Hadronic Light by Light Contributions to the Muon Anomalous Magnetic Moment With Physical Pions

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23 Mar 2016

INT

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- BNL E821 (0.54 ppm) and Standard Model Prediction
- Magnetic Moment in QFT
- Point Source Photon Method
  - → Summing Over x_{op}
  - → Zero Momentum Transfer Limit
  - → Difference with the General Magnetic Moment Formula
  - → Reorder the Summation
- Simulations
  - → Finite Volume Effects in Muon Leptonic Light by Light
  - → $333\mathrm{MeV}$ Pion $24^3 \times 64$ Lattice
  - → $171\mathrm{MeV}$ Pion $32^3 \times 64$ Lattice
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- Finite Volume Effects - QCD box inside QED box
- Conclusions and Future Plans

$$\boldsymbol{\mu} = -g\frac{e}{2m}\mathbf{s} \tag{1}$$

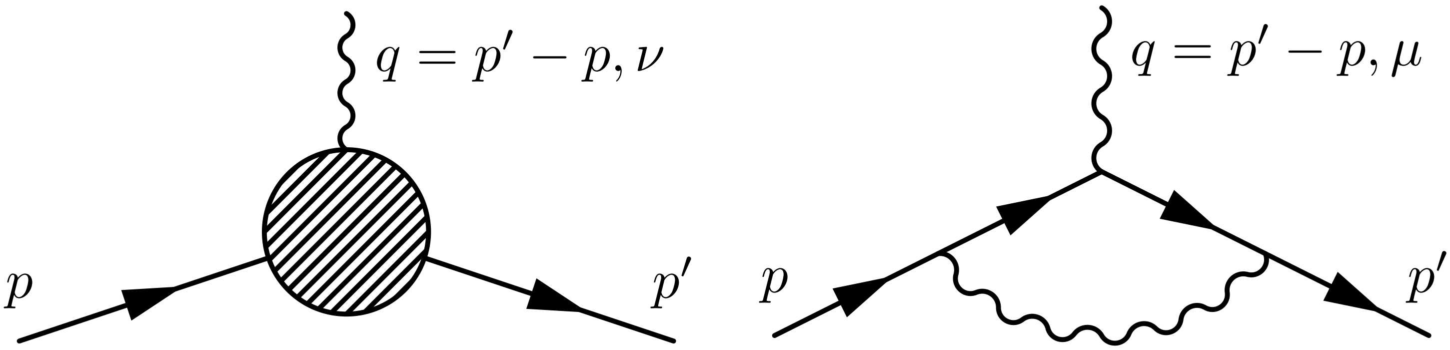

Figure 1. (L) Muon Vertex Function Diagram (R) Schwinger Term Diagram.

$$\bar{u}(\mathbf{p}')\Gamma_\nu(p',p)u(\mathbf{p}) = \bar{u}(\mathbf{p}')\Big[F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu,\gamma_\rho]q_\rho\Big]u(\mathbf{p}) \tag{2}$$

$$F_2(0) = \frac{g-2}{2} \equiv a \tag{3}$$

Use Euclidean convention by default, the relation is

$$\gamma_4 = (\gamma^0)^M \quad \boldsymbol{\gamma} = -i\boldsymbol{\gamma}^M \tag{4}$$

Figure 2. The headstone of Julian Schwinger at Mt Auburn Cemetery in Cambridge, MA.

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BNL E821 (0.54 ppm) and Standard Model Prediction

	Value $\pm$ Error	Reference
Experiment (0.54 ppm)	116592089 ± 63	E821, The $g-2$ Collab. 2006
Standard Model	116591815 ± 50	Particle Data Group, 2014
Difference ($\mathrm{Exp} - \mathrm{SM}$)	274 ± 80	
HVP LO ($e^+e^- \to \mathrm{hadrons}$)	6923 ± 42	Davier et al. 2011
	6949 ± 43	Hagiwara et al. 2011
Hadronic Light by Light	105 ± 26	Glasgow Consensus, 2007

Table 1. Standard model theory and experiment comparison [in units 10^{-11}]

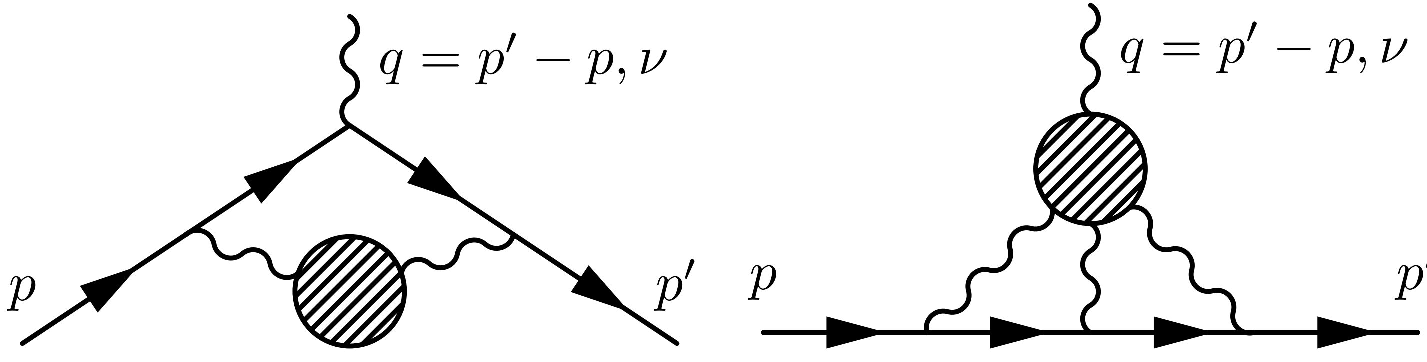

Figure 3. (L) Vaccum polarization diagram. (R) Light by light diagram.

There is 3.5 standard deviations!

Future Fermilab E989 (0.14 ppm)

Figure 4. The 50-foot-wide Muon g-2 electromagnet being driven north on I-355 between Lemont and Downers Grove, Illinois, shortly after midnight on Thursday, July 25, 2013. *Credit: Fermilab.*

Almost 4 times more accurate then the previous experiment.

J-PARC E34 also plans to measure muon $g-2$ with similar precision.

- In this talk, we focus on the calculation of connected light by light amplitude on lattice.
- This subject is started by T. Blum, S. Chowdhury, M. Hayakawa, T. Izubuchi more than 5 years ago. Phys. Rev. Lett. 114, 012001 (2015).

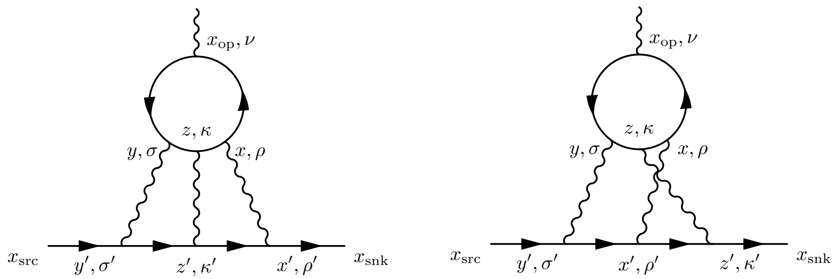

Figure 5. Light by Light diagrams. There are 4 other possible permutations.

$$
\begin{aligned}
\mathcal{M}_\nu^{\rm LbL} \;=\; & -(-ie)^6 \sum_{x,y,z} \left\langle \sum_q e_q^4 \,{\rm tr}[\gamma_\nu S_q(x_{\rm op}, x)\gamma_\rho S_q(x,z)\gamma_\kappa S_q(z,y)\gamma_\sigma S_q(y, x_{\rm op})] \right\rangle_{\rm QCD} \\
\cdot & \sum_{x',y',z'} G_{\rho\rho'}(x,x')G_{\sigma\sigma'}(y,y')G_{\kappa\kappa'}(z,z') \\
\cdot & \Big[S(x_{\rm snk}, x')\gamma_{\rho'} S(x',z')\gamma_{\kappa'} S(z',y')\gamma_{\sigma'} S(y', x_{\rm src}) \\
& + S(x_{\rm snk}, z')\gamma_{\kappa'} S(z',x')\gamma_{\rho'} S(x',y')\gamma_{\sigma'} S(y', x_{\rm src}) \\
& + \text{other 4 permutations} \Big] \qquad (5)
\end{aligned}
$$

NOT Yet Disconnected Diagrams

- One diagram (the biggest diagram below) do not vanish even in the SU(3) limit.

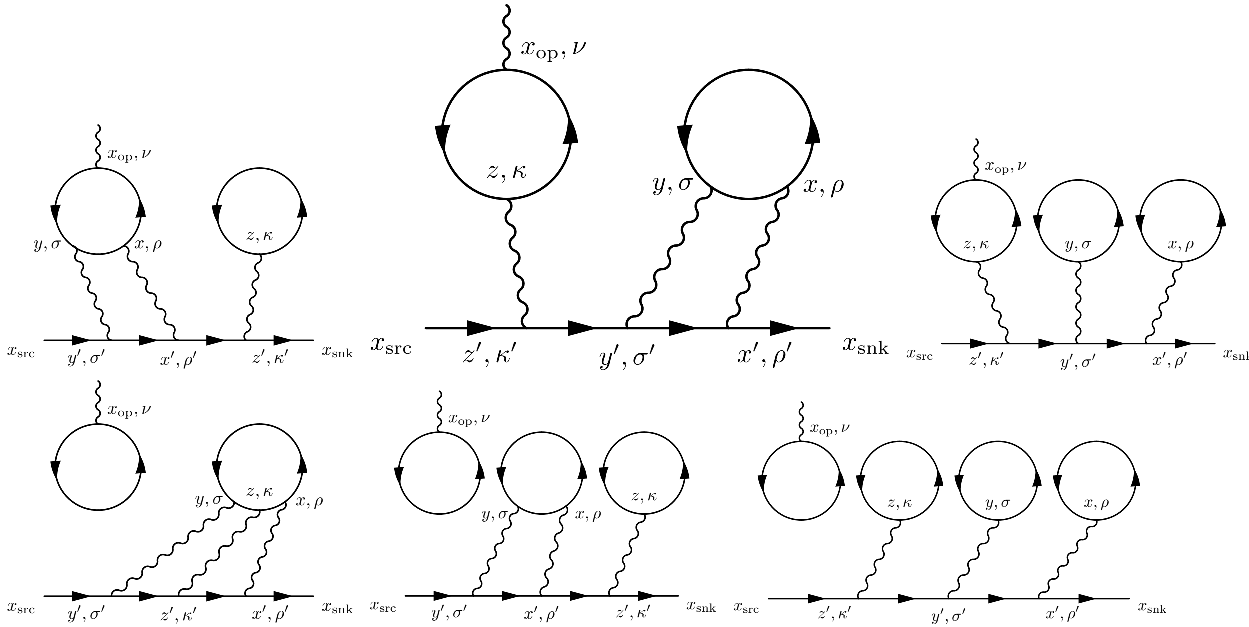

Figure 6. All possible disconnected diagrams. Permutations of the three internal photons are not shown.

- There should be gluons exchange between and within the quark loops, but are not drawn.
- We need to make sure that the loops are connected by gluons by “vacuum” subtraction. So the diagrams are 1-particle irreducible.

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Classicaly, magnetic moment is simply

$$\boldsymbol{\mu} \;=\; \int \frac{1}{2}\mathbf{x}\times\mathbf{j}\, d^3x \tag{6}$$

However, this formula is not correct in Quantum Mechanics, because the magnetic moment result from the spin are also needed to be included.

In QFT, magnetic moment of a fermion is often defined via vertex functions and form factors.

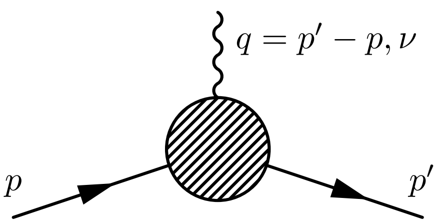

Figure 7. Muon Vertex Function Diagram

$$\begin{aligned}\langle \mathbf{p}', s'| j_\nu(\mathbf{x}_{\rm op}=0)|\mathbf{p}, s\rangle \;&=\; \langle \mathbf{p}', s'|\bar\psi(\mathbf{x}_{\rm op}=0)\,\gamma_\nu\,\psi(\mathbf{x}_{\rm op}=0)|\mathbf{p}, s\rangle \\ &=\; \bar u_{s'}(\mathbf{p}')\Big[F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)}{4m}[\gamma_\nu,\gamma_\rho]q_\rho\Big]u_s(\mathbf{p})\end{aligned} \tag{7}$$

$$\boldsymbol{\mu} \;=\; -g\frac{e}{2m}\mathbf{s} = -e\,\frac{F_1(q^2=0)+F_2(q^2=0)}{m}\,\mathbf{s} \tag{8}$$

Question: Is there a similar formula describe the magnetic moment in terms of current?

Study the spatial component,

$$\langle \mathbf{p}', s' | i\,\mathbf{j}(\mathbf{x}_{\rm op}=0) | \mathbf{p}, s\rangle \;=\; \bar{u}_{s'}(\mathbf{p}')\Big[F_1(q^2)\, i\,\boldsymbol{\gamma} - \frac{F_2(q^2)}{m}\, i\,\mathbf{q}\times\frac{\boldsymbol{\Sigma}}{2}\Big]u_s(\mathbf{p}) \tag{9}$$

$$[\gamma_i, \gamma_j] \;=\; 2\,i\,\epsilon_{ijk}\Sigma_k \tag{10}$$

With Gordon identity

$$\bar{u}_{s'}(\mathbf{p}')\, i\boldsymbol{\gamma}\, u_s(\mathbf{p}) \;=\; \bar{u}_{s'}(\mathbf{p}')\Big(\frac{\mathbf{p}'+\mathbf{p}}{2m} - \frac{1}{m}\, i\,\mathbf{q}\times\frac{\boldsymbol{\Sigma}}{2}\Big)u_s(\mathbf{p}) \tag{11}$$

$$\langle \mathbf{p}', s' | i\,\mathbf{j}(\mathbf{x}_{\rm op}=0) | \mathbf{p}, s\rangle \;=\; \bar{u}_{s'}(\mathbf{p}')\Big[F_1(q^2)\,\frac{\mathbf{p}'+\mathbf{p}}{2m} - \frac{F_1(q^2)+F_2(q^2)}{m}\, i\,\mathbf{q}\times\frac{\boldsymbol{\Sigma}}{2}\Big]u_s(\mathbf{p}) \tag{12}$$

Consider a normalized state $|\psi\rangle = |\mathbf{p},\, s\rangle\psi_s(\mathbf{p})$, with the momentum almost zero and the expectation value of the current exactly zero,

$$\int d^3x_{\rm op} \langle\psi| i\,\mathbf{j}(\mathbf{x}_{\rm op})|\psi\rangle \;=\; 0 \tag{13}$$

We will then consider the following amplitude with extremely small $\mathbf{q} \ll \Delta\mathbf{p} \sim 1/\Delta\mathbf{x}$.

$$\mathcal{M} \;=\; \int d^3x_{\rm op}\, \exp(i\,\mathbf{q}\cdot\mathbf{x}_{\rm op})\langle\psi| i\,\mathbf{j}(\mathbf{x}_{\rm op})|\psi\rangle \tag{14}$$

On one hand, we can safely subtract zero

$$\begin{aligned} \mathcal{M} &= \int d^3 x_{\rm op}\, [\exp(i\,\mathbf{q}\cdot\mathbf{x}_{\rm op}) - 1]\langle\psi|i\,\mathbf{j}(\mathbf{x}_{\rm op})|\psi\rangle \\ &\approx \int d^3 x_{\rm op}\, i\,\mathbf{q}\cdot\mathbf{x}_{\rm op}\,\langle\psi|i\,\mathbf{j}(\mathbf{x}_{\rm op})|\psi\rangle \end{aligned} \tag{15}$$

On the other hand, with momentum conservation

$$\begin{aligned} \mathcal{M} &= \int \frac{d^3p}{(2\pi)^3}\,\psi^*_{s'}(\mathbf{p}+\mathbf{q}/2)\psi_s(\mathbf{p}-\mathbf{q}/2) \\ \cdot & \quad \bar{u}_{s'}(\mathbf{p}+\mathbf{q}/2)\left[F_1(q^2)\,\frac{\mathbf{p}}{m} - \frac{F_1(q^2)+F_2(q^2)}{m}\,i\,\mathbf{q}\times\frac{\mathbf{\Sigma}}{2}\right]u_s(\mathbf{p}-\mathbf{q}/2) \end{aligned} \tag{16}$$

The second term is explicitly $\mathcal{O}(\mathbf{q})$, the first term must be at most $\mathcal{O}(\mathbf{q})$ as well. The magnetic moment of a fermion could result from its orbital angular momentum even if its momentum is small, because of large size. One way we can eliminate that is to require $\psi^*_s(\mathbf{p}) = \psi_s(\mathbf{p})$, so that the $\mathcal{O}(\mathbf{q})$ part of the first term vanishes. Only keep the leading $\mathcal{O}(\mathbf{q})$ term, we obtain

$$\mathcal{M} \approx -\frac{F_1(q^2=0)+F_2(q^2=0)}{m}\,i\,\mathbf{q}\times\left\langle\frac{\mathbf{\Sigma}}{2}\right\rangle \tag{17}$$

$$\left\langle\frac{\mathbf{\Sigma}}{2}\right\rangle = \int \frac{d^3p}{(2\pi)^3}\psi^*_{s'}(\mathbf{p})\psi_s(\mathbf{p})\bar{u}_{s'}(\mathbf{p})\frac{\mathbf{\Sigma}}{2}u_s(\mathbf{p}) \tag{18}$$

$$-\frac{F_1(q^2=0)+F_2(q^2=0)}{m}\, i\,\mathbf{q}\times\left\langle\frac{\mathbf{\Sigma}}{2}\right\rangle \;\approx\; \int d^3x_{\rm op}\, i\,\mathbf{q}\cdot\mathbf{x}_{\rm op}\,\langle\psi|i\,\mathbf{j}(\mathbf{x}_{\rm op})|\psi\rangle \tag{19}$$

Cancel the $\mathbf{q}$, we obtain

$$-\frac{F_1(q^2=0)+F_2(q^2=0)}{m}\,\epsilon_{ijk}\left\langle\frac{\Sigma_k}{2}\right\rangle \;=\; \int d^3x_{\rm op}\,(x_{\rm op})_j\,\langle\psi|i\,j_i(\mathbf{x}_{\rm op})|\psi\rangle \tag{20}$$

Finally

$$\frac{F_1(q^2=0)+F_2(q^2=0)}{m}\left\langle\frac{\mathbf{\Sigma}}{2}\right\rangle \;=\; \left\langle\psi\middle|\int\frac{1}{2}\,\mathbf{x}_{\rm op}\times i\,\mathbf{j}(\mathbf{x}_{\rm op})\,d^3x_{\rm op}\middle|\psi\right\rangle \tag{21}$$

Remarks

- $i\,\mathbf{j}$ is the conventional Minkovski spatial current, because of our γ matrix convention.
- The right hand generate the total magnetic momentum for the entire system, including magnetic moment from spin.
- Above formula applies to near zero momentum, normalizable state with zero total current and its wave function in momentum space being real.

Not practical on lattice because it need extremely large volume to evaluate.

$$L \;\gg\; \Delta\mathbf{x}_{\rm op}\sim 1/\Delta\mathbf{p} \tag{22}$$

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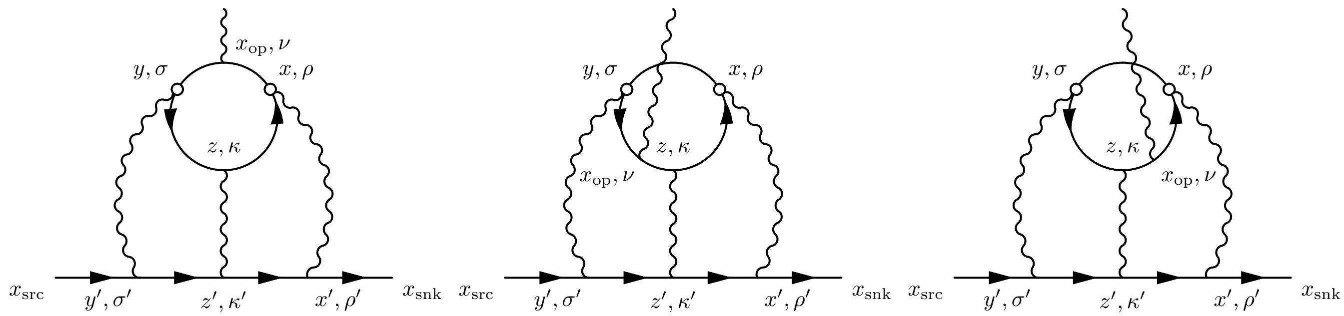

Figure 8. All three different possible insertions for the external photon. 5 other possible permutations of the three internal photons are not shown.

- It is not possible to exactly evaluation the full expression because of too many loops. As a trade off, we can use two point source photons at x and y, which are chosen randomly. It is a very standard 8-dimentional Monte Carlo integral over two space-time points.
- Major contribution comes from the region where x and y are not far separated. Importance sampling is needed. In fact, we can evaluate all possible (upto discrete symmetries) relative positions for distance less than a certain value r_{min}, which is normally $3 \sim 5$ lattice units.

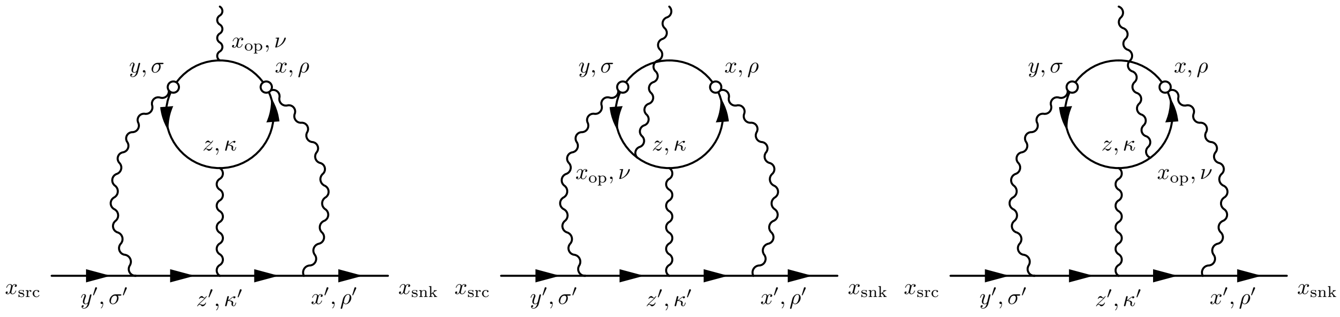

Figure 9. All three different possible insertions for the external photon. 5 other possible permutations of the three internal photons are not shown.

$$
\begin{aligned}
\mathcal{M}_\nu^{\text{LbL}}(\mathbf{q}) &= \exp(i\,\mathbf{q}\cdot\mathbf{x}_{\text{op}})\,\mathcal{M}_\nu^{\text{LbL}}(\mathbf{q}; x_{\text{op}}) && (23)\\
&= \sum_{x,y}\left[\sum_z \exp(i\,\mathbf{q}\cdot\mathbf{x}_{\text{op}})\,\mathcal{F}_\nu(\mathbf{q}; x, y, z, x_{\text{op}})\right] && (24)\\
&= \sum_{x,y}\sum_z \exp\left(i\,\mathbf{q}\cdot\left(\mathbf{x}_{\text{op}} - \frac{\mathbf{x}+\mathbf{y}}{2}\right)\right)\\
&\cdot\ \mathcal{F}_\nu\left(\mathbf{q}; \frac{x-y}{2}, -\frac{x-y}{2}, z - \frac{x+y}{2}, x_{\text{op}} - \frac{x+y}{2}\right) && (25)\\
&= \sum_r\left[\sum_{z, x_{\text{op}}} \exp(i\,\mathbf{q}\cdot\mathbf{x}_{\text{op}})\,\mathcal{F}_\nu\left(\mathbf{q}; -\frac{r}{2}, +\frac{r}{2}, z, x_{\text{op}}\right)\right] && (26)
\end{aligned}
$$

In the second last step, we use the translational covariance of $\mathcal{F}$.

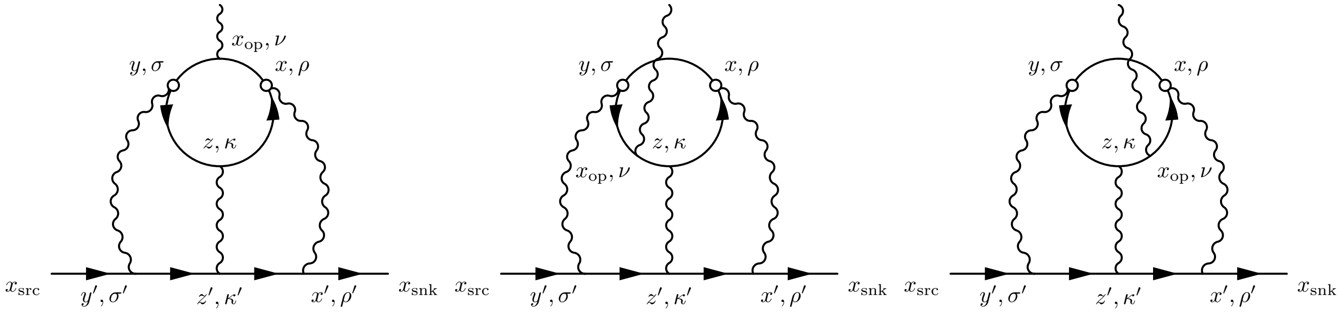

Figure 10. All three different possible insertions for the external photon. 5 other possible permutations of the three internal photons are not shown.

$$\mathcal{M}_\nu^{\rm LbL}(\mathbf{q}) = \sum_r \left[\sum_{z, x_{\rm op}} \exp(i\,\mathbf{q}\cdot\mathbf{x}_{\rm op})\,\mathcal{F}_\nu\Big(\mathbf{q}; -\frac{r}{2}, +\frac{r}{2}, z, x_{\rm op}\Big)\right] \tag{27}$$

$$= \sum_r \left[\sum_{z, x_{\rm op}} (1 + i\,\mathbf{q}\cdot\mathbf{x}_{\rm op})\,\mathcal{F}_\nu\Big(\mathbf{q}; -\frac{r}{2}, +\frac{r}{2}, z, x_{\rm op}\Big)\right] \tag{28}$$

$$= \sum_r \left[\sum_{z, x_{\rm op}} i\,\mathbf{q}\cdot\mathbf{x}_{\rm op}\,\mathcal{F}_\nu\Big(\mathbf{q}; -\frac{r}{2}, +\frac{r}{2}, z, x_{\rm op}\Big)\right] \tag{29}$$

- In the second last step we can drop the "1" term because of current conservation for the quark loop.

Zero Momentum Transfer Limit

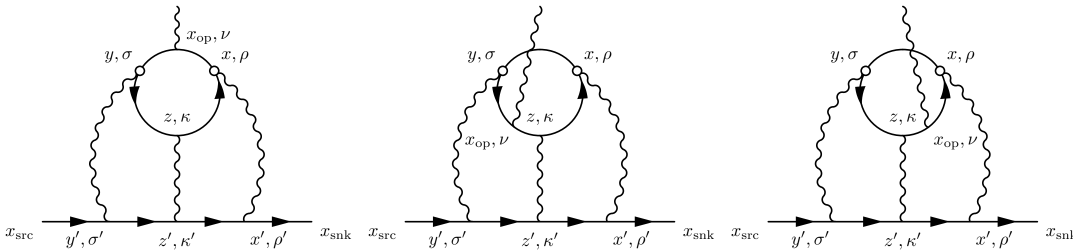

Figure 11. All three different possible insertions for the external photon. 5 other possible permutations of the three internal photons are not shown.

$$\bar{u}_{s'}(\mathbf{q}/2)\mathcal{M}_\nu^{\rm LbL}(\mathbf{q})u_s(-\mathbf{q}/2) \;=\; \bar{u}_{s'}(\mathbf{q}/2)\left(i\frac{F_2(q^2)}{4m}[\gamma_\nu, \gamma_\rho]q_\rho\right)u_s(-\mathbf{q}/2) \tag{30}$$

$$\mathcal{M}_\nu^{\rm LbL}(\mathbf{q}) \;=\; \sum_r \left[\sum_{z, x_{\rm op}} i\,\mathbf{q}\cdot\mathbf{x}_{\rm op}\,\mathcal{F}_\nu\left(\mathbf{q}; -\frac{r}{2}, +\frac{r}{2}, z, x_{\rm op}\right)\right] \tag{31}$$

Taking $q \to 0$ limit by computing derivative with respect to q, we obtained the familiar magnetic moment formula.

$$\frac{F_2(0)}{m}\,\bar{u}_{s'}(\mathbf{0})\frac{\vec{\Sigma}}{2}u_s(\mathbf{0})$$
$$= \sum_r \left[\sum_{z, x_{\rm op}} \frac{1}{2}\,\vec{x}_{\rm op} \times \bar{u}_{s'}(\mathbf{0})\, i\vec{\mathcal{F}}\left(\mathbf{0}; x = -\frac{r}{2}, y = +\frac{r}{2}, z, x_{\rm op}\right) u_s(\mathbf{0})\right] \tag{32}$$

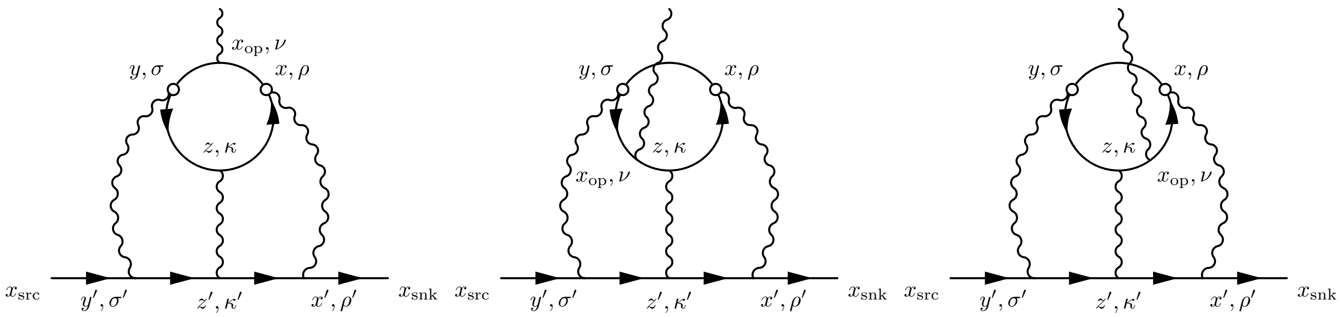

Figure 12. All three different possible insertions for the external photon. 5 other possible permutations of the three internal photons are not shown.

$$\frac{F_2(0)}{m}\,\bar{u}_{s'}(\mathbf{0})\frac{\vec{\Sigma}}{2}u_s(\mathbf{0}) = \sum_r \left[\sum_{z,x_{\rm op}} \frac{1}{2}\,\vec{x}_{\rm op}\times\bar{u}_{s'}(\mathbf{0})\,i\vec{\mathcal{F}}\Big(\mathbf{0}; x=-\frac{r}{2}, y=+\frac{r}{2}, z, x_{\rm op}\Big)\,u_s(\mathbf{0})\right] \tag{33}$$

- Recall the definition for $\mathcal{F}_\mu$, we sum all the internal points over the entire space time except we fix $x+y=0$.
- The time coordinate of the current, $(x_{\rm op})_0$ is integrated instead of being held fixed.
- The initial and final muon states are plane waves instead of properly normalized states.

These features allow us to perform the lattice simulation efficiently in finite volume.

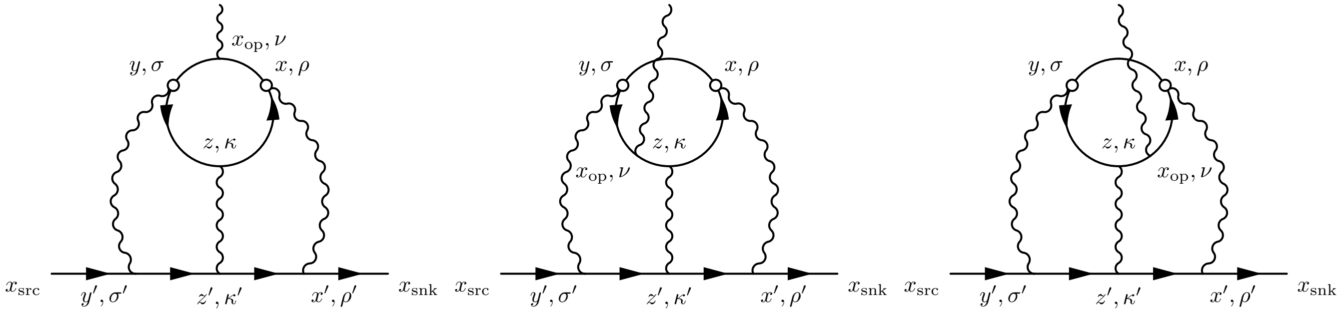

Figure 13. All three different possible insertions for the external photon. 5 other possible permutations of the three internal photons are not shown.

- The three internal photons are equivalent, that is $\mathcal{F}_\nu(\mathbf{q};\, x,\, y,\, z,\, x_{\rm op}) = \mathcal{F}_\nu(\mathbf{q};\, y,\, z,\, x,\, x_{\rm op}) = \mathcal{F}_\nu(\mathbf{q}; z, x, y, x_{\rm op})$.

$$\sum_{x,y,z} \mathcal{F}_\nu(\mathbf{q}; x, y, z, x_{\rm op}) = \sum_{\substack{x,y,z \\ |x-y|<\min(|x-z|,|y-z|)}} 3\,\mathcal{F}_\nu(\mathbf{q}; x, y, z, x_{\rm op}) \quad (34)$$

- Since we sum over z, but sample over $r = y - x$. It is beneficial to keep r small, where the fluctuation is small and sampling can be complete.

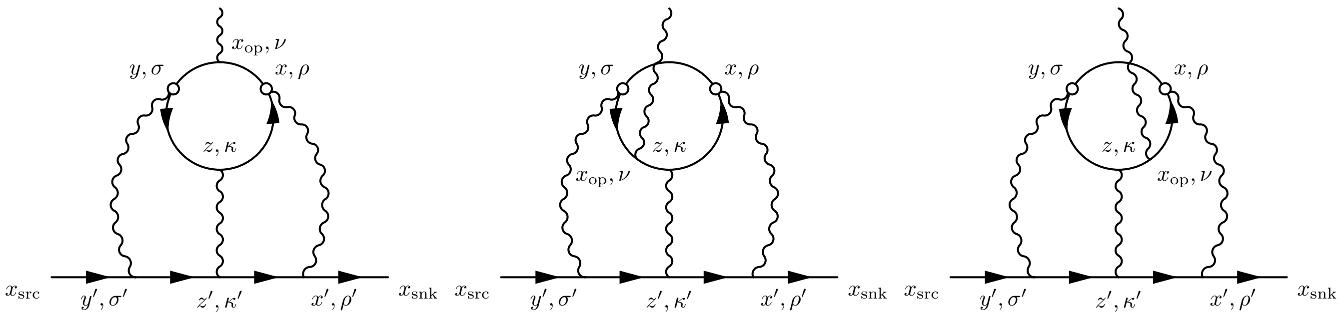

Figure 14. All three different possible insertions for the external photon. 5 other possible permutations of the three internal photons are not shown.

- Repeat our previous derivation, we can obtain a slightly different formula.

$$\frac{F_2(0)}{m}\,\bar{u}_{s'}(\mathbf{0})\frac{\vec{\Sigma}}{2}u_s(\mathbf{0}) = \sum_{r} \sum_{\substack{z \\ |r|<\min\left(|z-\frac{r}{2}|,|z+\frac{r}{2}|\right)}} 3\sum_{x_{\rm op}} \frac{1}{2}\vec{x}_{\rm op}\times i\vec{\mathcal{F}}\Big(\mathbf{0};-\frac{r}{2},+\frac{r}{2},z,x_{\rm op}\Big) \tag{35}$$

- We perform complete sum over z, but only sample over $r=y-x$.
- We have **moved** most of the contribution into the small r region, where the fluctuation is small and sampling can be complete.

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- Study of finite volume effect in **muon leptonic light by light** contribution to muon $g-2$.

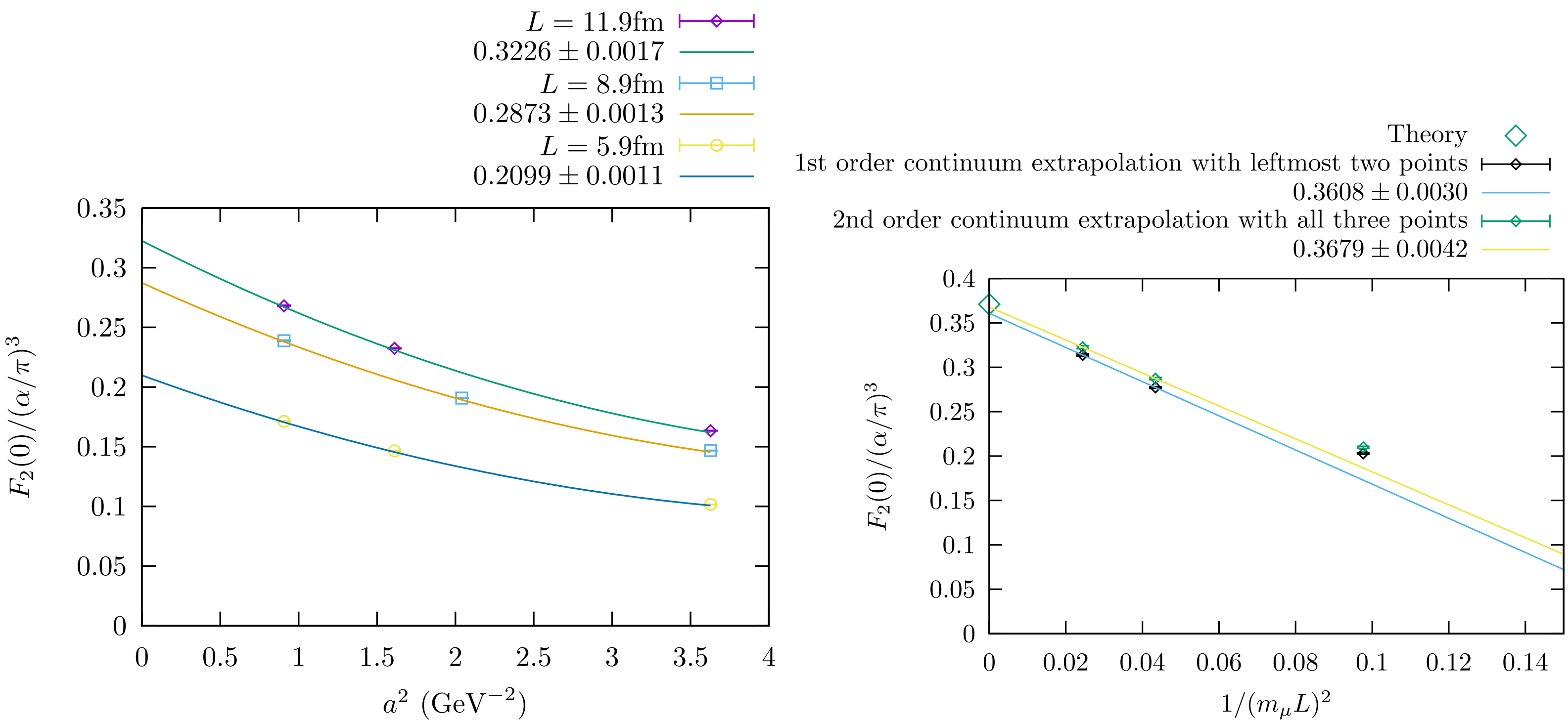

Figure 15. **Pure QED computation**. Muon leptonic light by light contribution to muon $g-2$. Phys.Rev. D93 (2016) 1, 014503. arXiv:1510.07100.

- $\mathcal{O}(1/L^2)$ finite volume effect, because the photons are emitted from a conserved loop.

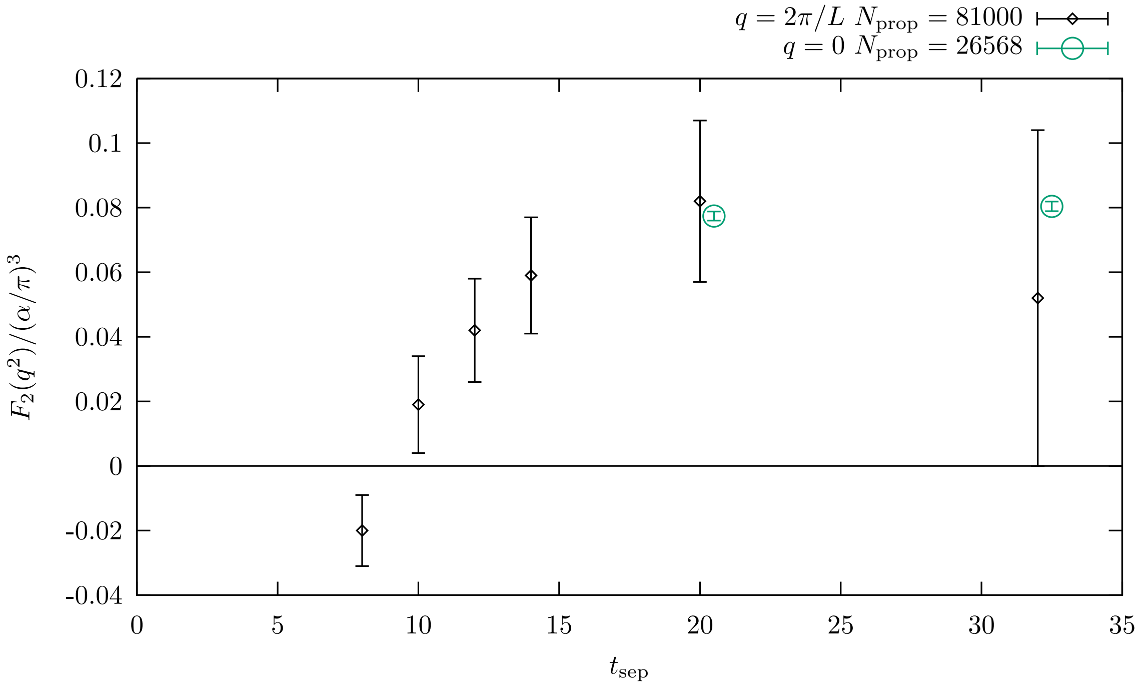

Figure 16. Phys.Rev.Lett. 114 (2015) 1, 012001. arXiv:1407.2923. Compare with the latest method and result. Phys.Rev. D93 (2016) 1, 014503. arXiv:1510.07100.

- $24^3 \times 64$ lattice with $a^{-1} = 1.747\text{GeV}$ and $m_\pi = 333\text{MeV}$. $m_\mu = 175\text{MeV}$.
- For comparison, at physical point, model estimation is 0.08 ± 0.02. The agreement is accidental, because the result has a strong dependence on m_μ, which is choose arbitrarily. The finite volume effects are also not taken into account.

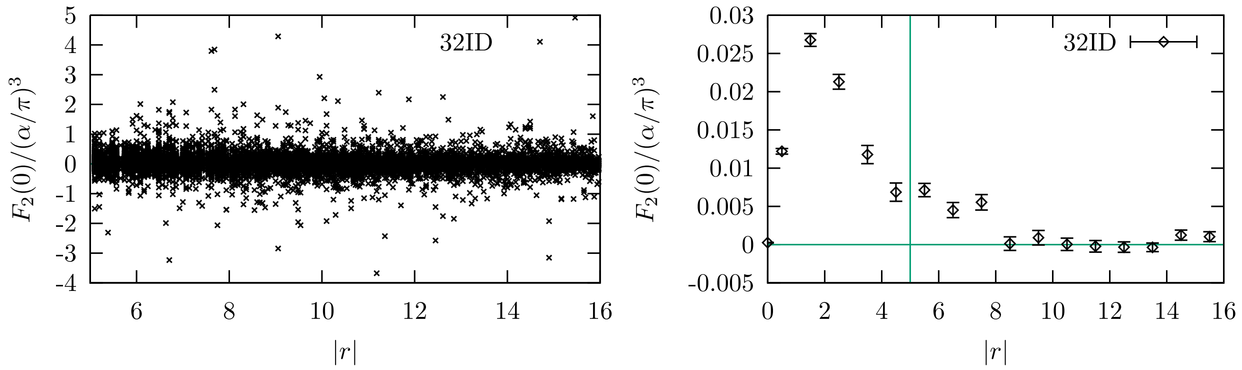

Figure 17. Phys.Rev. D93 (2016) 1, 014503. arXiv:1510.07100. $32^3 \times 64$ lattice, with $a^{-1} = 1.371\text{GeV}$, $m_\pi = 171\text{MeV}$, $m_\mu = 134\text{MeV}$.

- We use AMA technique to speed up the computations, above plots show the result from the sloppy solves. The corrections $(0.0060 \pm 0.0043)\,(\alpha/\pi)^3$ are then added to obtain the following result.

Label	size	$m_\pi L$	m_π/GeV	#qcdtraj	t_{sep}	$\frac{F_2}{(\alpha/\pi)^3}$	$\frac{\text{Cost}}{\text{BG/Q rack days}}$
32ID	$32^3 \times 64$	4.00	0.171	23	32	0.1054 ± 0.0054	13.2

Table 2. Central values and errors. $a^{-1} = 1.371\text{GeV}$. Muon mass and pion mass ratio is fixed at physical value. One BG/Q rack is composed of 1024×16 cores. For comparison, at physical point, with infinite volume, model estimation is 0.08 ± 0.02.

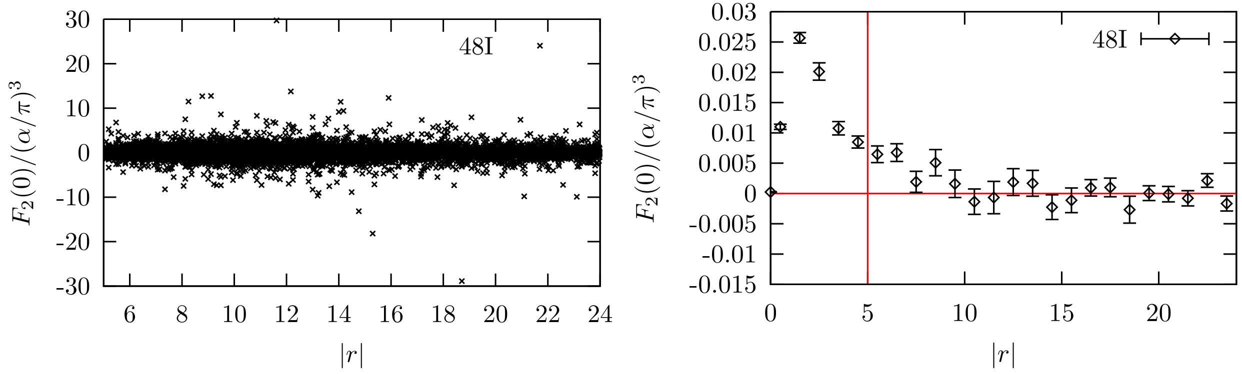

Figure 18. $48^3 \times 96$ lattice, with $a^{-1} = 1.73\text{GeV}$, $m_\pi = 139\text{MeV}$, $m_\mu = 106\text{MeV}$.

- We use AMA technique to speed up the computations, above plots show the result from the sloppy solves. The corrections $(-0.0018 \pm 0.0016)\,(\alpha/\pi)^3$ are then added to obtain the following result.

Label	size	$m_\pi L$	m_π/GeV	#qcdtraj	t_{sep}	$\frac{F_2}{(\alpha/\pi)^3}$	$\frac{\text{Cost}}{\text{BG/Q rack days}}$
48I	$48^3 \times 96$	3.86	0.139	69	48	0.0933 ± 0.0073	160

Table 3. Central values and errors. $a^{-1} = 1.371\text{GeV}$. Muon mass and pion mass ratio is fixed at physical value. One BG/Q rack is composed of 1024×16 cores. For comparison, at physical point, with infinite volume, model estimation is 0.08 ± 0.02.

Outline

- Muon Anomalous Magnetic Moment
- BNL E821 (0.54 ppm) and Standard Model Prediction
- Magnetic Moment in QFT
- Point Source Photon Method
  - → Summing Over x_{op}
  - → Zero Momentum Transfer Limit
  - → Difference with the General Magnetic Moment Formula
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- Simulations
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  - → $333\mathrm{MeV}$ Pion $24^3 \times 64$ Lattice
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  - → $139\mathrm{MeV}$ Pion $48^3 \times 96$ Lattice
- **Finite Volume Effects - QCD box inside QED box**
- Conclusions and Future Plans

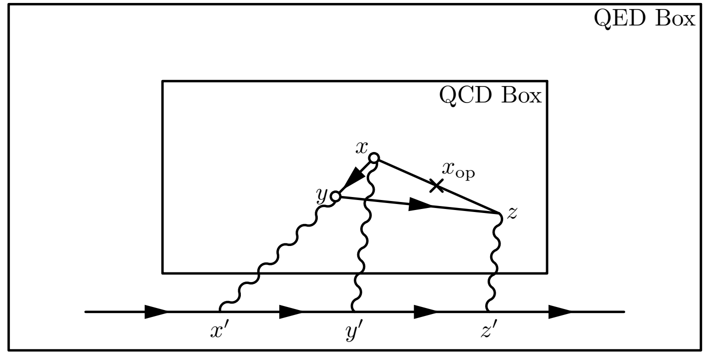

Figure 19. QCD box inside QED box illustration.

$$\sum_r \left[\sum_{z, x_{\mathrm{op}}} \frac{1}{2} \vec{x}_{\mathrm{op}} \times \bar{u}_{s'}(\mathbf{0}) \, i\vec{\mathcal{F}}\Big(\mathbf{0}; x = -\frac{r}{2}, y = +\frac{r}{2}; z, x_{\mathrm{op}}\Big) u_s(\mathbf{0}) \right] \quad (36)$$

- The integrand decreases exponentially if one of r, z, or x_{op} become large. The fact that the sum is limited within the lattice only has exponentially suppressed effect. We have use the moment method to take $q \to 0$ limit, eliminating that part of the "finite volume" effect.
- However, the integrand have implicit sum over x', y', and z'. Major finite volume effects result from these three variables are limited within lattice.
- Solution: do not limit x', y', and z' within the QCD box. We can sum over x', y', and z' in much larger QED boxes. We are also working on numerical strategies to compute the sum in infinite volume. This way, we can capture the major part of the finite volume effects with the QCD lattice just large enough to contain the quark loop.

$423\,\mathrm{MeV}$ Pion $16^3 \times 32$ Lattice V.S. $24^3 \times 64$ Lattice

Ensemble	$m_\pi L$	QCD Size	QED Size	$\frac{F_2(q^2=0)}{(\alpha/\pi)^3}$
16I	3.87	$16^3 \times 32$	$16^3 \times 32$	$0.1158(8)$
24I	5.81	$24^3 \times 64$	$24^3 \times 64$	$0.2144(27)$
16I-24		$16^3 \times 32$	$24^3 \times 64$	$0.1674(22)$

Table 4. arXiv:1511.05198. Finite volume effects studies. $a^{-1} = 1.747$ GeV, $m_\pi = 423$ MeV, $m_\mu = 332\,\mathrm{MeV}$.

- Large finite volume effects with these ensembles and muon mass.
- Increasing the QED box size help reducing the finite volume effect, but haven't completely fixed the problem.
- Suggesting significant QCD finite volume effect.
- The histogram plot may help us further investigating this QCD finite volume effect.

$423\,\mathrm{MeV}$ Pion $16^3 \times 32$ Lattice V.S. $24^3 \times 64$ Lattice

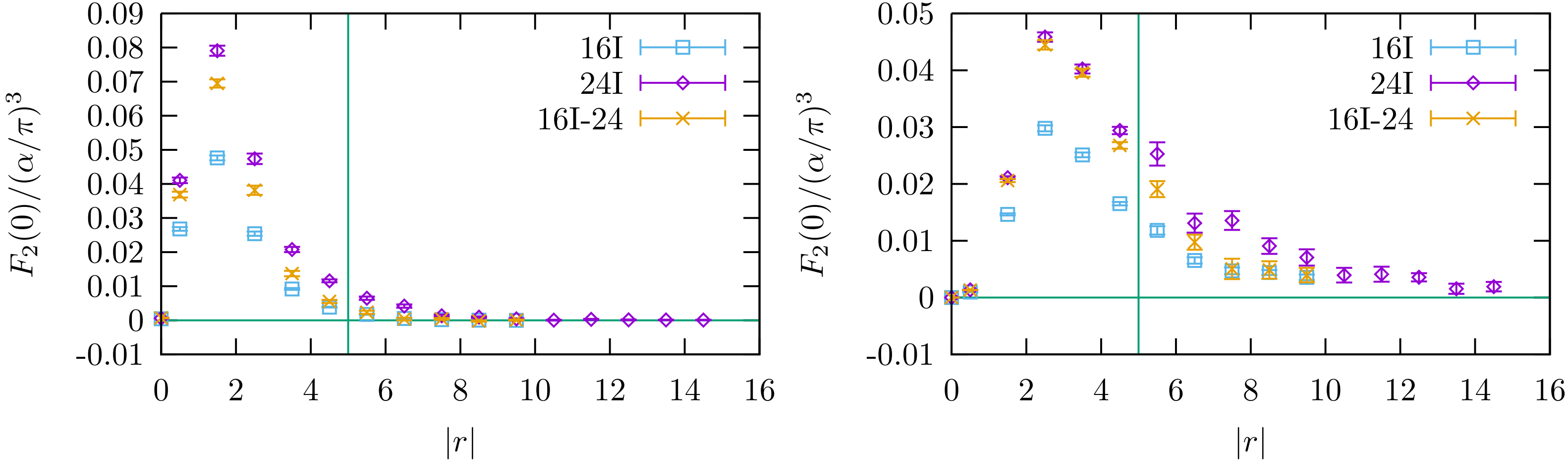

Figure 20. arXiv:1511.05198. Above plots show histograms of the contribution to F_2 from different separations $|r| = |x - y|$. The sum of all these points gives the final result for F_2. The vertical lines at $|r| = 5$ in the plots indicate the value of r_{max}. The left plot is evaluated with z sumed over longer distance region, so the small r region includes most of the contribution. The right plot is evaluated with z sumed over longer distance region, so the QCD finite volume is better controlled in the small r region.

$$\sum_{\substack{z \\ |x-y|<\min(|x-z|,|y-z|)}} \quad \text{v.s.} \quad \sum_{\substack{z \\ |x-y|>\max(|x-z|,|y-z|)}}$$

139MeV Pion $48^3 \times 96$ Lattice

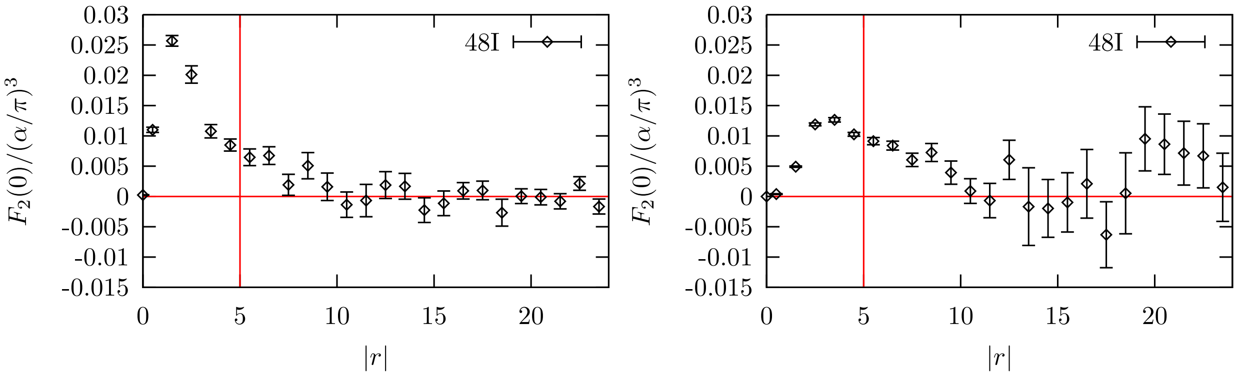

Figure 21. $48^3 \times 96$ lattice, with $a^{-1} = 1.73\text{GeV}$, $m_\pi = 139\text{MeV}$, $m_\mu = 106\text{MeV}$. The left plot is evaluated with z sumed over longer distance region, so the small r region includes most of the contribution. The right plot is evaluated with z sumed over longer distance region, so the QCD finite volume is better controlled in the small r region.

- Contribution vanishes long before reaching the boundary of the lattice.
- Suggesting the QCD finite volume effects be small in this case.
- Simply increasing the QED box will fix most of the finite volume effects.

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Conclusions

We have made significant improvements on the evaluation strategy.

- Eliminate the lower order $\mathcal{O}(e^4)$ noise and higher order contributions.
- Reduce the noise and keep it constant in infinite volume limit.
  i. Use analytical photon propagators instead of stochastic QED field configurations.
  ii. Make the external current exactly conserved configuration by configuration.
- Evaluate directly at zero momentum transfer limit.

With the improved method, we have computed LbL with serveral lattices. Most notablely

- Greater precision for $24^3 \times 64$ lattice where $L = 2.71\mathrm{fm}$, $m_\pi = 333\mathrm{MeV}$, $m_\mu = 175\mathrm{MeV}$.
- $32^3 \times 64$ lattice where $L = 4.6\mathrm{fm}$, $m_\pi = 171\mathrm{MeV}$, $m_\mu = 134\mathrm{MeV}$. We obtain that $F_2(0) = (0.1054 \pm 0.0054)(\alpha/\pi)^3$.
- $48^3 \times 96$ lattice where $L = 5.5\mathrm{fm}$, $m_\pi = 139\mathrm{MeV}$, $m_\mu = 106\mathrm{MeV}$. We obtain that $F_2(0) = (0.0933 \pm 0.0073)(\alpha/\pi)^3$.

Future Plans

- To address the discretization effect, we are currently running a simulation with a physical pion mass, 64 cube lattice with $5.4\mathrm{fm}$ volume, $a^{-1} = 2.36\mathrm{GeV}$.
- We also plan to address finite volume effect by evaluating LbL with $48^3 \times 96$ QCD box inside $96^3 \times 192$ QED box.
- Possible strategies for the calculation of all disconnected diagrams are being developed and we are currently doing numerical experiments.

Thank You!