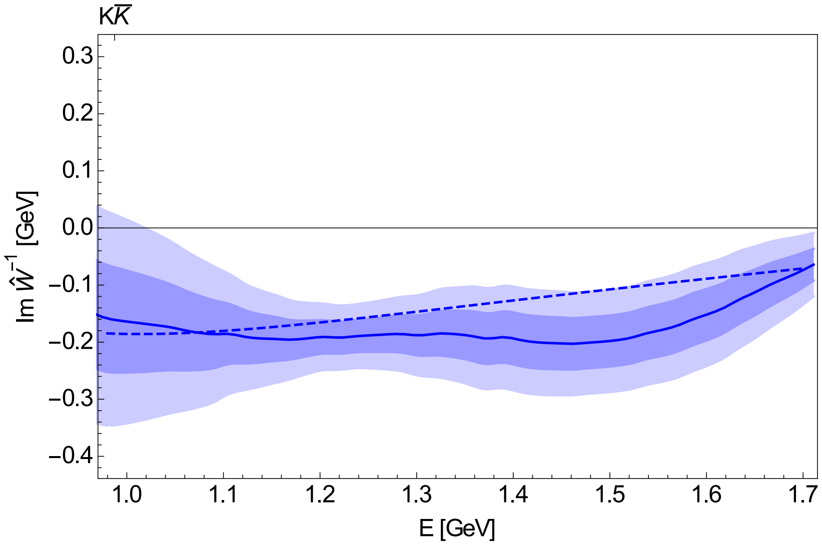
2 MeV error, Gaussian smearing





3 MeV error, Gaussian smearing





Conclusions, outlook

- Effective field theories in a finite volume provide a powerful tool for the extraction of hadronic observables in the scattering sector from the lattice QCD data

Infinite volume	:	QCD	$\to$	EFT (∞)
Finite volume	:	QCD on the lattice	$\to$	EFT (L)

$\Rightarrow$ Allow to study the limit $L \to \infty$ in the observables

$\Rightarrow$ Enable for a direct extraction of the optical potential on the lattice – multiparticle inelastic states allowed

- The method implies analytic continuation to the complex energies – otherwise, the limit $L \to \infty$ can not be performed
- Smoothing allows one to determine the optical potential at a finite L
- Twisted boundary conditions: a tool for the energy scan
- Forthcoming: testing in the ϕ^4 theory on the lattice