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Towards an Understanding of Clustering in Nuclei from First Principles

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- Results from nuclear lattice simulations
- Ab initio calculation of alpha-alpha scattering
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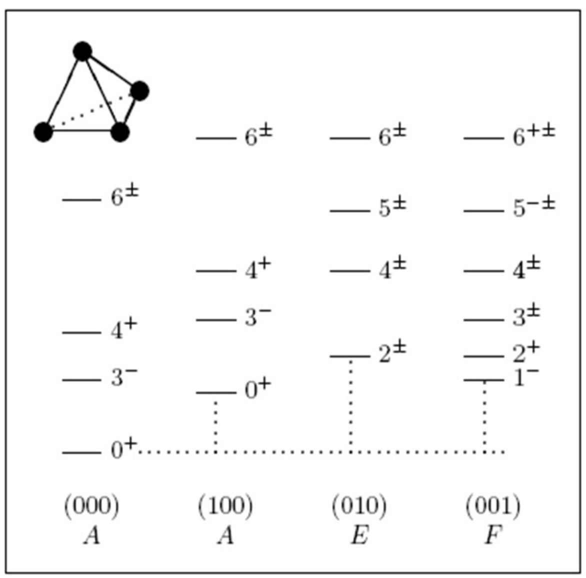
Short introduction

CLUSTERING in NUCLEI

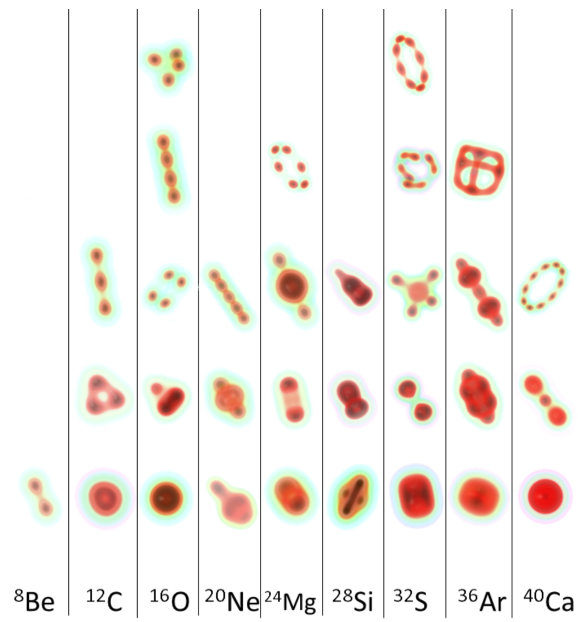
- Introduced theoretically by Wheeler already in 1937:

John Archibald Wheeler, “Molecular Viewpoints in Nuclear Structure,” Physical Review **52** (1937) 1083

- many works since then... Ikeda, Horiuchi, Freer, Schuck, Röpke, Khan, Zhou, Iachello, . . .



Bijker, Iachello (2014)



Ebran, Khan, Niksic, Vretenar (2014)

α-clusters

$\Rightarrow$ can we understand this phenomenon from *ab initio* calculations?

Basics of nuclear lattice simulations

for an easy intro, see: UGM, Nucl. Phys. News **24** (2014) 11

NUCLEAR LATTICE SIMULATIONS

Frank, Brockmann (1992), Koonin, Müller, Seki, van Kolck (2000) , Lee, Schäfer (2004), . . .
Borasoy, Krebs, Lee, UGM, Nucl. Phys. **A768** (2006) 179; Borasoy, Epelbaum, Krebs, Lee, UGM, Eur. Phys. J. **A31** (2007) 105

- *new method* to tackle the nuclear many-body problem

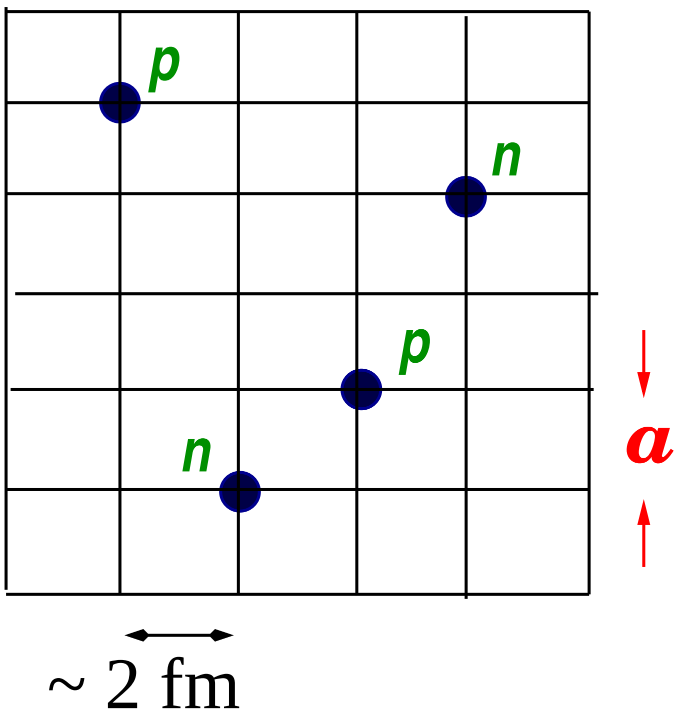
- discretize space-time $V = L_s \times L_s \times L_s \times L_t$:
  nucleons are placed on the sites

- discretized chiral potential w/ pion exchanges
  and contact interactions + Coulomb

  → see Epelbaum, Hammer, UGM, Rev. Mod. Phys. **81** (2009) 1773

- typical lattice parameters

$$\Lambda = \frac{\pi}{a} \simeq 300\,\text{MeV [UV cutoff]}$$

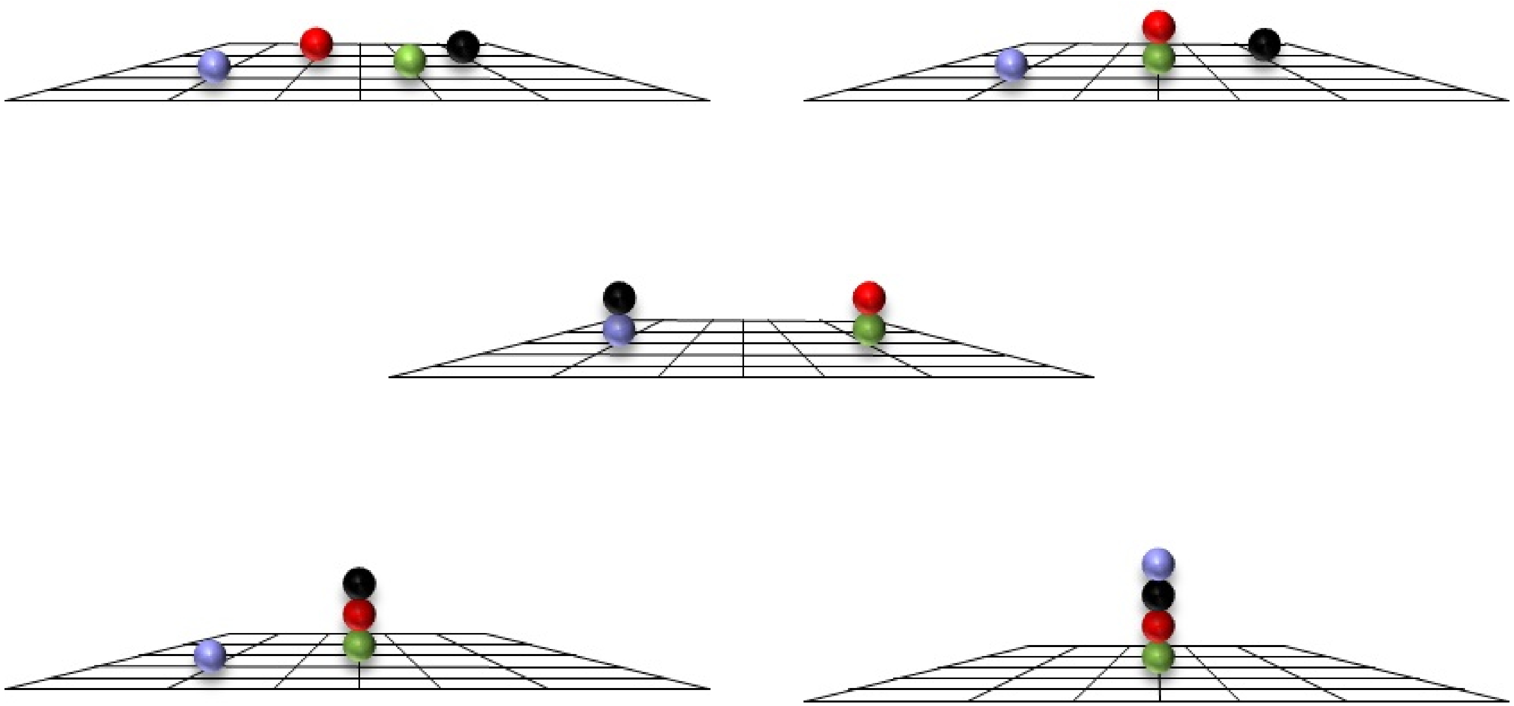


- strong suppression of sign oscillations due to approximate Wigner SU(4) symmetry

  J. W. Chen, D. Lee and T. Schäfer, Phys. Rev. Lett. **93** (2004) 242302, T. Lähde et al., EPJA **51** (2015) 92

- hybrid Monte Carlo & transfer matrix (similar to LQCD)

CONFIGURATIONS



⇒ all *possible* configurations are sampled

⇒ preparation of *all possible* initial/final states

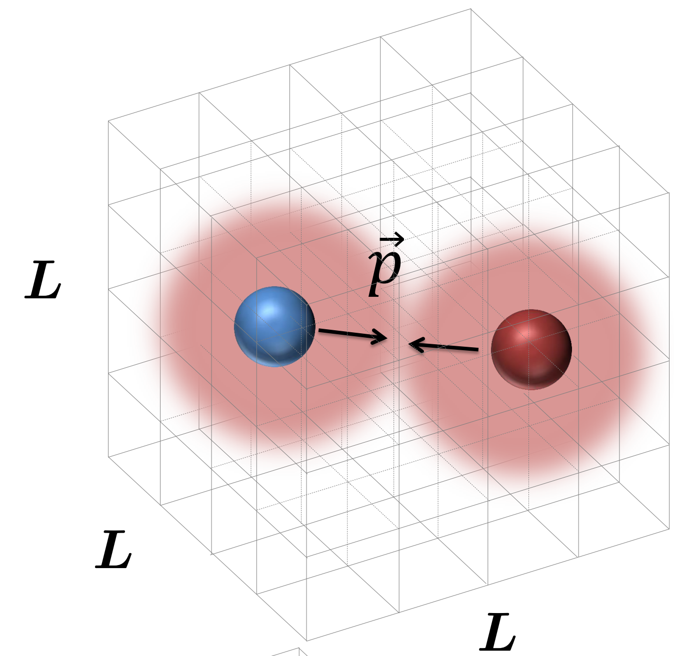
⇒ *clustering* emerges *naturally*

EXTRACTING PHASE SHIFTS on the LATTICE

- Lüscher's method:

  Two-body energy levels below the inelastic threshold on a periodic lattice are related to the phase shifts in the continuum

  Lüscher, Comm. Math. Phys 105 (1986) 153
  Lüscher, Nucl. Phys 354 (1991) 531



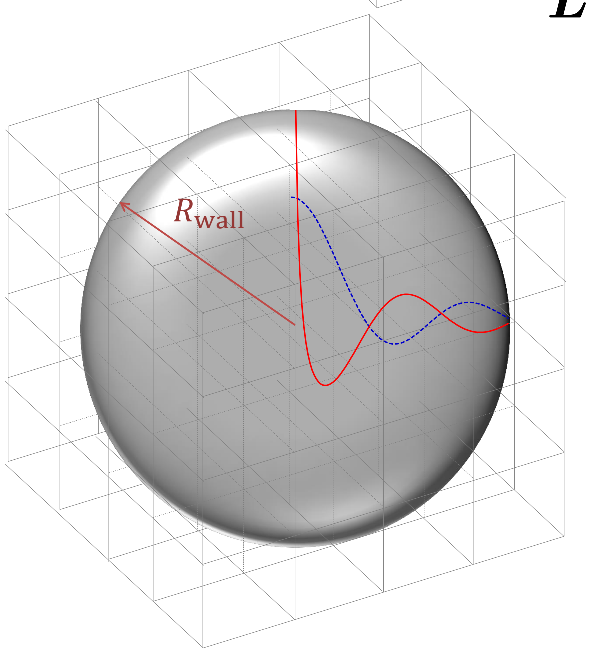
- Spherical wall method:

  Impose a hard wall on the lattice and use the fact that the wave function vanishes for $r = R_{\rm wall}$:

$$\psi_\ell(r) \sim [\cos\delta_\ell(p) F_\ell(pr) + \sin\delta_\ell(p) G_\ell(pr)]$$

  Borasoy, Epelbaum, Krebs, Lee, UGM, EPJA 34 (2007) 185

  Carlson, Pandharipande, Wiringa, NPA 424 (1984) 47

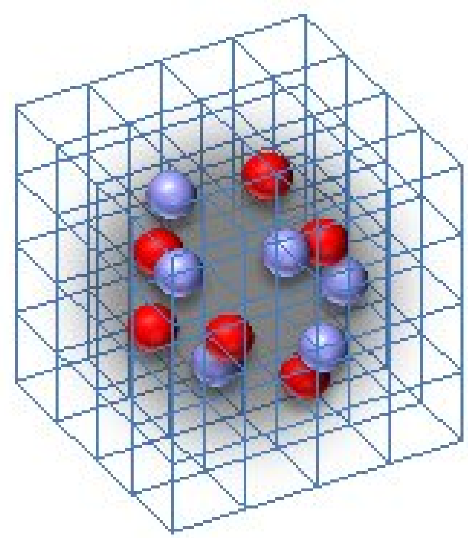


COMPUTATIONAL EQUIPMENT

- Present = JUQUEEN (BlueGene/Q)

6 Pflops

Lattice: some results



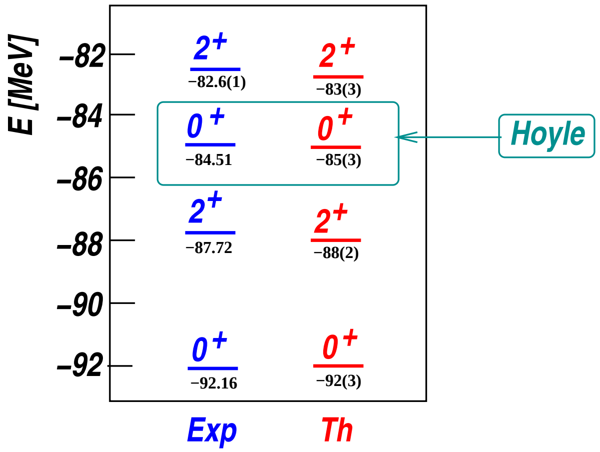
NLEFT

Epelbaum, Krebs, Lähde, Lee, Luu, UGM, Rupak + post-docs + students

RESULTS from LATTICE NUCLEAR EFT @ NNLO

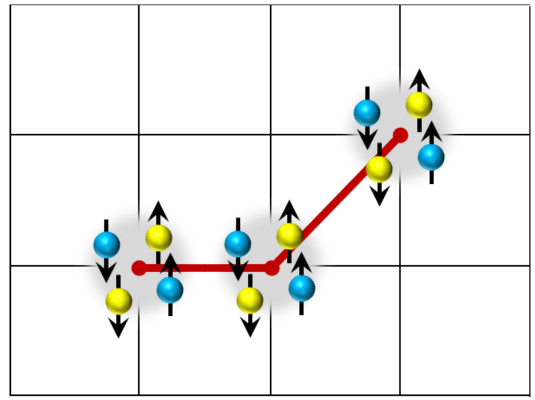
- Hoyle state in ^{12}C

PRL **106** (2011)



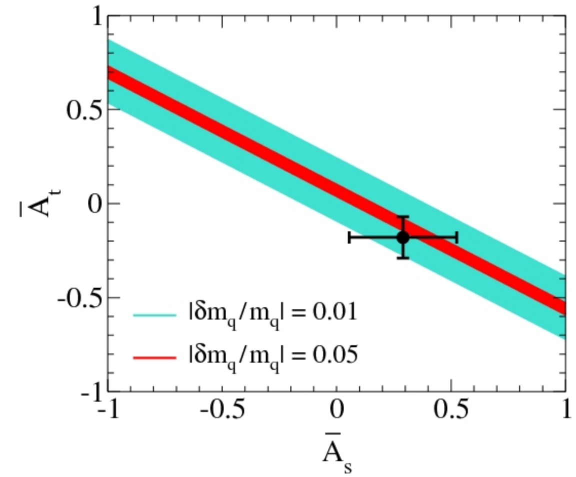
- Structure of the Hoyle state

PRL **109** (2012)



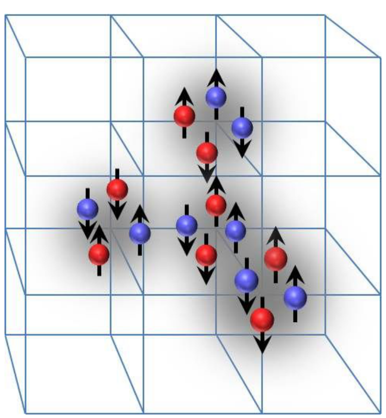
- Fate of carbon-based life

PRL **110** (2013), EPJ A**49** (2013)



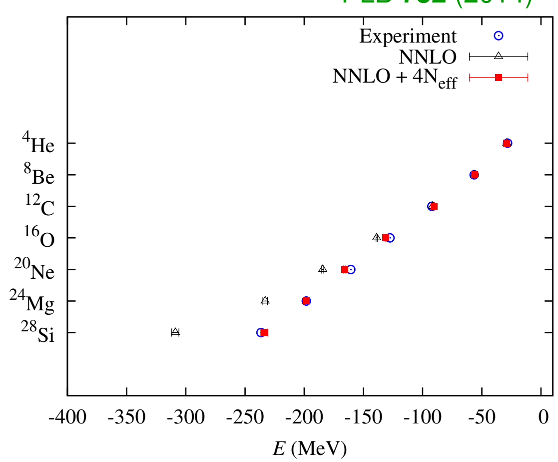
- Spectrum of ^{16}O

PRL **112** (2014)



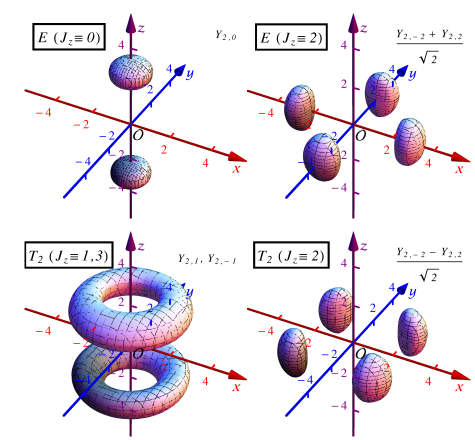
- Going up the α-chain

PLB **732** (2014)



- Rot. symmetry breaking

PRD **90** (2014), PRD **92** (2015)



STRUCTURE of ^{16}O

- Mysterious nucleus, despite modern ab initio calcs

Hagen et al. (2010), Roth et al. (2011), Hergert et al. (2013), Jansen et al. (2014), Cipollone et al. (2015)

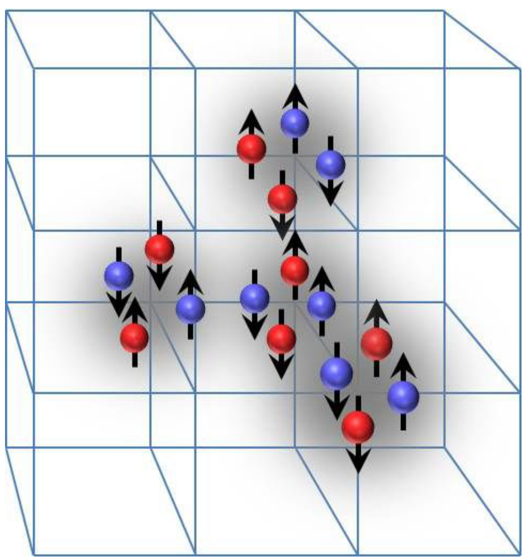
- Alpha-cluster models since decades, some exp. evidence

Wheeler (1937), Dennison (1954), Robson (1979), . . ., Freer et al. (2005)

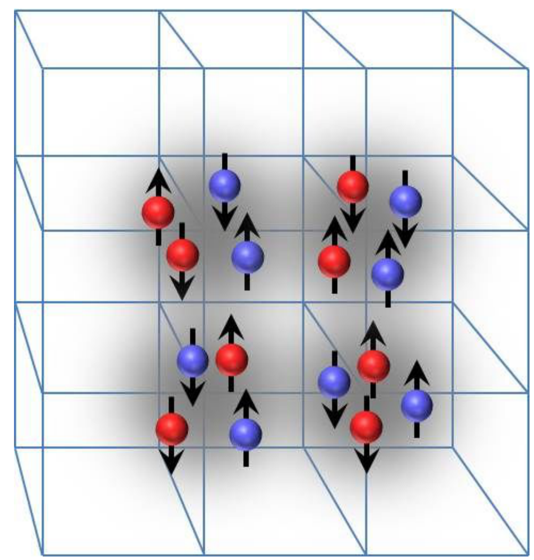
- Spectrum very close to tetrahedral symmetry group — Bijker & Iachello (2014)

- Relevant configurations in lattice simulations:

Tetrahedron (A)



Square (narrow (B) and wide (C))



DECODING the STRUCTURE of ^{16}O

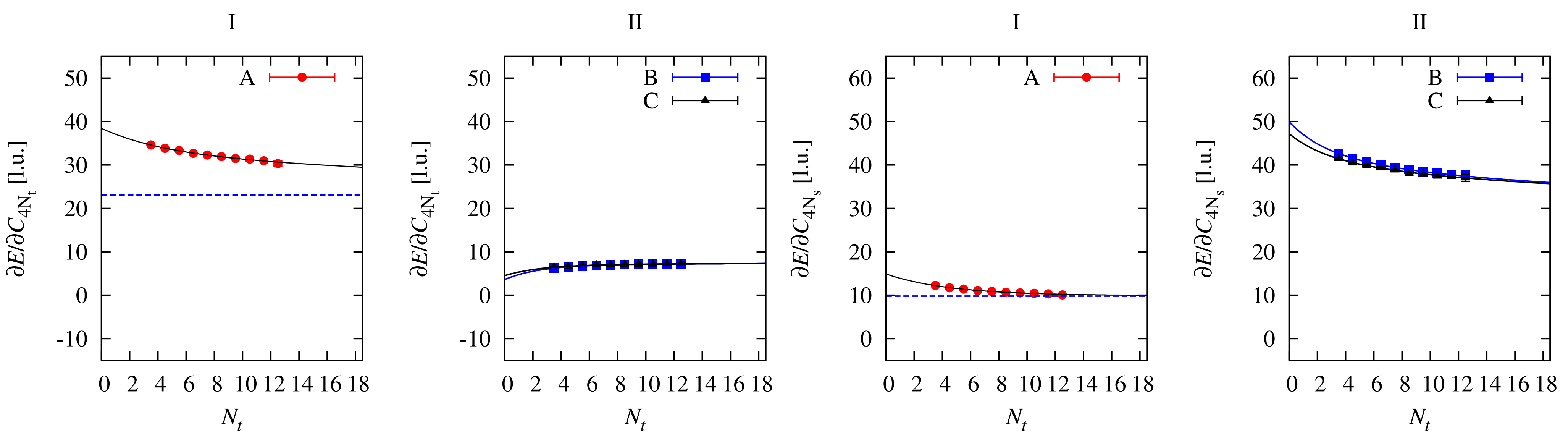
Epelbaum, Krebs, Lähde, Lee, UGM, Rupak, Phys. Rev. Lett. **112** (2014) 102501

- measure the 4N density, where each of the nucleons is placed at adjacent points

$\Rightarrow$ $\mathbf{0_1^+}$ ground state: mostly tetrahedral config

$\Rightarrow$ $\mathbf{0_2^+}$ excited state: mostly square configs
$\mathbf{2_1^+}$ excited state: rotational excitation of the $\mathbf{0_2^+}$

overlap w/ tetrahedral config. overlap w/ square configs.



RESULTS for ^{16}O

- Spectrum:

	LO	NNLO	Exp.
0_1^+	$-147.3(5)$	$-131.3(5)$	-127.62
0_2^+	$-145(2)$	$-123(2)$	-121.57
2_1^+	$-145(2)$	$-123(2)$	-120.70

[NB: Eff. 4N term included]

- LO charge radius: $r(0_1^+) = 2.3(1)$ fm Exp. $r(0_1^+) = 2.710(15)$ fm

$\Rightarrow$ compensate for this by rescaling with appropriate units of $r/r_{\rm LO}$

- LO EM properties:

	LO	LO(r-scaled)	Exp.
$Q(2_1^+)$ [e fm^2]	10(2)	15(3)	—
$B(E2, 2_1^+ \to 0_2^+)$ [e^2 fm^4]	22(4)	46(8)	65(7)
$B(E2, 2_1^+ \to 0_1^+)$ [e^2 fm^4]	3.0(7)	6.2(1.6)	7.4(2)
$M(E0, 0_2^+ \to 0_2^+)$ [e fm^2]	2.1(7)	3.0(1.4)	3.6(2)

$\Rightarrow$ gives credit to the interpretation of the 2_1^+ as rotational excitation

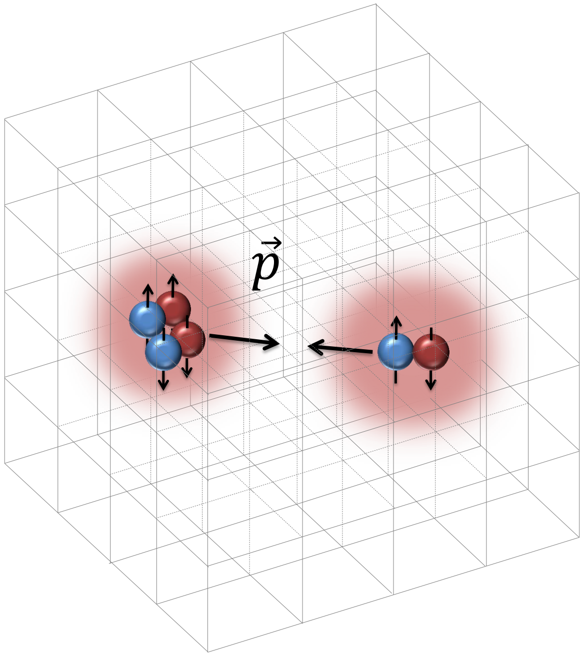
$\Rightarrow$ **results independent of the initial configurations! $\to$clustering**

Ab initio calculation of α-α scattering

Elhatisari, Lee, Rupak, Epelbaum, Krebs, Lähde, Luu, UGM,
Nature **528** (2015) 111 [arXiv:1506.03513]

NUCLEUS–NUCLEUS SCATTERING on the LATTICE

- Processes involving α-particles and α-type nuclei comprise a major part of stellar nucleosynthesis, and control the production of certain elements in stars



- Ab initio calculations of scattering and reactions suffer from computational scaling with the number of nucleons in the clusters

Lattice EFT computational scaling $\Rightarrow (A_1 + A_2)^2$

Rupak, Lee, Phys. Rev. Lett. **111** (2013) 032502
Pine, Lee, Rupak, Eur. Phys. J. A**49** (2013) 151
Elhatisari, Lee, Phys. Rev. C**90** (2014) 064001
Elhatisari et al., Phys.Rev. C**92** (2015) 054612
Elhatisari, Lee, UGM, Rupak, arXiv:1603.02333

ADIABATIC PROJECTION METHOD

- Basic idea to treat scattering and inelastic reactions:
  split the problem into two parts

First part:

use Euclidean time projection to construct an *ab initio* low-energy cluster Hamiltonian, called the **adiabatic Hamiltonian**

Second part:

compute the two-cluster scattering phase shifts or reaction amplitudes using the adiabatic Hamiltonian

ADIABATIC PROJECTION METHOD II

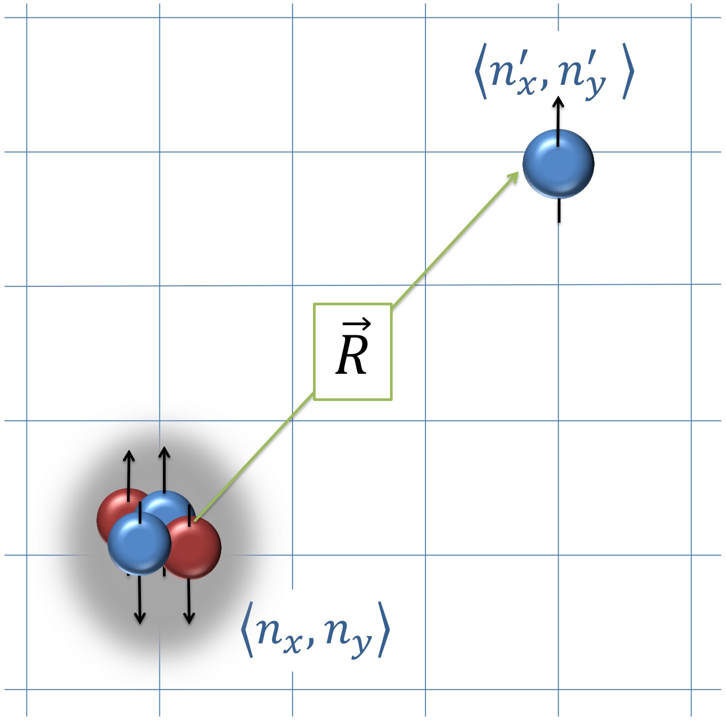
- Construct a low-energy effective theory for clusters

- Use initial states parameterized by the relative separation between clusters

$$|\vec{R}\rangle = \sum_{\vec{r}} |\vec{r} + \vec{R}\rangle \otimes \vec{r}$$

- project them in Euclidean time with the chiral EFT Hamiltonian H

$$|\vec{R}\rangle_\tau = \exp(-H\tau)|\vec{R}\rangle$$



$\rightarrow$ “dressed cluster states” (polarization, deformation, Pauli)

- The adiabatic projection in Euclidean times gives a systematically improvable description of the low-lying scattering states

- In the limit of large Euclidean time, the description becomes exact

ADIABATIC HAMILTONIAN

- Construct the adiabatic Hamiltonian from the dressed cluster states:

$$[H_\tau]_{\vec{R}\vec{R}'} = {}_\tau\langle\vec{R}|H|\vec{R}'\rangle_\tau$$

- States are i.g. not normalized, require *norm matrix*:

$$[N_\tau]_{\vec{R}\vec{R}'} = {}_\tau\langle\vec{R}|\vec{R}'\rangle_\tau$$

- construct the full adiabatic Hamiltonian:

$$\left[H^a_\tau\right]_{\vec{R}\vec{R}'} = \sum_{\vec{R}_n\vec{R}_m} \left[N_\tau^{-1/2}\right]_{\vec{R}\vec{R}_n} \left[H_\tau\right]_{\vec{R}_n\vec{R}_m} \left[N_\tau^{-1/2}\right]_{\vec{R}_m\vec{R}'}$$

→ The structure of the adiabatic Hamiltonian is similar to the Hamiltonian matrix used in recent ab initio NCSM/RGM calculations

Navratil, Quaglioni, Phys. Rev. C **83** (2011) 044609
Navratil, Roth, Quaglioni, Phys. Lett. B **704** (2011) 379
Navratil, Quaglioni, Phys. Rev. Lett. **108** (2012) 042503

SCATTERING CLUSTER WAVE FUNCTIONS

- During Euclidean time interval τ_ε,
  each cluster undergoes spatial diffusion:

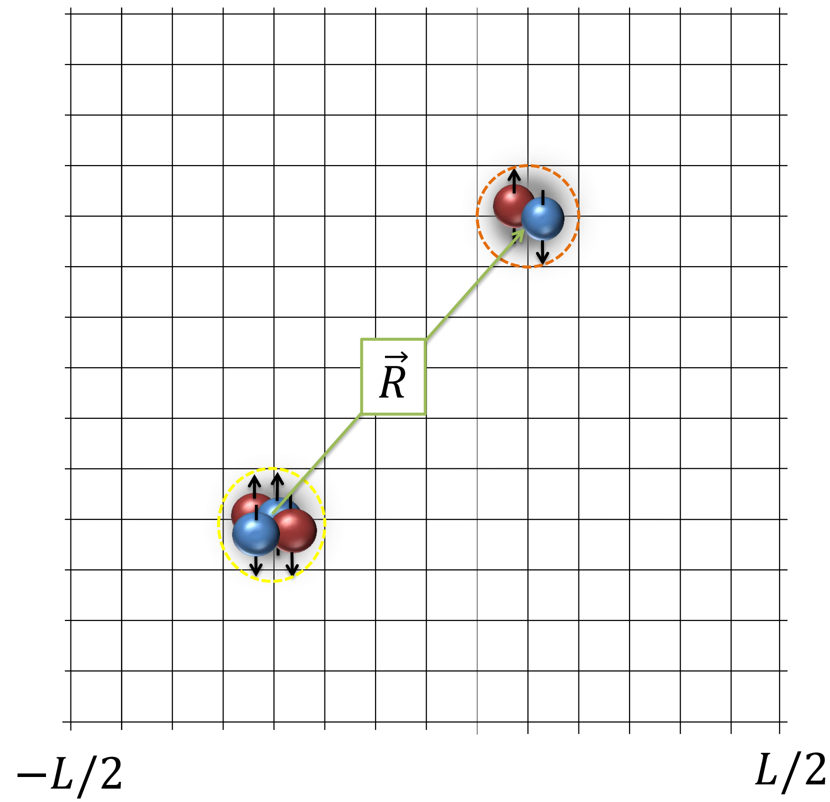
$$d_{\varepsilon,i} = \sqrt{\tau_\varepsilon / M_i}$$

- Only non-overlapping clusters if

$$|\vec{R}| \gg d_{\varepsilon,i} \quad \Rightarrow \quad |\vec{R}\rangle_{\tau_\varepsilon}$$

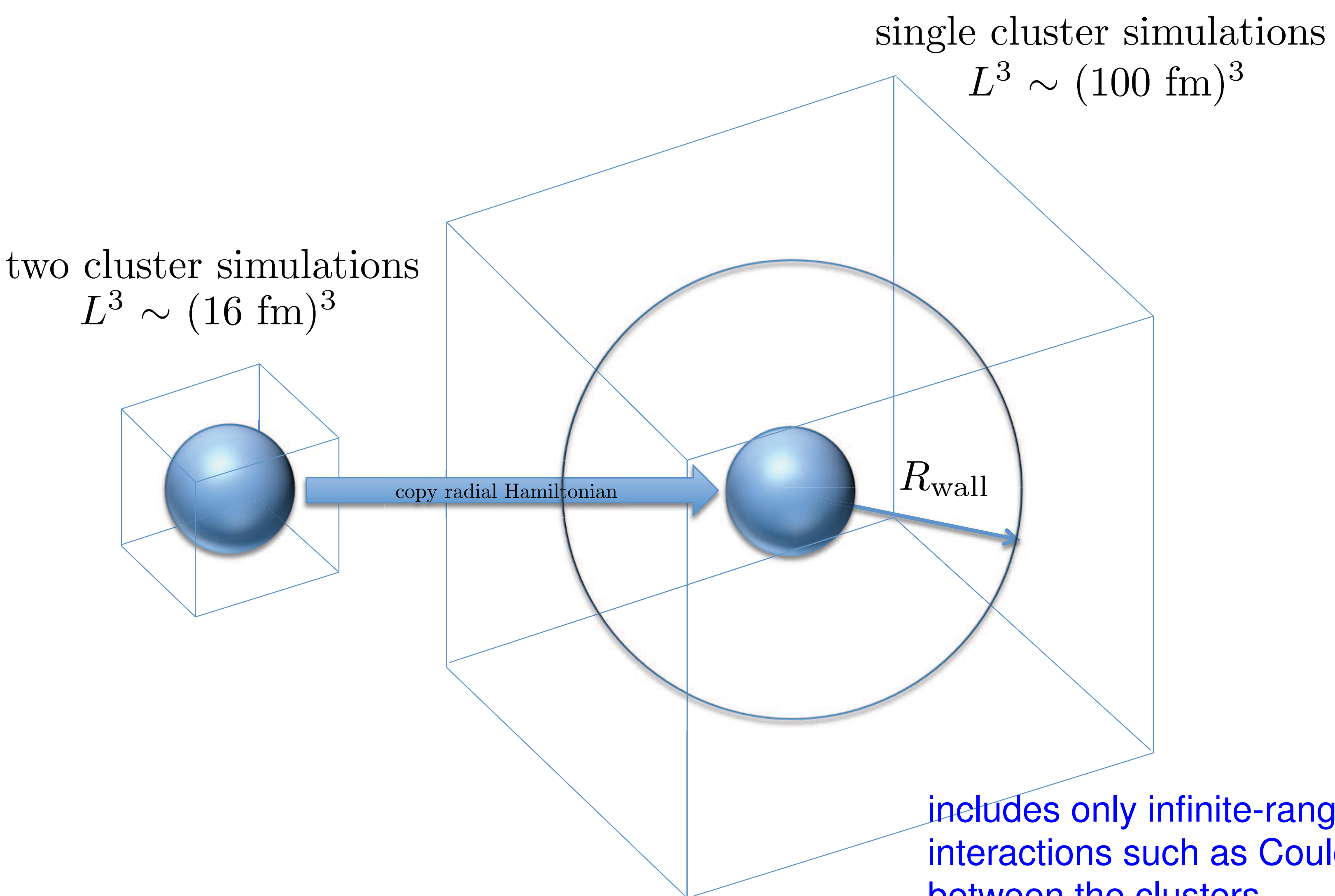
- Defines asymptotic region, where the amount of overlap between clusters is less than ε

$$|\vec{R}| > R_\varepsilon$$



$\Rightarrow$ In the asymptotic region we can describe the system in terms of an effective cluster Hamiltonian (the free lattice Hamiltonian for two clusters) plus infinite-range interactions (like the Coulomb int.)

ADIABATIC HAMILTONIAN plus COULOMB



includes only infinite-range interactions such as Coulomb between the clusters

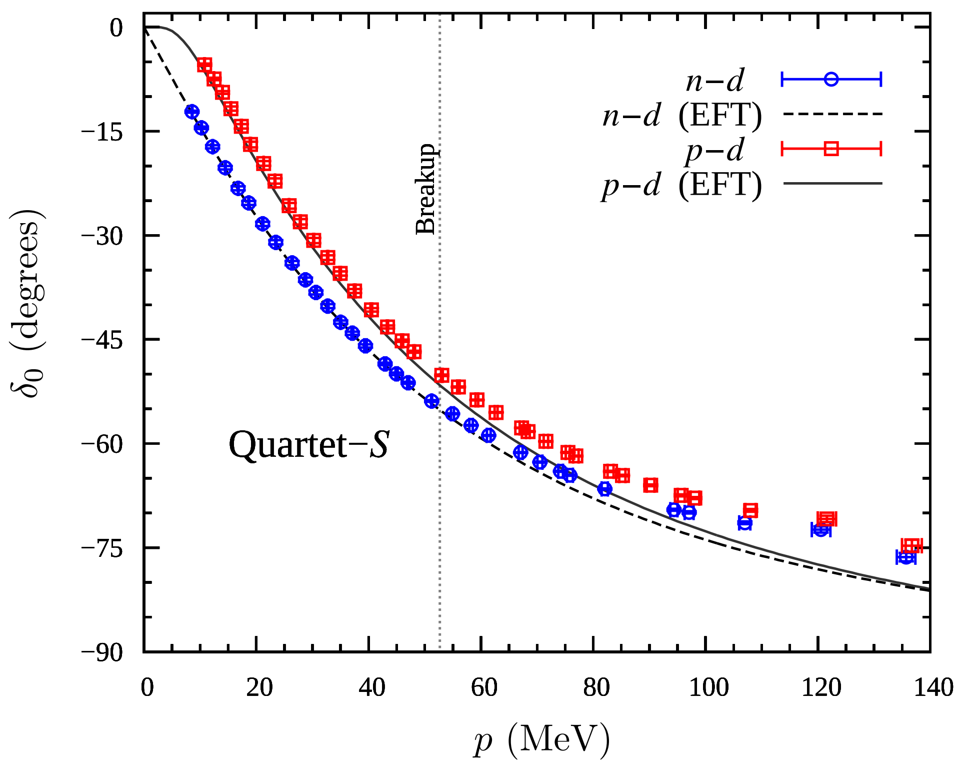
25

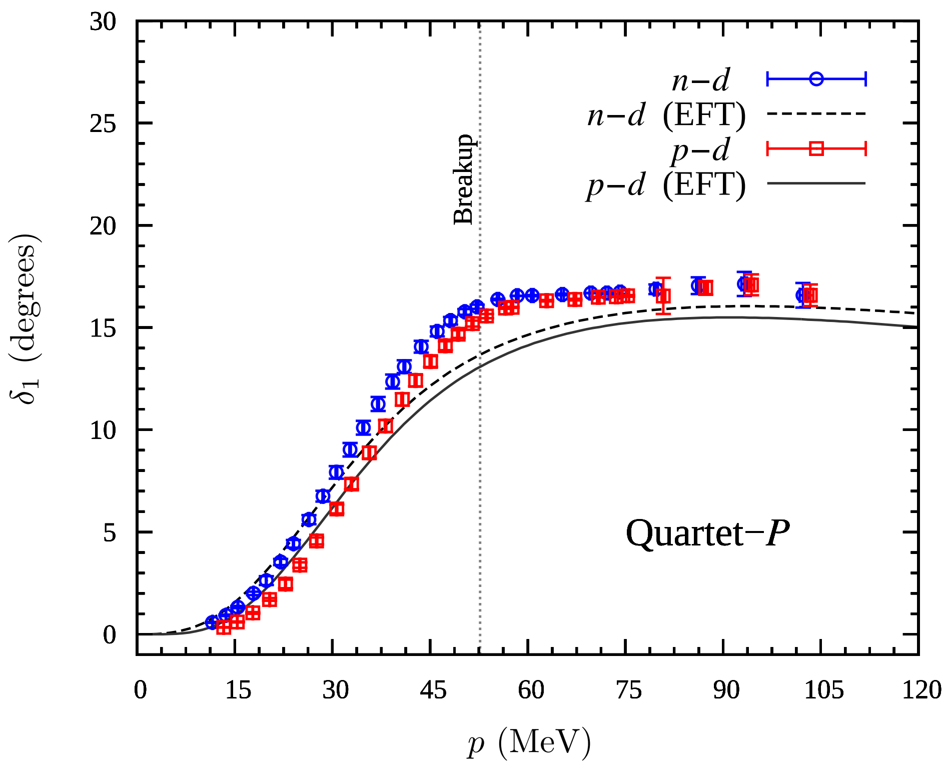
A TEST: NUCLEON–DEUTERON SCATTERING

Elhatisari, Lee, UGM, Rupak, arXiv:1603.02333, EPJA (2016)

- Use improved methods (cluster states projected on sph. harmonics, etc.) & algorithmic improvements
- Precision calculation of proton-deuteron and neutron-deuteron scattering @ LO

Pionless EFT: König, Hammer, Gabbiani, Bedaque, Rupak, Griesshammer, van Kolck, 1998-2011

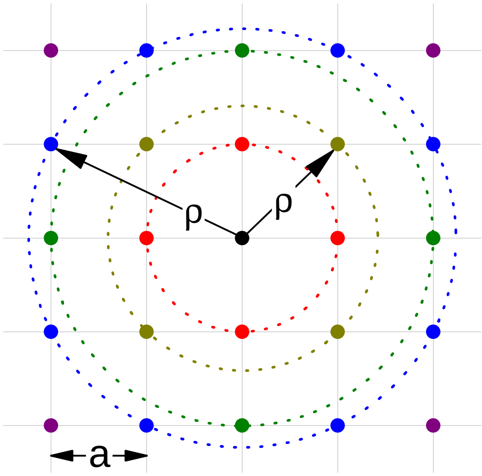




ALPHA–ALPHA SCATTERING

- same lattice action as for the Hoyle state in ^{12}C and the structure of ^{16}O
- (9+2) NN + 2 3N LECs, coarse lattice $a = 1.97$ fm, $N = 8$
- new algorithm for Monte Carlo updates and alpha clusters
- adiabatic projection method to construct a two-alpha Hamiltonian
- spherical wall method to extract the phase shifts using radial Hamiltonian

$$|R\rangle^{(\ell),(\ell_z)} = \sum_{\vec{R}'} Y_{\ell,\ell_z}(\vec{R}')\,\delta_{R,|\vec{R}'|}\,|\vec{R}'\rangle$$

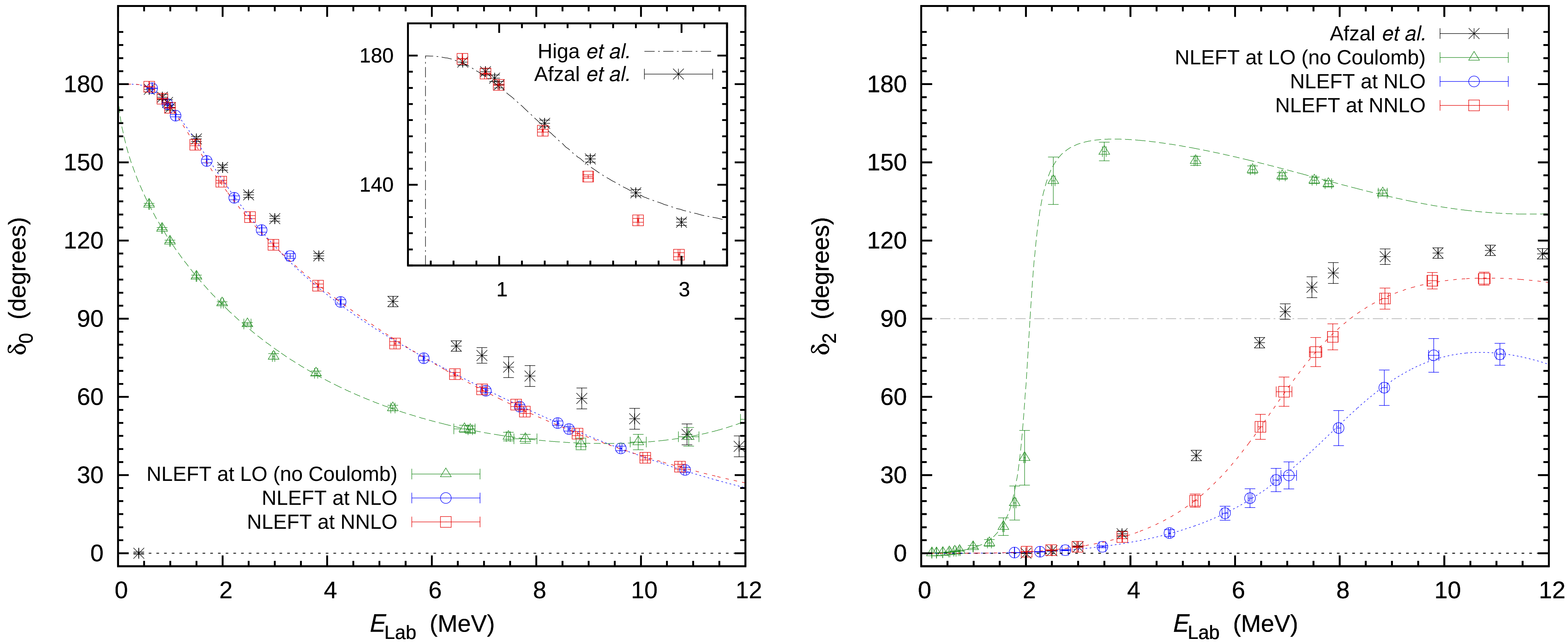


$\rightarrow$ precise extraction of phase shifts & mixing angles

Lu, Lähde, Lee, UGM, arXiv:1506.05652
Moinard et al., work in progress
Elhatisari, Lee, UGM, Rupak, arXiv:1603.02333

PHASE SHIFTS

- S-wave and D-wave phase shifts (LO has no Coulomb)



$E_R^{\rm NNLO} = -0.11(1)$ MeV $[+0.09$ MeV$]$

$E_R^{\rm NNLO} = 3.27(12)$ MeV $[\,2.92(18)$ MeV$]$

$\Gamma_R^{\rm NNLO} = 2.09(16)$ MeV $[1.35(50)$ MeV$]$

Afzal et al., Rev. Mod. Phys. **41** (1969) 247 [data]; Higa et al., Nucl.Phys. A**809** (2008) 171 [halo EFT]

Nuclear binding near a quantum phase transition

Elhatisari, Li, Rokash, Alarcon, Du, Klein, Lu, UGM, Epelbaum, Krebs, Lähde, Lee, Rupak, arXiv:1602.04539

GENERAL CONSIDERATIONS

- *Ab initio* chiral EFT is an excellent theoretical framework

- not guaranteed to work well with increasing A

$\rightarrow$ possible sources of problems:

higher-body forces, higher orders, cutoff dependence, . . .

- very many ways of formulating chiral EFT at any given order (smearing etc.)

$\rightarrow$ use not only NN scattering and light nuclei BEs
but also light nucleus-nucleus scattering data
to pin down the pertinent interactions

$\rightarrow$ troublesome corrections might be small

$\rightarrow$ investigate these issues using two seemingly equivalent interactions

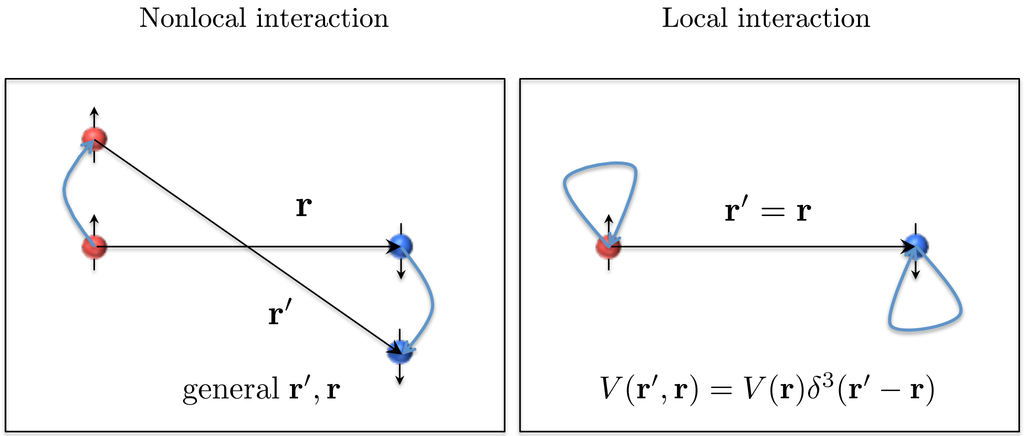
[not a precision study!]

LOCAL and NON–LOCAL INTERACTIONS

- General potential: $V(\vec{r}, \vec{r}\,')$
- Two types of interactions:

  local: $\vec{r} = \vec{r}\,'$

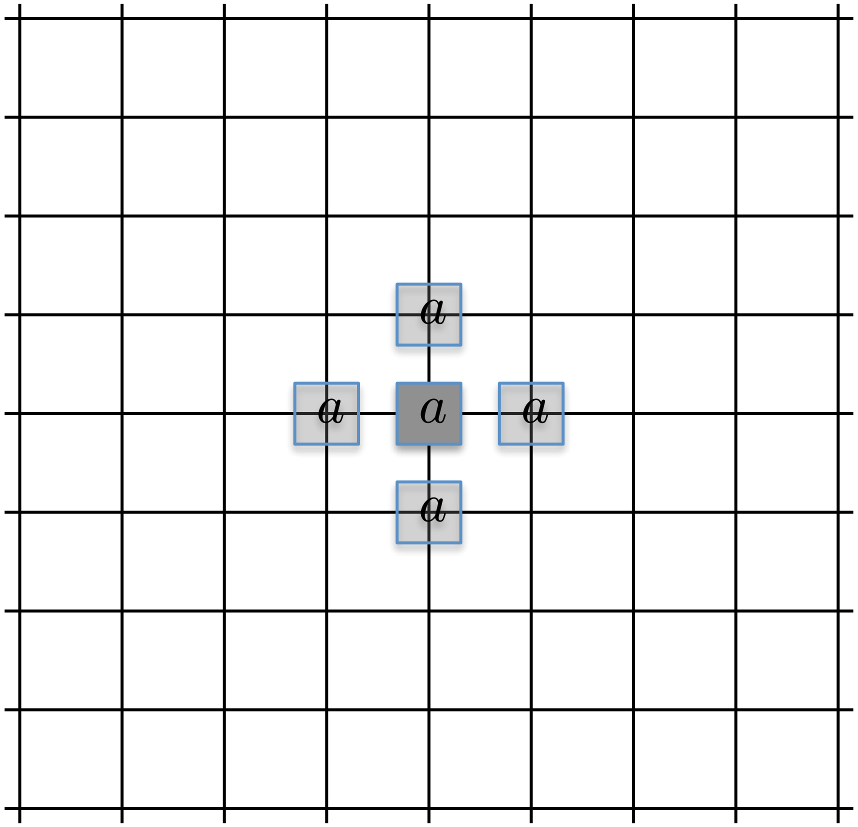
  non-local: $\vec{r} \neq \vec{r}\,'$



- Taylor two very different interactions:

Interaction A at LO (+ Coulomb)	Interaction B at LO (+ Coulomb)
Non-local short-range interactions	Non-local short-range interactions
+ One-pion exchange interaction	+ Local short-range interactions
(+ Coulomb interaction)	+ One-pion exchange interaction
	(+ Coulomb interaction)

LOCAL/NON–LOCAL INTERACTIONS on the LATTICE



- Local operators/densities:

$a(\mathbf{n}), a^\dagger(\mathbf{n})$ [$\mathbf{n}$ denotes a lattice point]

$$\rho_{\rm L}(\mathbf{n}) = a^\dagger(\mathbf{n})a(\mathbf{n})$$

- Non-local operators/densities:

$$a_{\rm NL}(\mathbf{n}) = a(\mathbf{n}) + s_{\rm NL}\sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle} a(\mathbf{n}')$$

$$a^\dagger_{\rm NL}(\mathbf{n}) = a^\dagger(\mathbf{n}) + s_{\rm NL}\sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle} a^\dagger(\mathbf{n}')$$

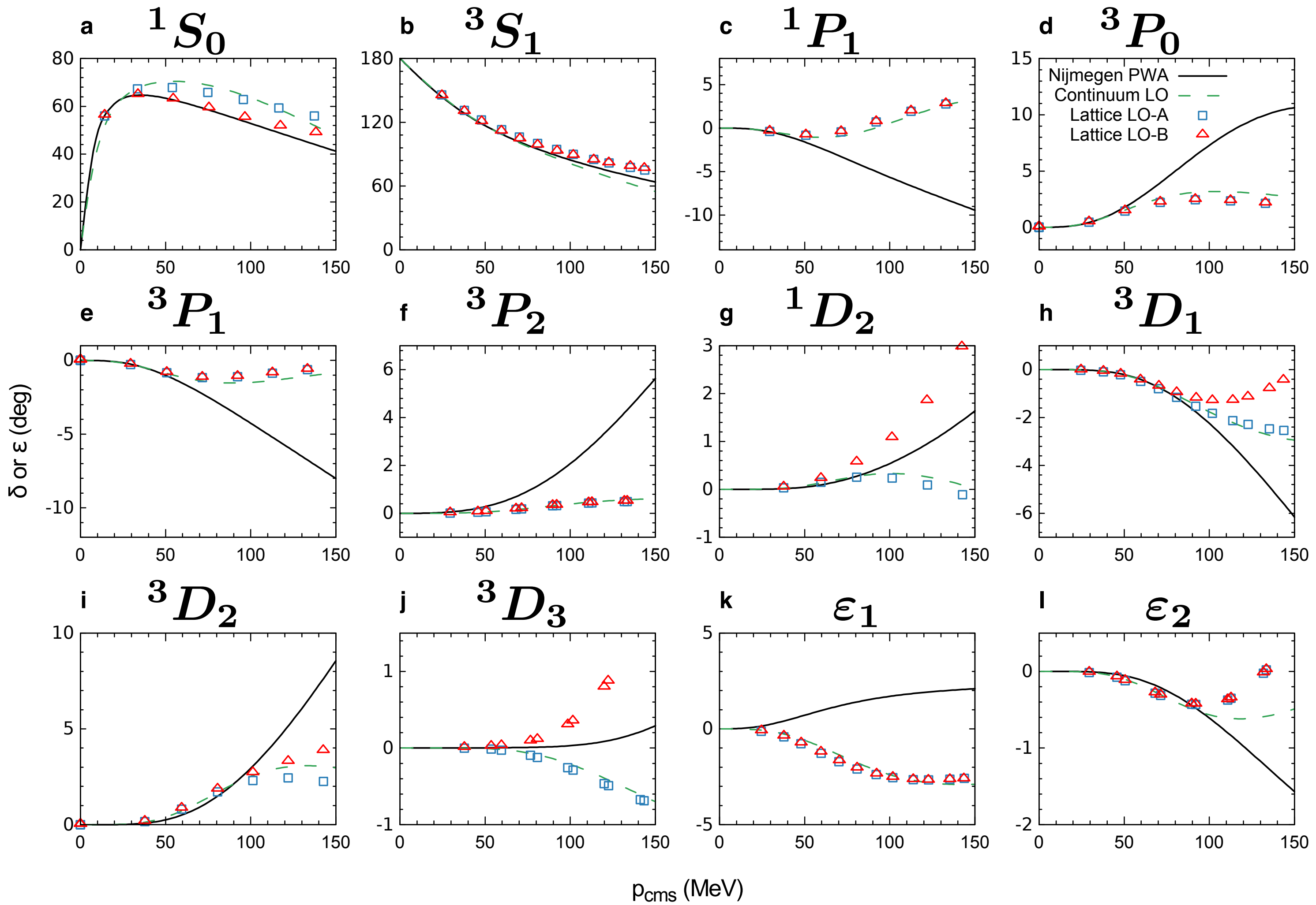
$$\rho_{\rm NL}(\mathbf{n}) = a^\dagger_{\rm NL}(\mathbf{n})a_{\rm NL}(\mathbf{n})$$

$\to$ where $\sum_{\langle \mathbf{n}'\,\mathbf{n}\rangle}$ denotes the sum over nearest-neighbor lattice sites of $\mathbf{n}$

$\to$ the smearing parameter $s_{\rm NL}$ is determined when fitting to the phase shifts

NUCLEON–NUCLEON PHASE SHIFTS

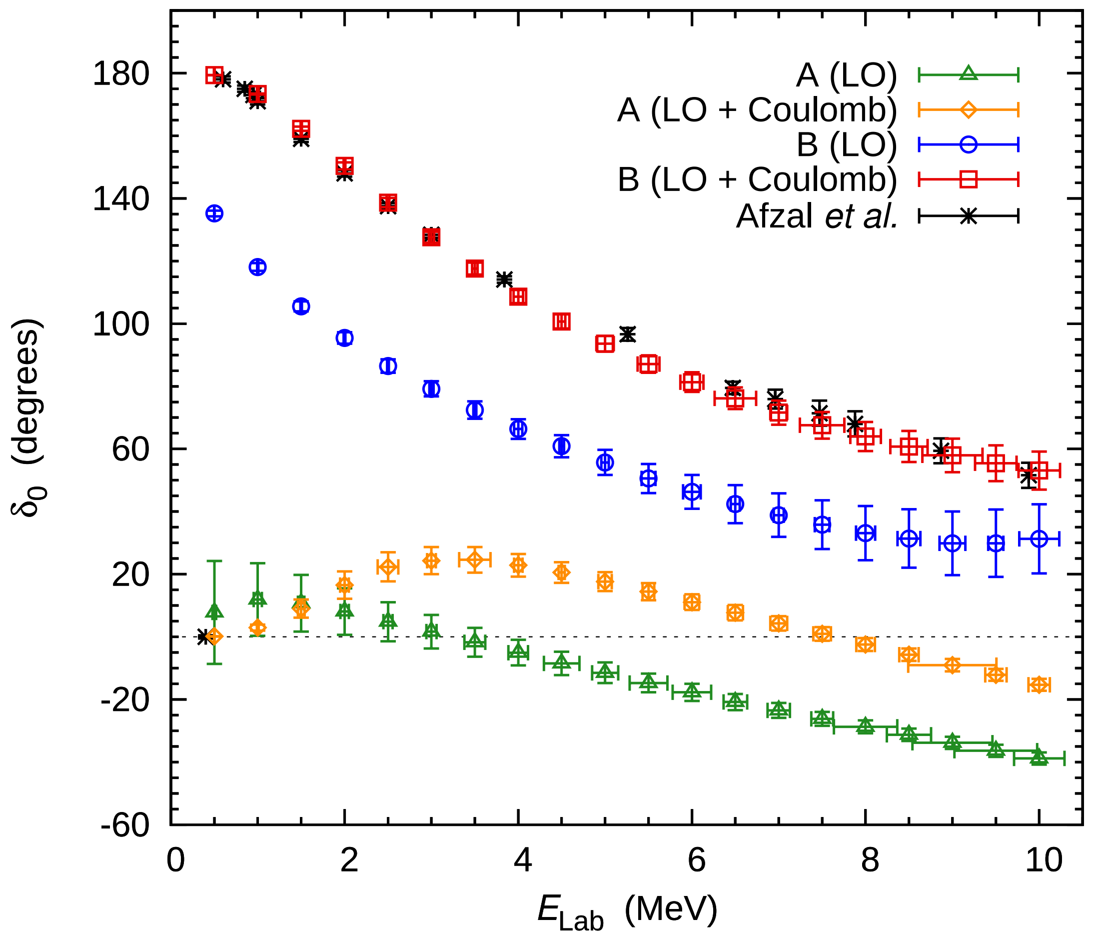
- Show results for NN [and α-α] phase shifts for both interactions:



$\rightarrow$ both interactions very similar

ALPHA–ALPHA PHASE SHIFTS

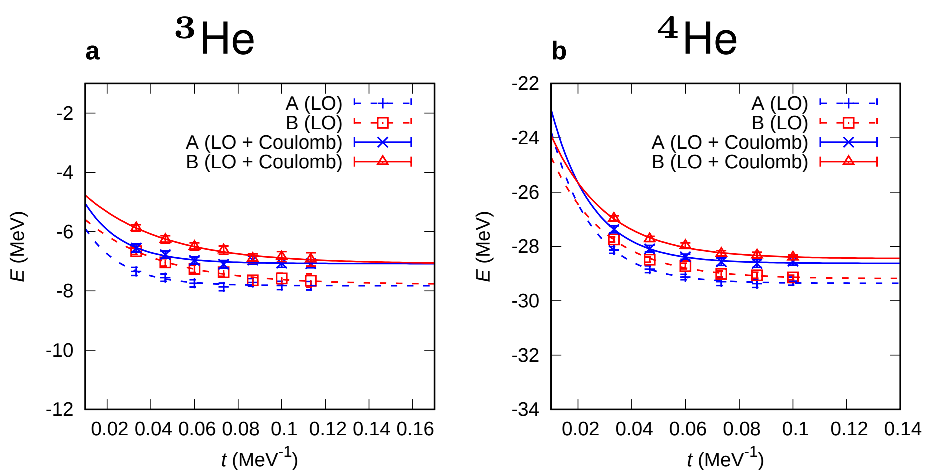
- Show results for [NN and] α-α phase shifts for both interactions:

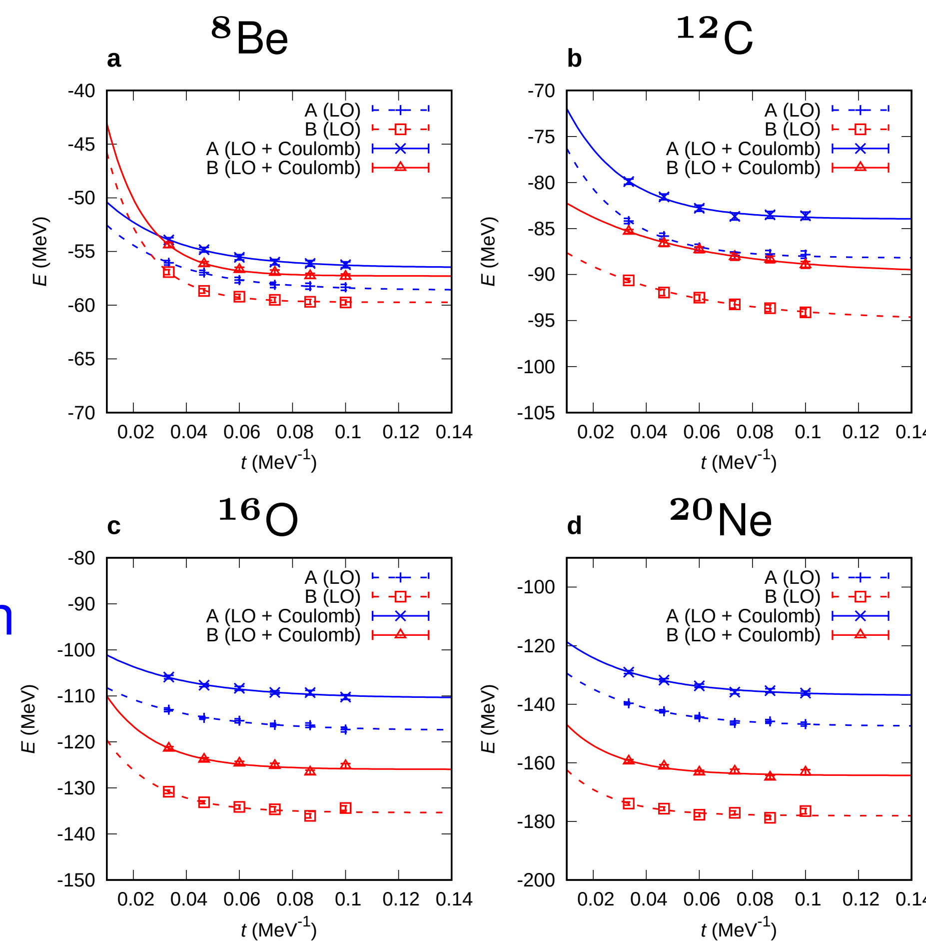


$\rightarrow$ Interaction A fails, interaction B fitted $\hookrightarrow$ consequences for nuclei?

GROUND STATE ENERGIES I

- Ground state energies for alpha-type nuclei plus ^{3}He:





- small errors, excellent thermalization

$\Rightarrow$ Both interactions significantly reduce the Monte Carlo sign oscillations (original motivation)

GROUND STATE ENERGIES I

- Ground state energies for alpha-type nuclei (in MeV):

	A (LO)	A (LO+C.)	B (LO)	B (LO+C.)	Exp.
^{4}He	$-29.4(4)$	$-28.6(4)$	$-29.2(1)$	$-28.5(1)$	-28.3
^{8}Be	$-58.6(1)$	$-56.5(1)$	$-59.7(6)$	$-57.3(7)$	-56.6
^{12}C	$-88.2(3)$	$-84.0(3)$	$-95.0(5)$	$-89.9(5)$	-92.2
^{16}O	$-117.5(6)$	$-110.5(6)$	$-135.4(7)$	$-126.0(7)$	-127.6
^{20}Ne	$-148(1)$	$-137(1)$	$-178(1)$	$-164(1)$	-160.6

- B (LO+Coulomb) quite close to experiment (within 2% or better)

- A (LO) describes a Bose condensate of particles:

$$E(^8\mathrm{Be})/E(^4\mathrm{He}) = 1.997(6) \qquad E(^{12}\mathrm{C})/E(^4\mathrm{He}) = 3.00(1)$$

$$E(^{16}\mathrm{O})/E(^4\mathrm{He}) = 4.00(2) \qquad E(^{20}\mathrm{Ne})/E(^4\mathrm{He}) = 5.03(3)$$

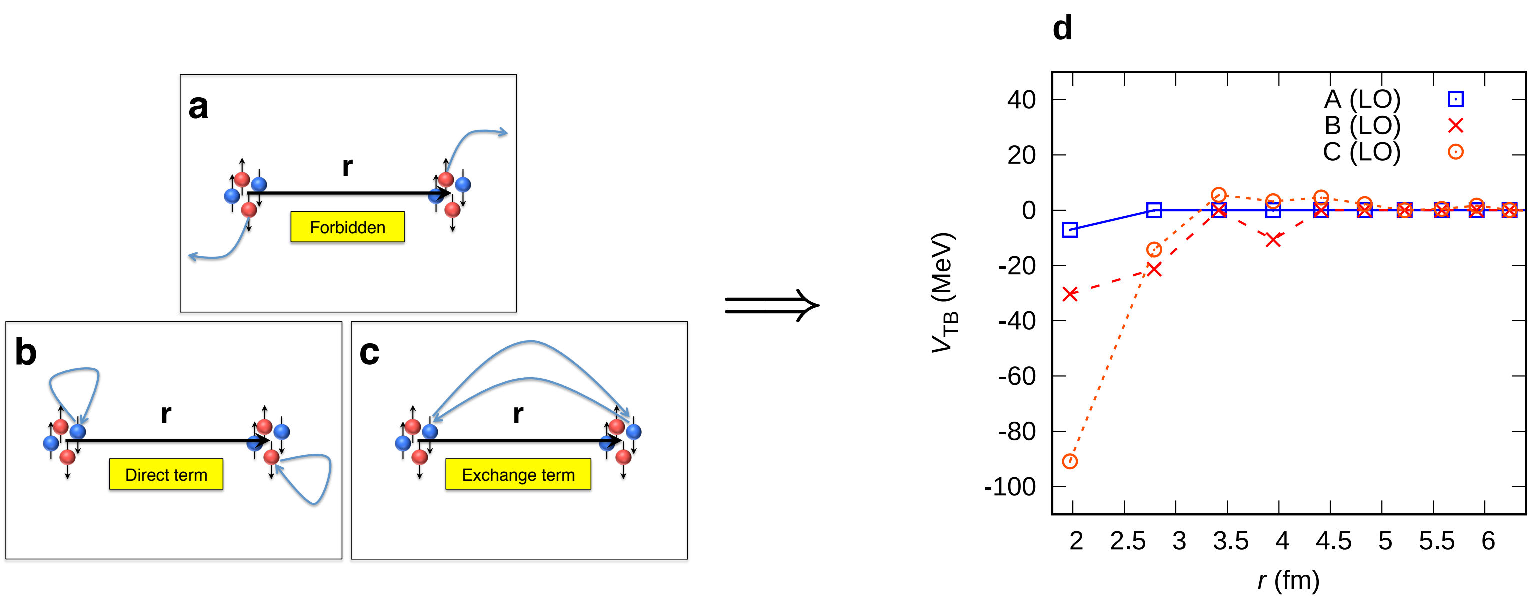
FIRST INSIGHT

- Interaction B was tuned to the nucleon-nucleon phase shifts, the deuteron binding energy, and the S-wave α-α phase shift

- Interaction A starts from interaction B, but *all* local short-distance interactions are switched off, then the LECs of the non-local terms are refitted to describe the nucleon-nucleon phase shifts and the deuteron binding energy

$\rightarrow$ The alpha-alpha interaction is sensitive to the degree of locality of the NN int.

$\rightarrow$ Qualitative understanding: tight-binding approximation (eff. α-α int.)



CONSEQUENCES for NUCLEI and NUCLEAR MATTER

- Define a one-parameter family of interactions that interpolates between the interactions A and B:

$$\boxed{V_\lambda = (1-\lambda)\, V_A + \lambda\, V_B}$$

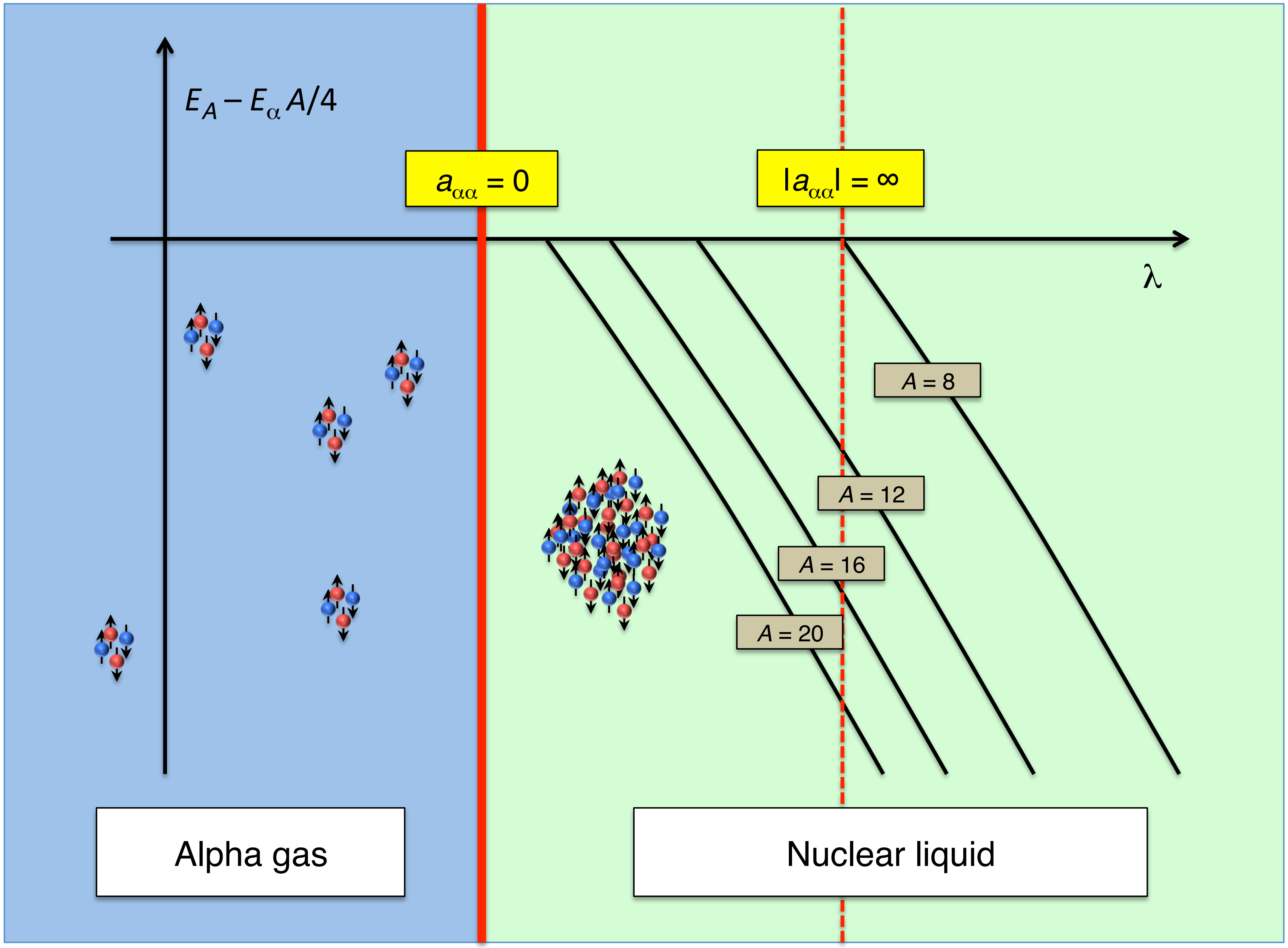
- To discuss the many-body limit, we turn off the Coulomb interaction and explore the zero-temperature phase diagram

- As a function of λ, there is a quantum phase transition at the point where the alpha-alpha scattering length vanishes

Stoff, Phys. Rev. A **49** (1994) 3824

- The transition is a first-order transition from a Bose-condensed gas of alpha particles to a nuclear liquid

ZERO–TEMPERATURE PHASE DIAGRAM



FURTHER CONSEQUENCES

- By adjusting the parameter λ in *ab initio* calculations, one can move the of any α-cluster state up and down to alpha separation thresholds.

$\rightarrow$ This can be used as a new window to view the structure of these exotic nuclear states

- In particular, one can tune the α-α scattering length to infinity!

$\rightarrow$ In the absence of Coulomb interactions, one can thus make contact to **universal Efimov physics**:

for a review, see Braaten, Hammer, Phys. Rept. 428 (2006) 259

Hoyle state of ^{12}C	$\xrightarrow{\lambda}$	Universal Efimov trimer
Second 0^+ of ^{16}O	$\xrightarrow{\lambda}$	Universal Efimov tetramer

Beyond alpha-cluster nuclei

Lähde, Luu, Lee, UGM, Epelbaum, Krebs, Rupak, EPJA **51**: 92 (2015)

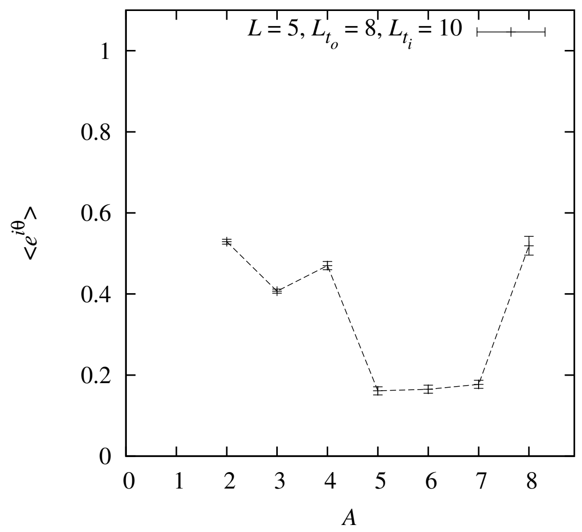
SYMMETRY-SIGN EXTRAPOLATION METHOD

Lähde, Luu, Lee, UGM, Epelbaum, Krebs, Rupak, EPJA **51**: 92 (2015)

- so far: nuclei with $N = Z$, and $A = 4 \times \mathrm{int}$
  as these have the least sign problem
  due to the approximate SU(4) symmetry

$$\langle \mathrm{sign} \rangle = \langle \exp(i\theta) \rangle = \frac{\det M(t_o, t_i, \ldots)}{|\det M(t_o, t_i, \ldots)|}$$

$M(t_o, t_i, \ldots)$ is the transition matrix



Borasoy et al. (2007)

- Symmetry-sign extrapolation (SSE) method: control the sign oscillations

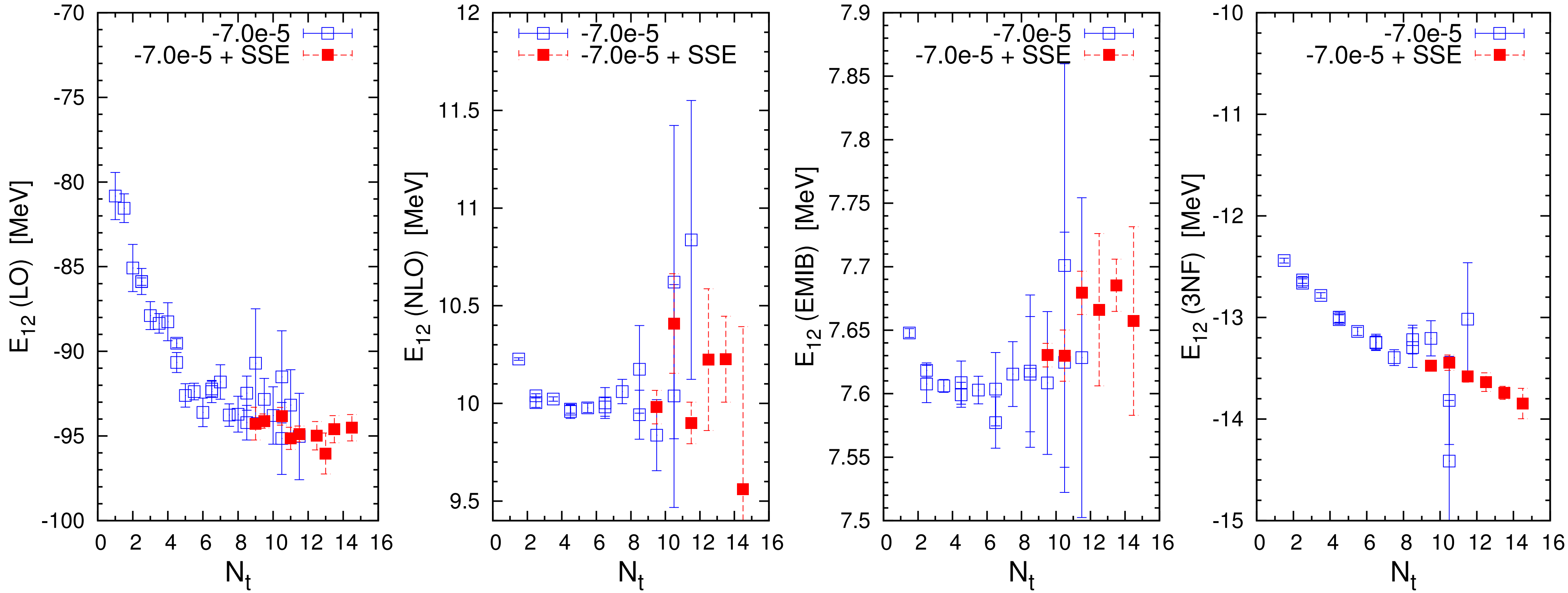
$$H_{d_h} = d_h \cdot H_{\mathrm{phys}} + (1 - d_h) \cdot H_{\mathrm{SU(4)}} \qquad H_{\mathrm{SU(4)}} = \frac{1}{2} C_{\mathrm{SU(4)}} \, (N^\dagger N)^2$$

$\hookrightarrow$ family of solutions for different SU(4) couplings $C_{\mathrm{SU(4)}}$
that converge on the physical value for $d_h = 1$

- cf. Shell Model MC in the mid 1990ties — Alhassid et al. (1994), Coonin et al. (1997)

RESULTS for ^{12}C

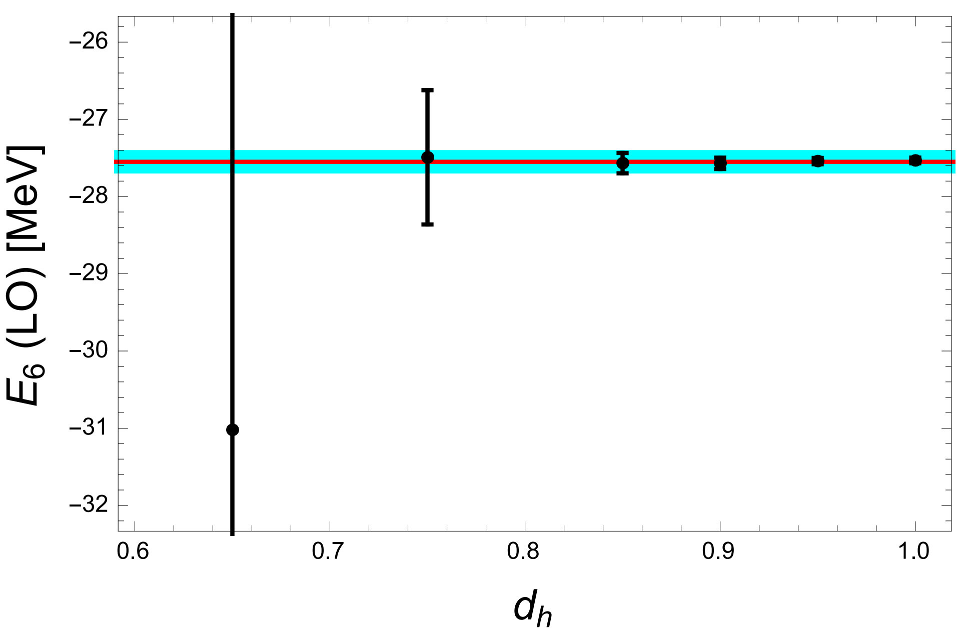
- generate a few more MC data at large N_t using SSE

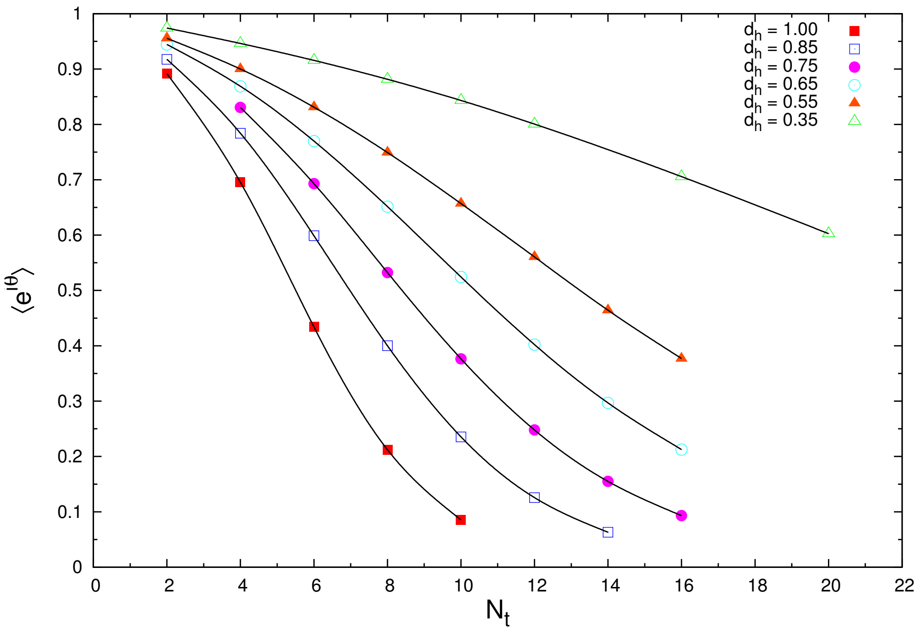


- promising results → no more exponential deterioration of the MC data
- results w/ small uncertainties for $d_h \geq 0.8$

RESULTS for $A = 6$

- Simulations for ^{6}He and ^{6}Be





$\Rightarrow$ methods works for nuclei with $A \neq Z$

$\Rightarrow$ neutron/proton-rich nuclei can now be systematically explored (larger volumes)

SUMMARY & OUTLOOK

- Nuclear lattice simulations as a novel quantum many-body approach
  - $\rightarrow$ based on the successful continuum nuclear chiral EFT
  - $\rightarrow$ clustering emerges naturally, α-cluster nuclei
  - $\rightarrow$ ab initio study of α-α scattering: promising results
  - $\rightarrow$ holy grail of nuclear astrophysics (α+^{12}C $\rightarrow$ ^{16}O+γ) in reach
  - $\rightarrow$ new window to nuclear structure:
    non-local interactions and fitting to nucleus-nucleus data
  - $\rightarrow$ sign-symmetry extrapolation to explore neutron/proton-rich nuclei

- Some on-going activities:
  - $\rightarrow$ improving the forces (N3LO, sph. harmonics)
  - $\rightarrow$ systematic studies of a-independence $\hookrightarrow$ Klein, Lee, Liu, UGM, PLB**747** (2015) 511
  - $\rightarrow$ finite size effects/averaging procedures
    $\hookrightarrow$Lu, Lähde, Lee, UGM, Phys. Rev. D**90** (2014) 034507 & D**92** (2015) 014506
  - $\rightarrow$ and much more . . .

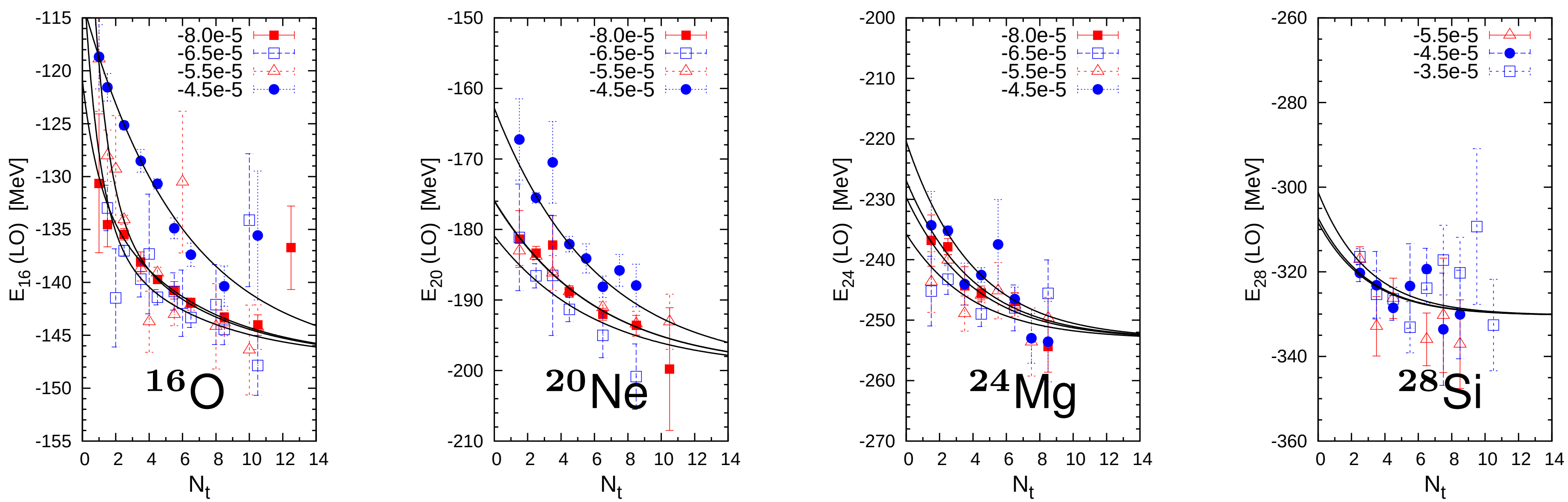
SPARES

GOING up the ALPHA CHAIN

- Consider the α ladder ^{12}C, ^{16}O, ^{20}Ne, ^{24}Mg, ^{28}Si as $t_{\rm CPU} \sim A^2$
- Improved "multi-state" technique to extract ground state energies

  $\Rightarrow$ higher A, better accuracy

  $\Rightarrow$ overbinding at LO beyond $A = 12$ persists up to NNLO



$$E = -131.3(5) \quad [-127.62]$$

$$E = -165.9(9) \quad [-160.64]$$

$$E = -232(2) \quad [-198.26]$$

$$E = -308(3) \quad [-236.54]$$

REMOVING the OVERBINDING

Lähde, Epelbaum, Krebs, Lee, UGM, Rupak, Phys. Lett. B 732 (2014) 110

- Overbinding is due to four α clusters in close proximity

  $\Rightarrow$ remove this by an effective 4N operator [long term: N3LO]

$$V^{(4\mathrm{N}_{\mathrm{eff}})} = D^{(4\mathrm{N}_{\mathrm{eff}})} \sum_{1 \le (\vec{n}_i - \vec{n}_j)^2 \le 2} \rho(\vec{n}_1)\rho(\vec{n}_2)\rho(\vec{n}_3)\rho(\vec{n}_4)$$

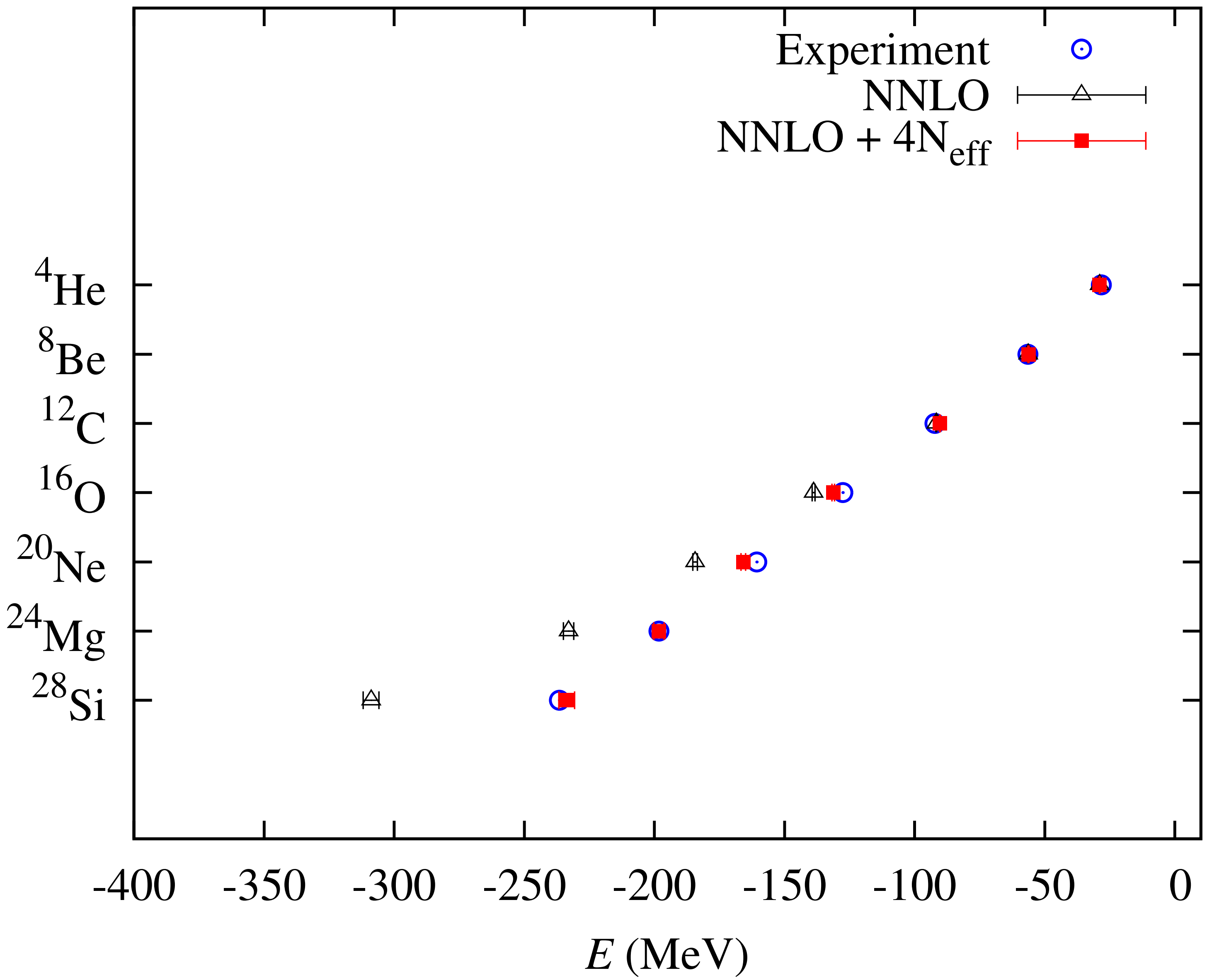
- fix the coefficient $D^{(4\mathrm{N}_{\mathrm{eff}})}$ from the BE of ^{24}Mg

  $\Rightarrow$ excellent description of the ground state energies

A	12	16	20	24	28
Th	$-90.3(2)$	$-131.3(5)$	$-165.9(9)$	$-198(2)$	$-233(3)$
Exp	-92.16	-127.62	-160.64	-198.26	-236.54

$\rightarrow$ ultimately, reduce lattice spacing [interaction more repulsive] & N^{3}LO

GROUND STATE ENERGIES



TESTING the ADIABATIC HAMILTONIAN

- Consider fermion-dimer scattering:

Microscopic Hamiltonian $\Rightarrow$ Two-cluster adiabatic Hamitltonian

$$L^{3(A-1)} \times L^{3(A-1)} \qquad\qquad L^3 \times L^3$$

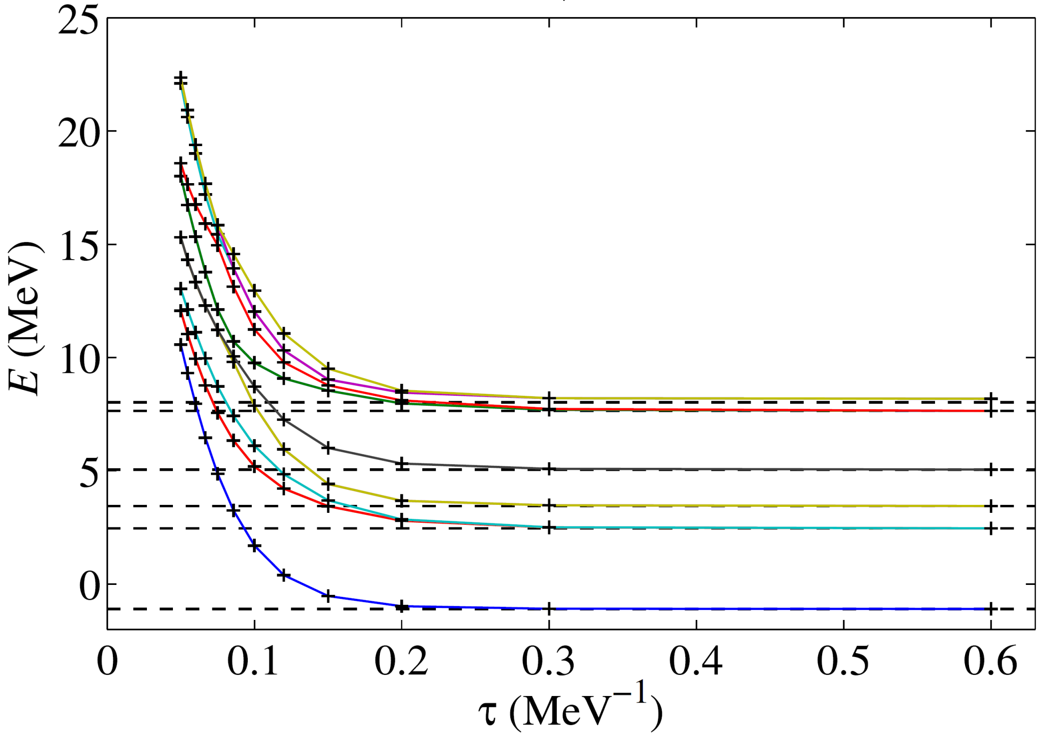
- calculation of a 7^3 lattice, lattice spacing $a = 1.97$ fm

Pine, Lee, Rupak, EPJA **49** (2013) 151

exact Lanzcos: black dashed lines

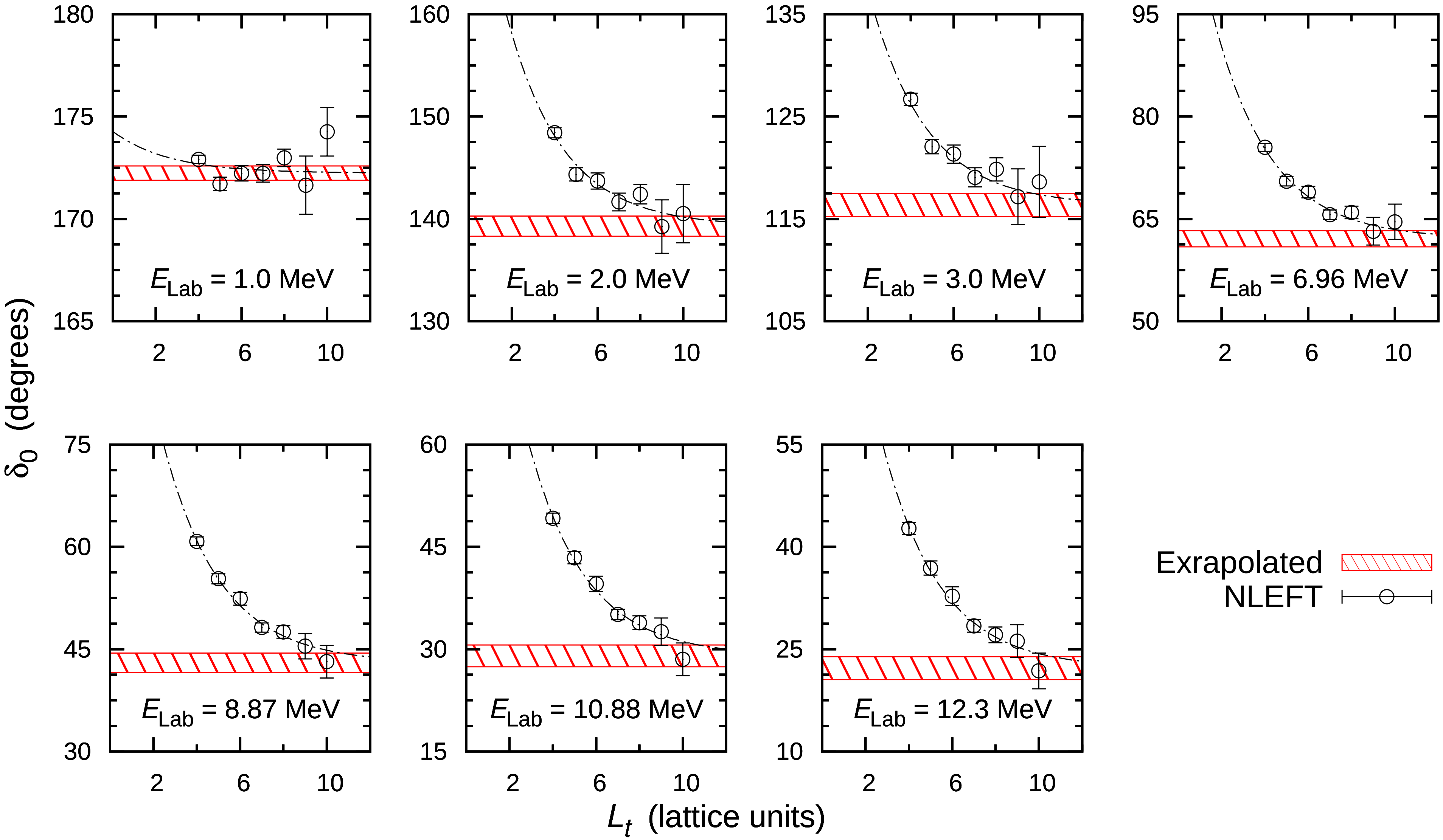
adiab. Ham.: solid colored lines

lattice size is 7^3 , $a = 1.93$ fm



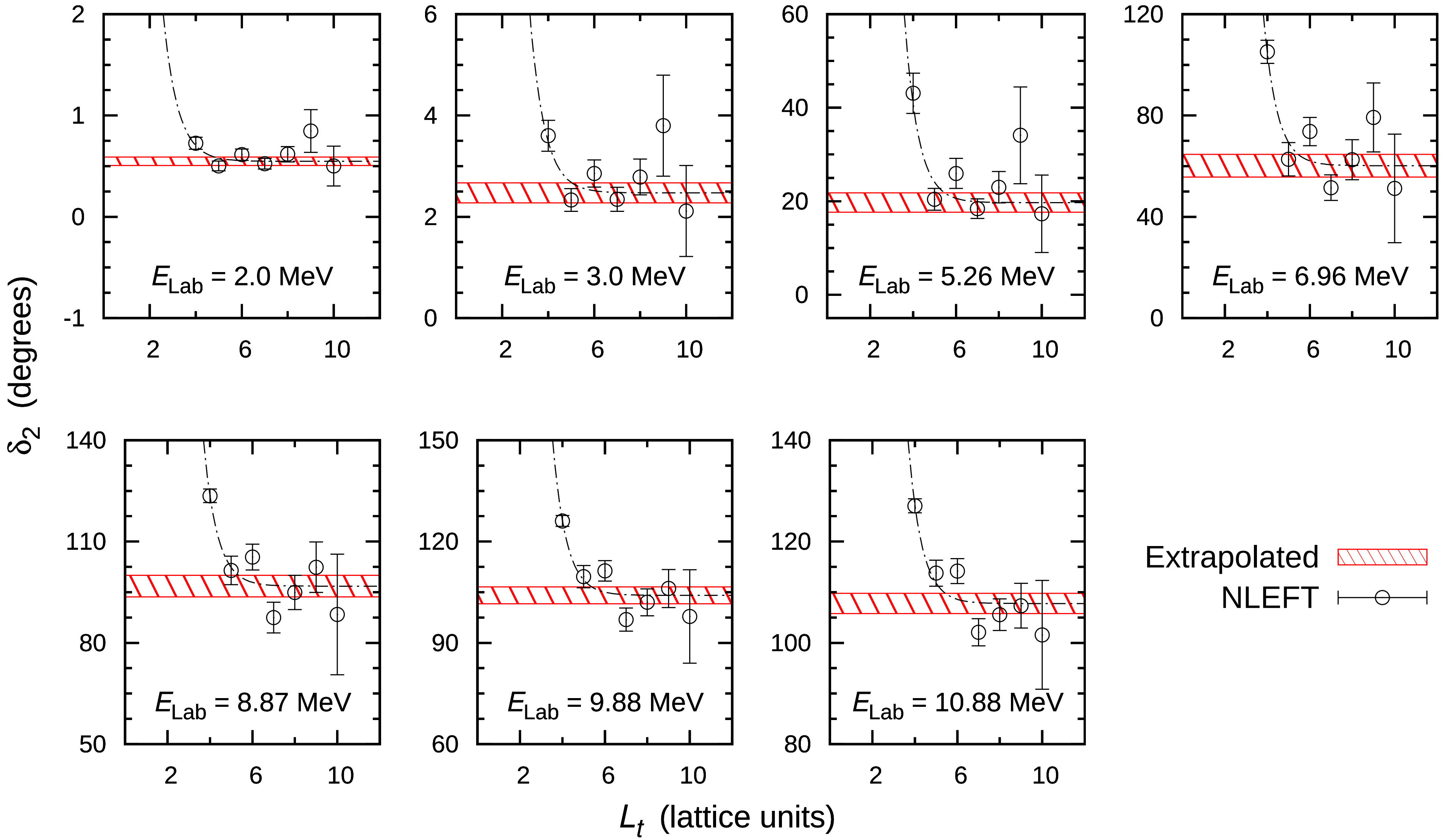
LATTICE DATA I

- Show data for the S-wave:



LATTICE DATA II

- Show data for the D-wave:



PHASE SHIFTS: ERROR BANDS

- S-wave and D-wave phase shifts: Error estimate (not the whole story)