Baryon interactions from Lüscher's finite volume method and HAL QCD method

Takumi IRITANI (Stony Brook Univ.)
for HAL QCD Collaboration

Apr. 18, 2016 @ INT-16-1 Nuclear Physics from Lattice QCD

Ref. TI for HAL QCD Coll., PoS(Lattice2015), arXiv:1511.05246[hep-lat].

- S. Aoki, S. Gongyo, D. Kawai, T. Miyamoto (YITP)
- T. Doi, T. Hatsuda, Y. Ikeda (RIKEN)
- T. Inoue (Nihon Univ.)
- N. Ishii, K. Murano (RCNP Osaka Univ.)
- H. Nemura, K. Sasaki (Univ. Tsukuba)
- F. Etminan (Univ. Birjand)

In This Talk: We Solve Lüscher vs HAL QCD Puzzle

1. QCD to Nuclear Physics: Puzzle between Lüscher's method and HAL
2. Baryon Interaction from Lattice QCD
   - Lattice Formulations and Setup
   - Lüscher's Method: Energy Shift in Finite Volume
   - HAL QCD Method: Potential
   - Energy Eigenvalue in Finite Volume
3. HAL and Lüscher — Origin of Fake Plateau
4. Summary and Outlook

2 Methods for Hadron Interaction from Lattice QCD

first principle calculations based on QCD

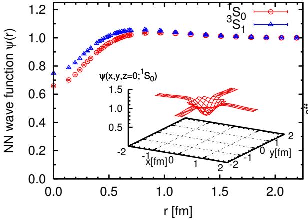
1. **Lüscher's finite volume method** — Lüscher '86, '91

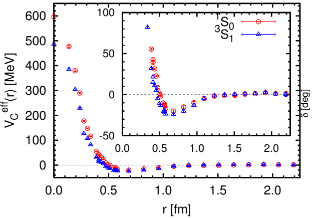
   1. energy shift of two-particle system in **"box"** ▶ 2. phase shift

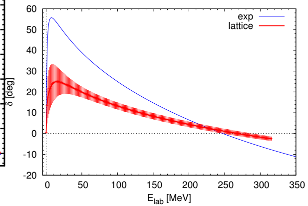
$$\Delta E_L = 2\sqrt{k^2 + m^2} - 2m \quad \Longrightarrow \quad k \cot\delta(k) = \frac{1}{\pi L} \sum_{\boldsymbol{n}\in\mathbb{Z}^3} \frac{1}{|\boldsymbol{n}|^2 - (kL/2\pi)^2}$$

2. **HAL QCD method** — Ishii-Aoki-Hatsuda '07

   1. NBS wave function ▶ 2. potential ▶ 3. phase shift







Consistency of Lüscher and HAL — $I = 2$ $\pi\pi$ scattering

$\Rightarrow$ Good

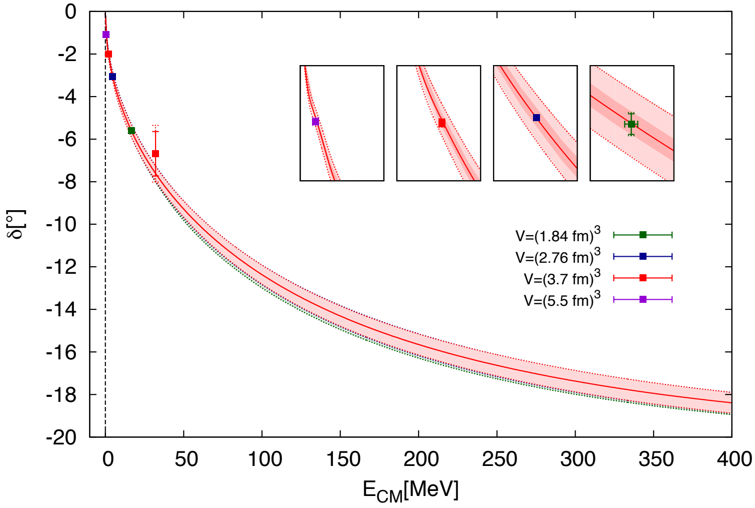


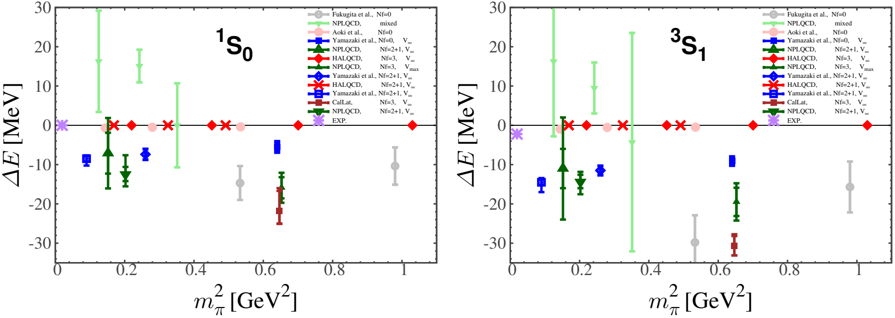
Figure: Kurth-Ishii-Doi-Aoki-Hatsuda '13

NN Interactions from Lattice QCD

	Lüscher		HAL QCD	phys. point
$^1S_0$	**bound**	$\Leftrightarrow$	**unbound**	**unbound**
$^3S_1$	**bound**	$\Leftrightarrow$	**unbound**	**bound**

$\Rightarrow$ **contradictions** between Lüscher and HAL QCD

- What is the origin of the difference ?



there are systematic discrepancies between

Lüscher's finite volume method and **HAL QCD method**

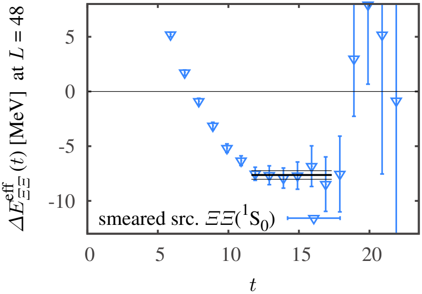
but those works used **different** quark mass, source (smeared and wall), ...

☐ **check baryon interactions** from **both methods** with **the same setup**

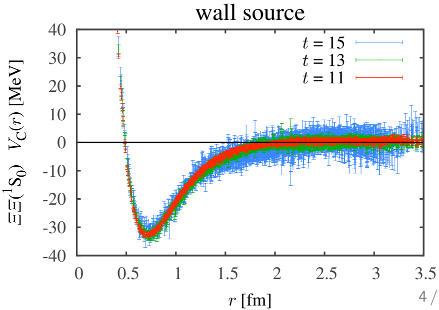
■ which is correct ? what is the origin of discrepancies ?

Lüscher vs HAL

■ measure energy shift



■ analyze potential



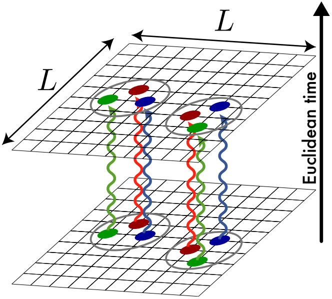
Lüscher's Finite Volume Method

- "**energy shift**" **in finite box** L^3

$$\Delta E_L = E_{BB} - 2m_B = 2\sqrt{k^2 + m_B^2} - 2m_B$$

$\Rightarrow$ **phase shift** $\delta(k)$

$$k \cot \delta(k) = \frac{1}{\pi L} \sum_{\boldsymbol{n} \in \mathbb{Z}^3} \frac{1}{|\boldsymbol{n}|^2 - (kL/2\pi)^2}$$



- **measurement**: plateau in **effective mass**

$$\Delta E_{\text{eff}}(t) = \log \frac{R(t)}{R(t+1)} \to \Delta E_L$$

$$R(t) = \frac{G_{BB}(t)}{\{G_B(t)\}^2} \to \exp\left[-\left(E_{BB} - 2m_B\right) t\right]$$

with $G_{BB}(t)(G_B(t))$: BB(B) correlators

- NN($^1\mathrm{S}_0$) energy shift

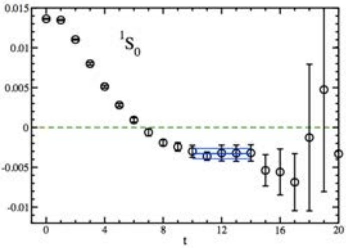


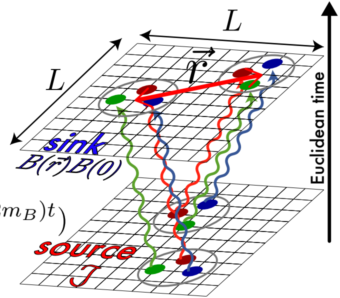
Fig. Yamazaki et al. '12

- effective mass plot

$\Longrightarrow$ **standard method** in "single particle state"

Time-dependent HAL QCD Method

■ Nambu-Bethe-Salpeter correlation function



$$R(\vec{r},t) \equiv \frac{\langle 0|T\{B(\vec{x}+\vec{r},t)B(\vec{x},t)\}\bar{\mathcal{J}}(0)|0\rangle}{\{G_B(t)\}^2}$$
$$= \sum_n A_n \psi_n(\vec{r}) e^{-(E_n-2m_B)t} + \mathcal{O}(e^{-(E_{\rm th}-2m_B)t})$$

▶ in Lüscher's method: $\sum_{\vec{r}} R(\vec{r},t) = R(t) \Longrightarrow \exp\left[-(E_{BB}-2m_B)t\right]$

☐ each $\psi_n(\vec{r})e^{-E_n t} \equiv \langle 0|T\{B(\vec{x}+\vec{r},t)B(\vec{x},t)\}|2B,n\rangle$ satisfies

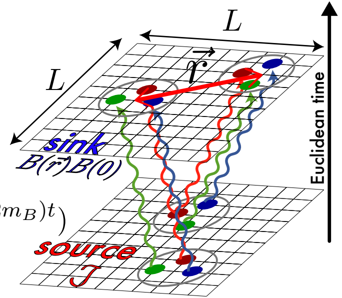
$$\left[\frac{k_n^2}{m_B} - H_0\right]\psi_n(\vec{r}) = \int d\vec{r'}\, U(\vec{r},\vec{r'})\psi_n(\vec{r'})$$

with non-local interaction kernel $U(\vec{r},\vec{r'})$

■ R-corr. satisfies t-dep. Schrödinger-like eq. with **elastic** saturation

$$\left[\frac{1}{4m_B}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right] R(\vec{r},t) = \int d\vec{r'}\, U(\vec{r},\vec{r'}) R(\vec{r'},t)$$

Time-dependent HAL QCD Method



- Nambu-Bethe-Salpeter correlation function

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$$\left[\frac{1}{4m_B}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right] R(\vec{r},t) = \int d\vec{r'} U(\vec{r},\vec{r'}) R(\vec{r'},t)$$

▶ **"potential"** using velocity expansion $U(r,r') \simeq V(r)\delta(r-r')$

$$V(\vec{r}) = \frac{1}{4m_B}\frac{(\partial/\partial t)^2 R(\vec{r},t)}{R(\vec{r},t)} - \frac{(\partial/\partial t)R(\vec{r},t)}{R(\vec{r},t)} - \frac{H_0 R(\vec{r},t)}{R(\vec{r},t)}$$

▶ **This method does not require the ground state saturation.**

Difficulties in Multi-Baryons

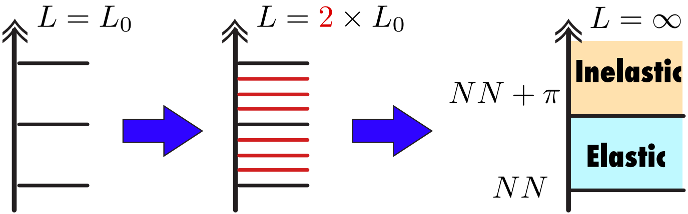
- Lüscher's method requires **ground state saturation**

$$G_{NN}(t) = c_0 \exp(-E_0^{(NN)} t) + c_1 \exp(-E_1^{(NN)} t) + \cdots \simeq c_0 \exp(-E_0^{(NN)} t)$$

  - S/N becomes worse as [mass number A] $\times$ [light quark] $\times$ [$t \to \infty$]

$$S/N \sim \exp\left[-A \times \left(m_N - \frac{3}{2} m_\pi\right) \times t\right]$$

  - smaller gaps: $\Delta E \sim \vec{p}^{\,2}/m \sim \mathcal{O}(1/L^2)$



- **HAL QCD method**
  - only **elastic states saturation** — E-indep potential

$$\left[\frac{1}{4m_B}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right] R(\vec{r}, t) = \int d\vec{r'} U(\vec{r}, \vec{r'}) R(\vec{r'}, t)$$

Lattice Setup

■ $2+1$ improved Wilson + Iwasaki gauge†

lattice spacing $a = 0.08995(40)$ fm, $a^{-1} = 2.194(10)$ GeV

$m_\pi = 0.51$ GeV, $m_N = 1.32$ GeV, $m_K = 0.62$ GeV, $m_\Xi = 1.46$ GeV

■ **2-type quark sources**

- **wall source**

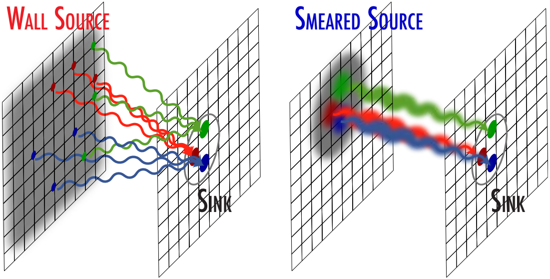
  standard of HAL QCD

- **smeared source**

  the same as Yamazaki *et al*†.

  → **NN int.** by Lüscher

  & **Helium** binding energy



□ analyze **ΞΞ($^1S_0$)-channel** — the same rep. as NN($^1S_0$) and better S/N with high stat. — ex. 48^4: (#smeared src.)/(*Yamazaki et al.*) $\simeq 5$

volume	# conf.	# smeared src.	# wall src.
$40^3 \times 48$	200	512	48
$48^3 \times 48$	200	4 × 256	4 × 48
$64^3 \times 64$	327	256	4 × 32

† Yamazaki-Ishikawa-Kuramashi-Ukawa, arXiv:1207.4277.

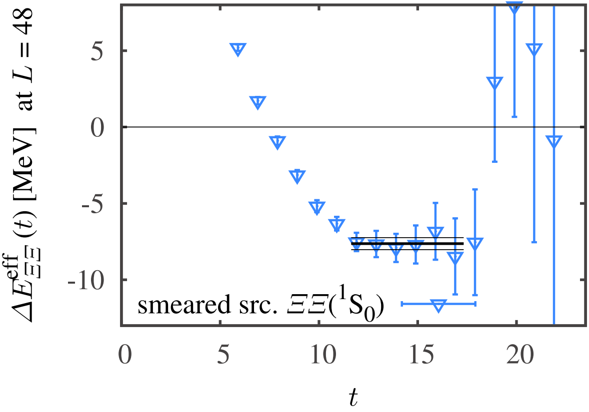
$\Xi\Xi(^1S_0)$ Effective Energy Shift Plot

effective mass plot

$$\Delta E^{\text{eff}}_{\Xi\Xi}(t) = \log R(t)/R(t+1) \to E_{\Xi\Xi} - 2m_{\Xi}$$

with $R(t) \equiv G_{\Xi\Xi}(t)/\{G_{\Xi}(t)\}^2$

- **smeared source**
  $\Delta E_L \sim -8$ MeV



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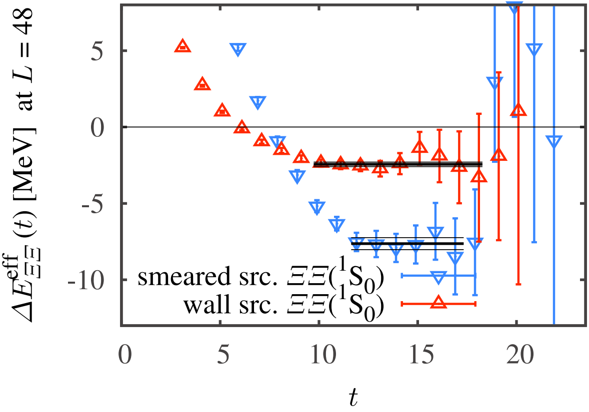
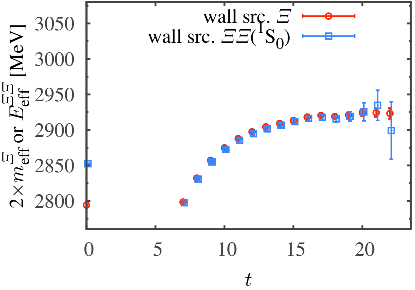
- **wall source**
  $\Delta E_L \sim -3$ MeV
- **smeared source**
  $\Delta E_L \sim -8$ MeV
- **plateau** depends on quark source

???

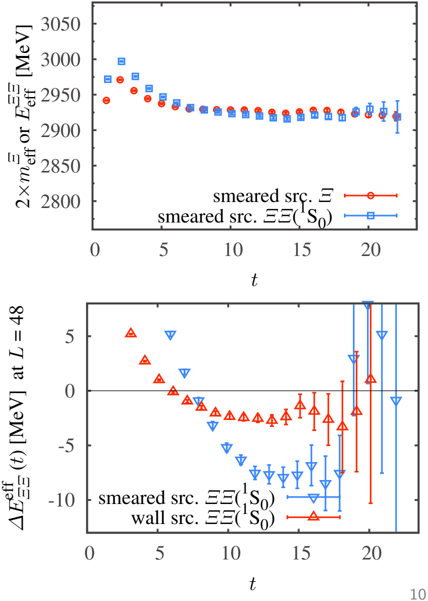


Effective Masses of ΞΞ and Ξ

wall source

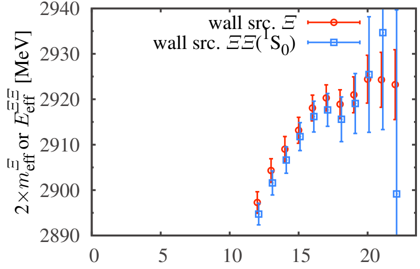


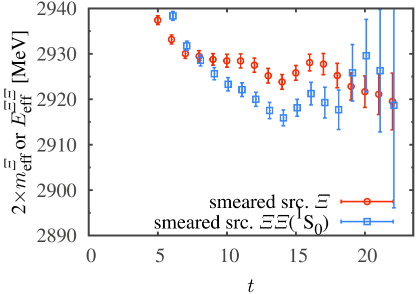
smeared source



Effective Masses of ΞΞ and Ξ

wall source / **smeared source**





- Be careful with effective mass plot

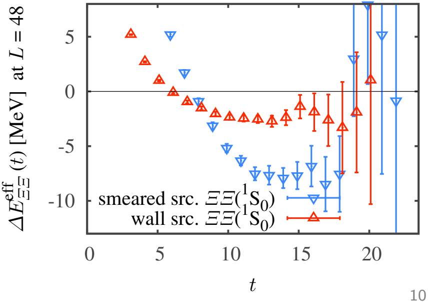
w/o plateaux in $E^{\mathrm{eff}}_{\Xi\Xi}(t)$ & $m^{\mathrm{eff}}_{\Xi}(t)$,

$$\varDelta E_{\mathrm{eff}}(t) = E^{\Xi\Xi}_{\mathrm{eff}}(t) - 2m^{\Xi}_{\mathrm{eff}}(t)$$

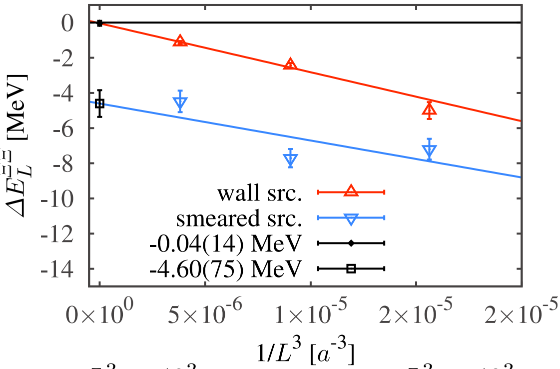
shows a **"fake" plateau** by cancellation

▶ we need much larger t,

and much more statistics



“Fake” Plateaux $\Longrightarrow$ Wrong Conclusion

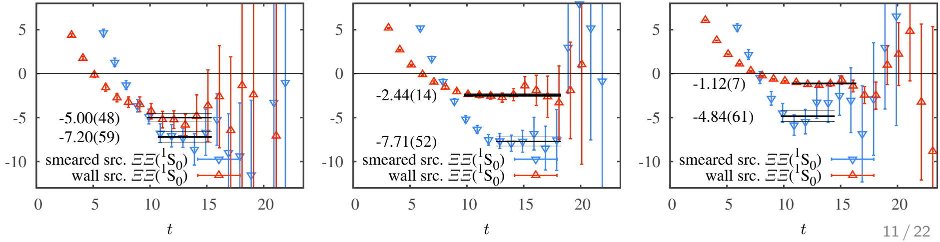


- **“direct method”**

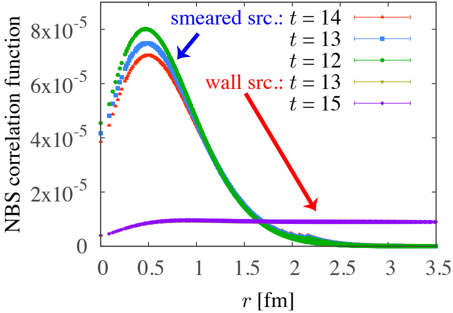
wall src. vs **smeared src.**

- different plateaux
- different conclusions
  unbound or **bound**

$L^3 = 40^3$ $\quad$ $L^3 = 48^3$ $\quad$ $L^3 = 64^3$



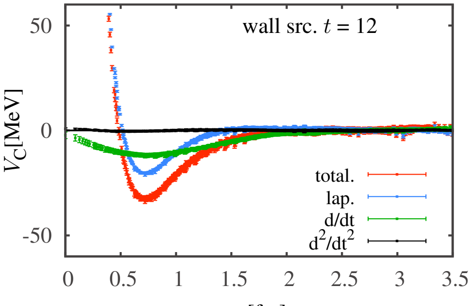
HAL: NBS Wave Function and $\Xi\Xi(^1S_0)$ Potential $V_c(\vec{r})$



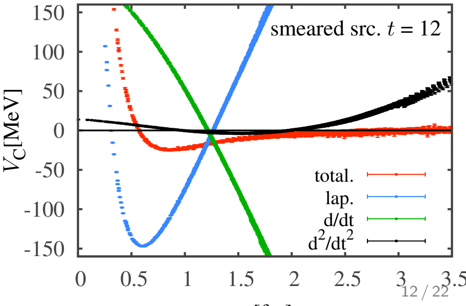
- ○ **wall src.** — weak t-dep.
- ○ **smeared. src.** — strong t-dep.
  - $\Rightarrow$ $\mathcal{O}(100)$ MeV of cancellation
- • time-dep. HAL method works well

$$V_c(\vec{r}) = -\frac{H_0 R}{R} - \frac{(\partial/\partial t)R}{R} + \frac{(\partial/\partial t)^2 R}{4mR}$$

■ **wall src.**



□ **smeared src.**



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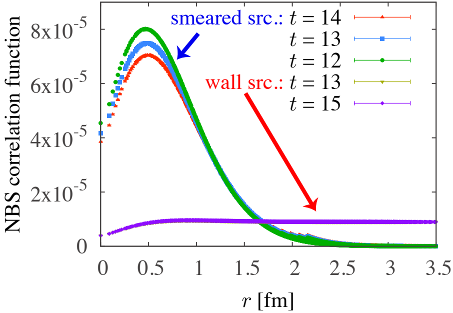

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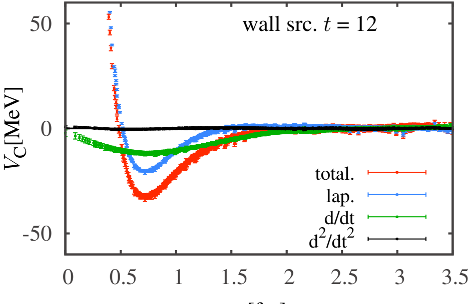
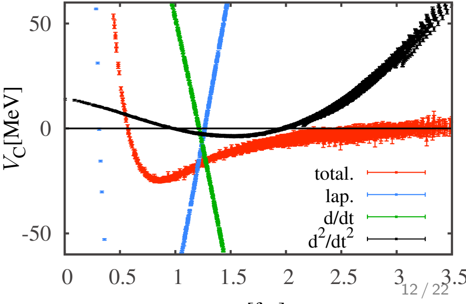
■ **wall src.**

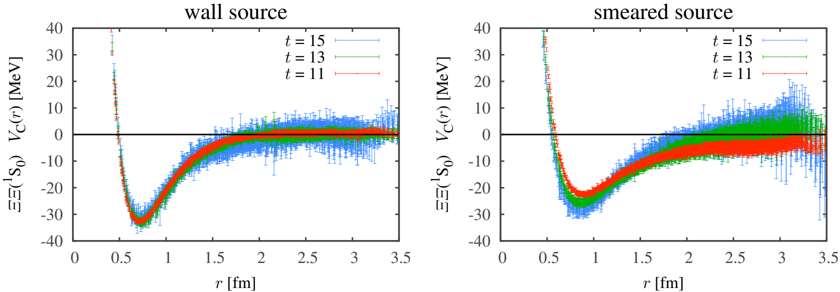


□ **smeared src.**

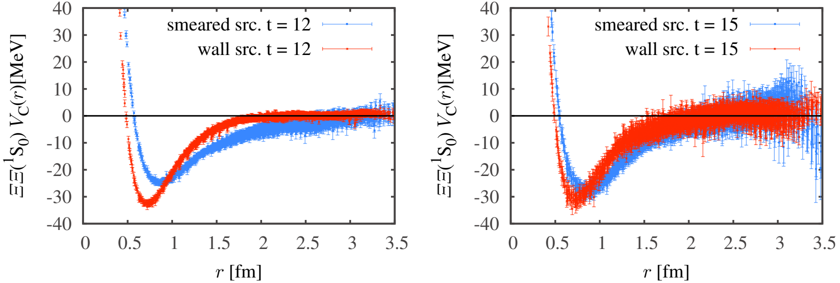


$\Xi\Xi(^1S_0)$ Potential: Wall Source vs Smeared Source

- **wall src.** "stable" $\Longleftrightarrow$ **smeared src.** t-depend



- **"smeared src."** converges to **"wall src."** — residual from NLO pot.



Residual Diff. of Pot.: Next Leading Order Correction

Derivative expansion: $U(r,r') = \{V_0(r) + V_1(r)\nabla^2\}\delta(r-r')$ (for $^1S_0$)

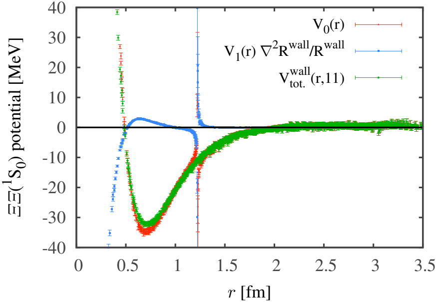
$$\left[\frac{1}{4m}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right] R(r,t) = \int d^3r' U(r,r') R(r',t)$$

$$\therefore \frac{1}{4m}\frac{(\partial^2/\partial t^2)R}{R} - \frac{(\partial/\partial t)R}{R} - \frac{H_0 R}{R} = V_0(r) + V_1(r)\frac{\nabla^2 R(r,t)}{R(r,t)} \equiv \tilde{V}^{\rm naive}_{\rm total}(r,t)$$

we have $R^{\rm smear}(r,t)$ and $R^{\rm wall}(r,t)$ $\Rightarrow$ let's solve $V_0(r)$ and $V_1(r)$

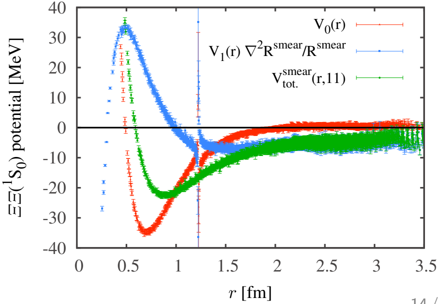
▶ **wall source**

- Good convergence of **LO pot.**



▶ **smeared source**

- **with NLO pot. correction**



HAL QCD meets Lüscher: Energy Shift from Potential

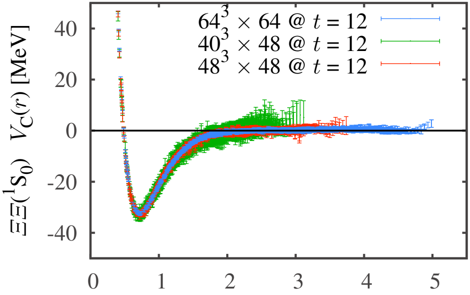
- HAL QCD gives reliable interaction **w/o g.s. saturation**
  quark source, time, and volume independent
- what is the true **“energy shift”** in finite volume

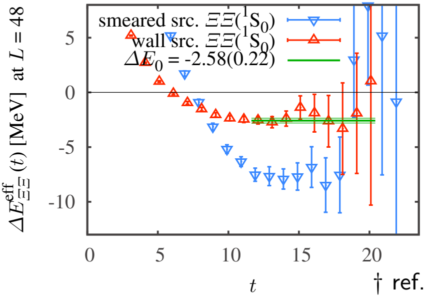
1 **INPUT**: potential $V(\vec{r})$

2 **SOLVE**: eigenvalue problem†

$$[H_0 + V]\,\psi = \Delta E \psi$$

in finite volume L^3





vol.	g.s. [MeV]	1st [MeV]
40^3	**-4.55(1.18)**	75.63(1.31)
48^3	**-2.58(22)**	52.87(33)
64^3	**-1.13(9)**	28.71(9)

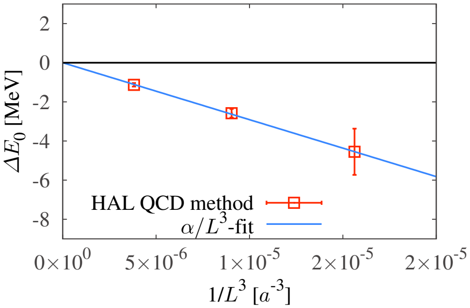
eigenvalues using $V_c(\vec{r}, t = 12)$

† ref. B. Charron for HAL Coll. arXiv:1312.1032.

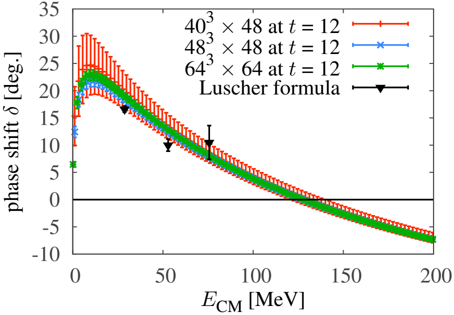
Conclusion: $\Xi\Xi(^1S_0)$ is Unbound at current mass

$$k \cot \delta(k) = \frac{1}{\pi L} \sum_{n \in \mathbf{Z}^3} \frac{1}{|n|^2 - (kL/2\pi)^2}, \qquad \Delta E = 2\sqrt{m^2 + k^2} - 2m$$

volume dep. of ΔE_0



phase shift δ



- wall src. pot. $\Rightarrow$ solve eigenvalue & Lüscher formula $\Rightarrow$ phase shift
- wall src. pot. $\Rightarrow$ fit & solve Schrödinger eq. $\Rightarrow$ phase shift
  - using 2 Gaussians + $(\text{Yukawa})^2$ ansatz for $V_c(\vec{r})$ fit

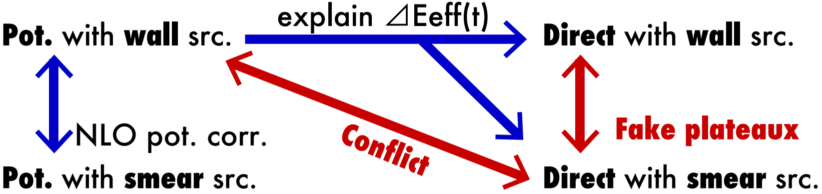
Short Summary: Consistency of 2 Methods

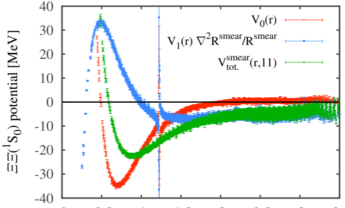
- **ground state saturation** of multi-baryon is extremely difficult
- **even without g.s. saturation** there may appear **plateau**

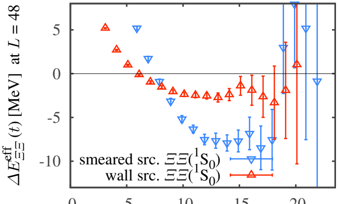
  ⇒ it should be checked by other source or variational method

  fake plateaux ⇒ wrong conclusion
- HAL QCD method gives source indep. results **w/o g.s. saturation**.

✓ ~~"Lüscher vs HAL"~~ ⇒ "Lüscher(**smeared**) vs Lüshcer(**wall**)"

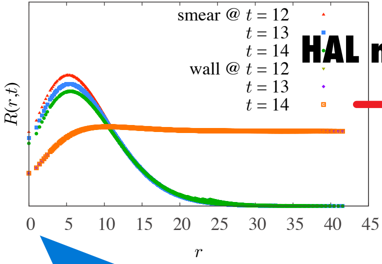






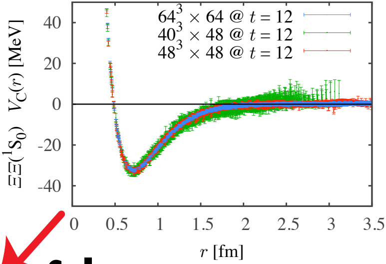
Wavefunction, Potential, Eigenvalues and Eienfunctions

NBS wave function



HAL method

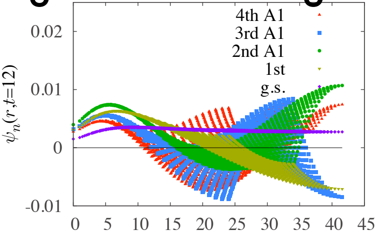
Potential



Solve $[H_0 + V]\psi = E\psi$

feed back

Eigenvalues & Eigenfunctions



decomposition
projection

ground state

& excited states

(elastic scattering)

Exited States in Wavefunction

▶ R-corr. decomposition by energy eigenmodes ◀ from **HAL pot.**

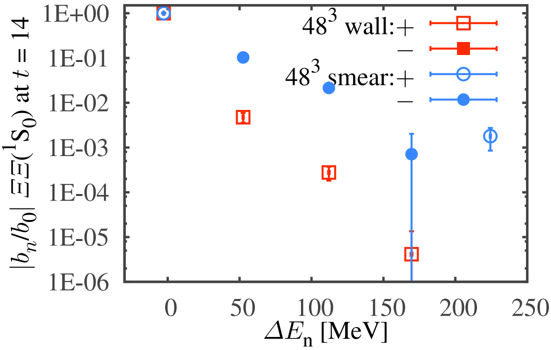
$$R^{\text{wall/smear}}(\vec{r},t) = \sum_n a_n^{\text{wall/smear}} \Psi_n(\vec{r},t) \exp\left(-\Delta E_n t\right)$$

$$\therefore R(\vec{p}=0,t) = \sum_r R(\vec{r},t) = \sum_n b_n^{\text{wall/smear}} e^{-\Delta E_n t}$$

"contamination" of excited states b_n/b_0

☐ ex. **1st excited state**

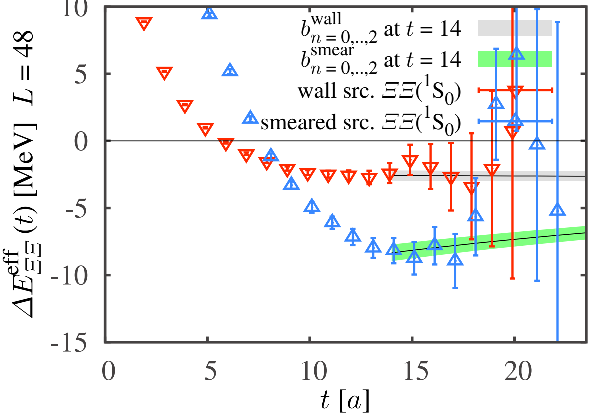
- **wall source**
  $b_1/b_0 \ll 0.01$
- **smeared source**
  $b_1/b_0 \simeq -0.1$
- with energy gap
  $E_1 - E_0 \simeq 50$ MeV
  for $L^3 = 48^3$



Excited States Contamination and Fake Plateau

$$\Delta E_{\text{eff}}(t) = \log \frac{\sum_n b_n \exp\left(-\Delta E_n t\right)}{\sum_n b_n \exp\left(-\Delta E_n (t+1)\right)}$$

- "direct measurement" — well reproduced by low-lying 3 eigenmodes†
  - **"fake plateau"** of **smeared src.** around $t = 14 - 18$



† eigenvalues ΔE_n, coefficients $b_n^{\text{smear/wall}}$ for $n = 0, 1, 2$, at $t = 14$.

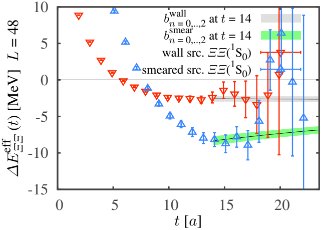
Excited States Contamination and Fake Plateau

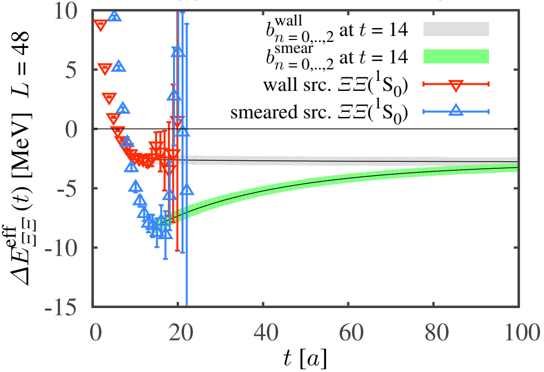
$$\Delta E_{\text{eff}}(t) = \log \frac{\sum_n b_n \exp\left(-\Delta E_n t\right)}{\sum_n b_n \exp\left(-\Delta E_n (t+1)\right)}$$

■ "direct measurement" — well reproduced by low-lying 3 eigenmodes†

▶ **"fake plateau"** of **smeared src.** around $t = 14 - 18$

☐ **g.s. saturation** of smeared source — **100 lattice units** $\sim$ **10 fm !!!**





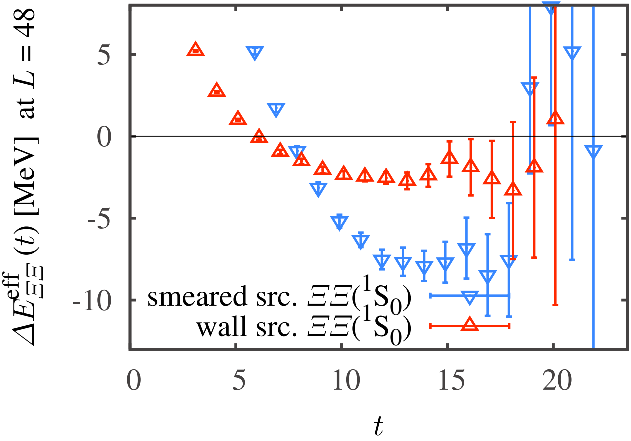
† eigenvalues ΔE_n, coefficients $b_n^{\text{smear/wall}}$ for $n = 0, 1, 2$, at $t = 14$.

Lüscher's Method by "Sink" Projection

introducing "some" function $f(\vec{r})$

$$\tilde{R}^{(f)}(t) \equiv \sum_{\vec{r}} f(\vec{r}) \sum_{\vec{x}} \langle 0 | B(\vec{r}+\vec{x},t) B(\vec{x}) \bar{\mathcal{J}}(0) | 0 \rangle = \sum_{\vec{r}} f(\vec{r}) R(\vec{r},t)$$

and effective energy shift — $\Delta\tilde{E}^{(f)}_{\rm eff}(t) = \log \tilde{R}^{(f)}(t)/\tilde{R}^{(f)}(t+1)$



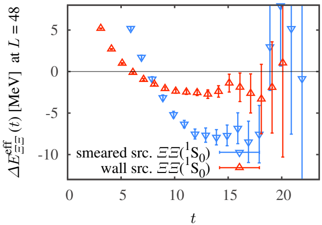
Lüscher's Method by "Sink" Projection

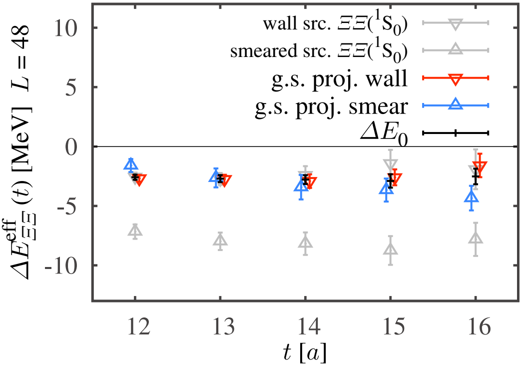
introducing "some" function $f(\vec{r})$

$$\tilde{R}^{(f)}(t) \equiv \sum_{\vec{r}} f(\vec{r}) \sum_{\vec{x}} \langle 0|B(\vec{r}+\vec{x},t)B(\vec{x})\bar{\mathcal{J}}(0)|0\rangle = \sum_{\vec{r}} f(\vec{r}) R(\vec{r},t)$$

and effective energy shift — $\Delta\tilde{E}^{(f)}_{\text{eff}}(t) = \log \tilde{R}^{(f)}(t)/\tilde{R}^{(f)}(t+1)$

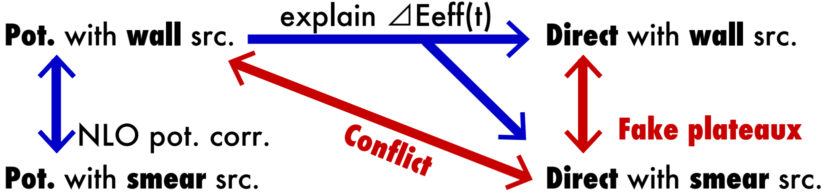
▶ **"correct plateau"** by using $f(\vec{r}) = \Psi^{\dagger}_{\text{gs}}(\vec{r})$ ◀ from **HAL pot.**





Summary: Lüscher ~~vs~~ and HAL QCD are Consitent

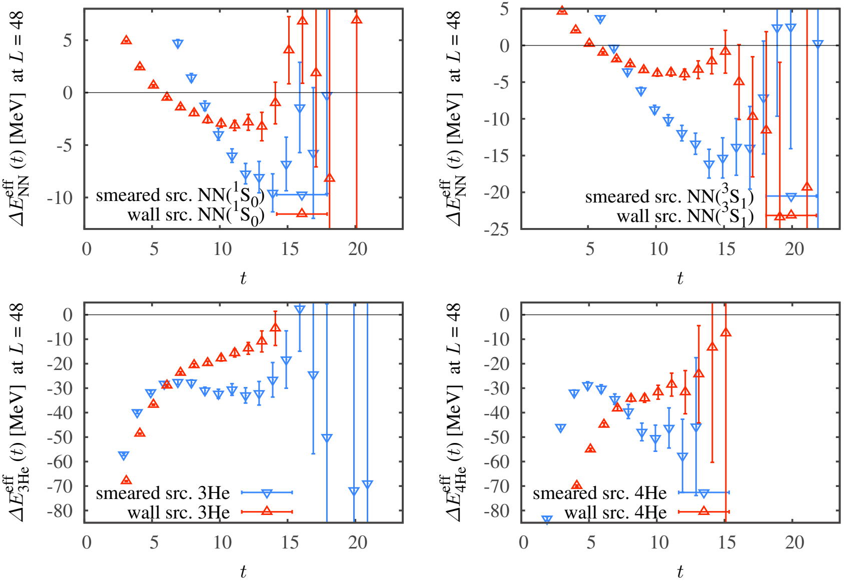
- "Direct method" — **ground state saturation** is extremely difficult
  $\Rightarrow$ *do not trust plateau*, it may be a **wrong signal!**
- HAL QCD works well **without g.s. saturation**.
- **NBS corr.** and **"potential"** clarify excited states contamination
  and origin of **fake plateau**.
- using **NBS corr.**, **"potential"** and **wavefunctions**
  ◦ improvement of Lüshcer's measurement — **"feed back method"**



5 Appendix

NN($^1S_0$), NN($^3S_1$), Triton and Helium

Do not trust plateaux.



- speed up for ^{3}He/^{4}He calc. — unified contraction algorithm Doi-Endres '13

So-Called "Plateau"

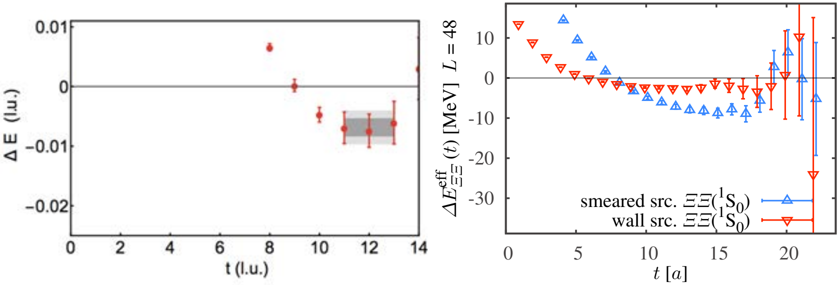
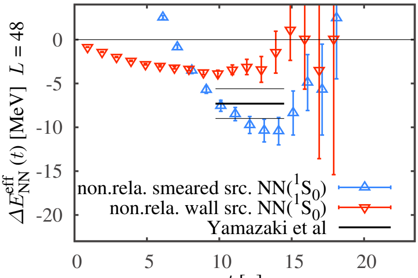


Figure: (a) NPL (2015) dineutron $L = 48^3$ (b) $\Xi\Xi(^1\mathrm{S}_0)$ $L^3 = 48^3$ with the same range of ΔE

Fit range and value in Yamazaki et al '12 and our re-analysis of $\mathrm{NN}(^1\mathrm{S}_0)$



Effective mass of Single Channel

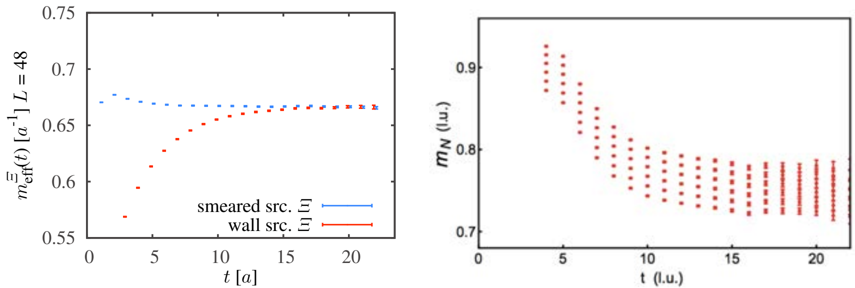
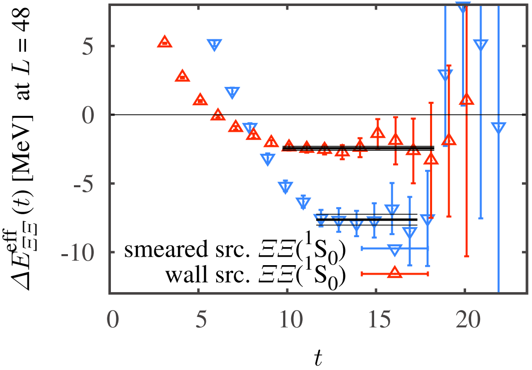


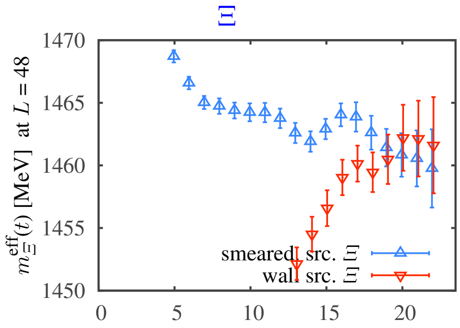
Figure: (a) NPL (2015) nucleon $L = 48^3$ (b) our data Ξ mass $L = 48^3$ with the same range of ΔE

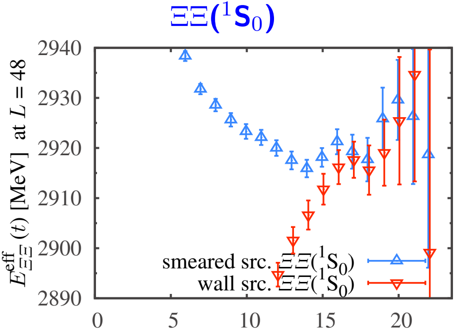
Effective Masses Plot

$$\Delta E^{\mathrm{eff}}_{\Xi\Xi}(t) = E^{\mathrm{eff}}_{\Xi\Xi}(t) - 2m^{\mathrm{eff}}_{\Xi}(t)$$

shows plateau
even without plateaux
in $m^{\mathrm{eff}}_{\Xi}(t)$ or $E^{\mathrm{eff}}_{\Xi\Xi}(t)$





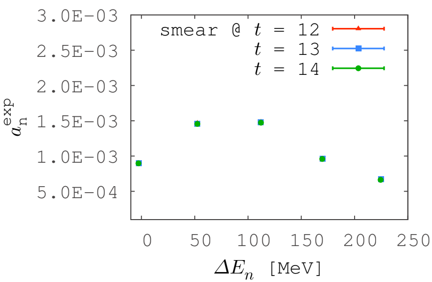


Weight of Eigenstates in Wavefunction

$$R^{\text{wall/smear}}(\vec{x},t) = \sum_n a_n^{\text{wall/smear}} \Psi_n(\vec{x},t) e^{-\Delta E_n t}$$

with eigenfunctions Ψ_n and eigenvalues ΔE_n

- smeared $\Longrightarrow$ higher states are significant $|a_0^{\text{exp.}}| \sim |a_1^{\text{exp.}}| \sim |a_2^{\text{exp.}}|$
- wall $\Longrightarrow$ g.s. dominant $|a_0^{\text{wall}}| \gg |a_1^{\text{wall}}| \gg |a_2^{\text{wall}}|$



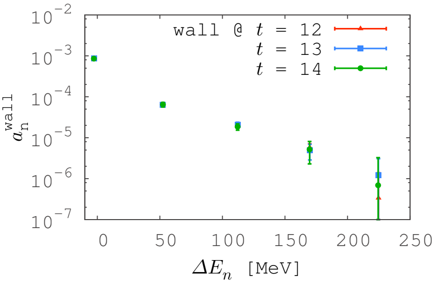
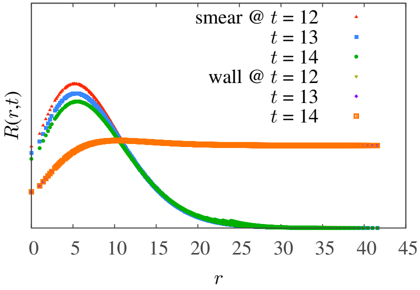
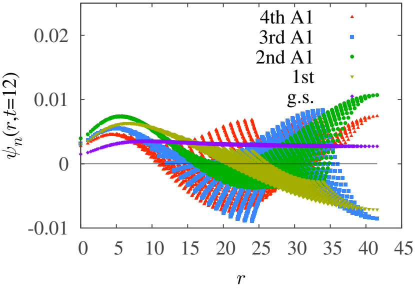


Figure: (a) smeared source. (b) wall source.

Wavefunction and Eigenfunctions below Threshold





n-th A1	ΔE_n [MeV]
0	-2.58
1	52.49
2	112.08
3	169.78
4	224.73

Table: 48^3 wall source @ $t = 12$

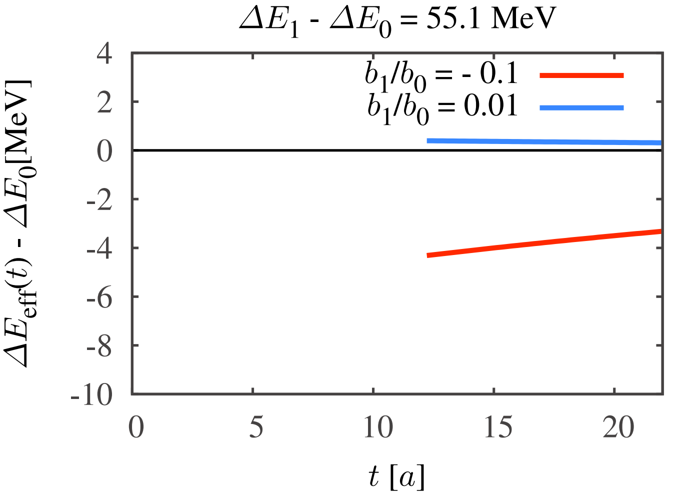
- **A1 states** ($\mathrm{SO}(3, \mathbf{Z})$ rep. of $L = 0$) below inelastic threshold (48^3 vol.)
  - $\Xi + \Xi \to \Xi^* + \Xi$ @ 0.29 GeV
  - $\Xi + \Xi \to \Omega + \Sigma$ @ 0.29 GeV

cf. $m_\pi = 0.51$ GeV

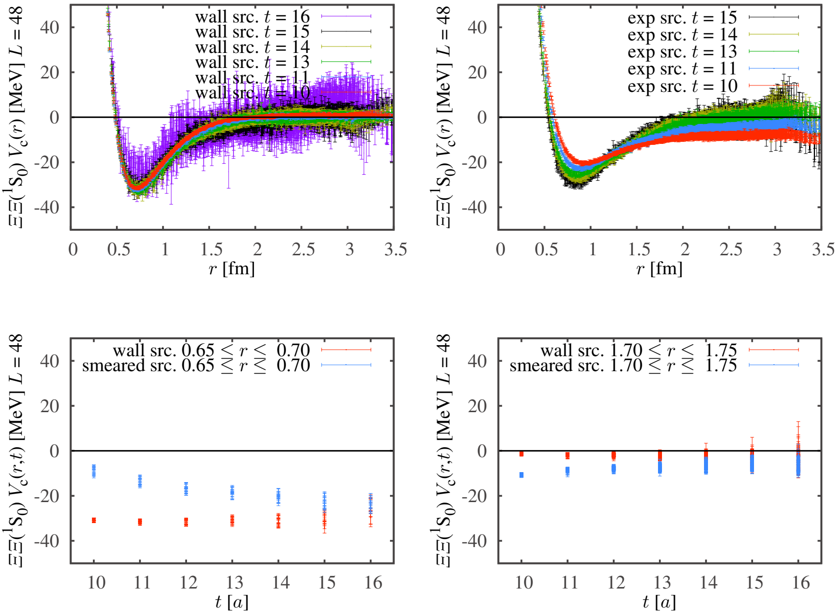
Excited States Contamination and "Fake-Plateau"

ex. $R(t) = b_0 \exp(-\Delta E_0 t) + b_1 \exp(-\Delta E_1 t)$

■ at $b_1/b_0 = -0.1$ — "plateau-like" behavior for $t = [12:20]$



t-dependence of Potential



Projection of Eigenstate: "Sink" Projection

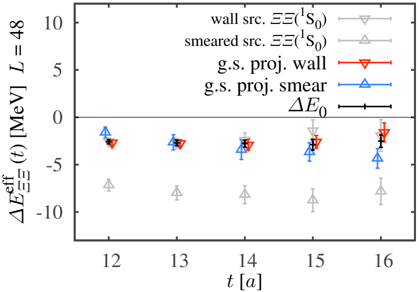
$$\tilde{R}^{(f)}(t) \equiv \sum_{\vec{r}} f(\vec{r}) \sum_{\vec{x}} \langle 0|B(\vec{r}+\vec{x},t)B(\vec{x})\bar{\mathcal{J}}(0)|0\rangle = \sum_{\vec{r}} f(\vec{r}) R(\vec{r},t)$$

and ($f(\vec{r})$: arbitrary func.)

$$\Delta\tilde{E}^{(f)}_{\mathrm{eff}}(t) = \log \frac{\tilde{R}^{(f)}(t)}{\tilde{R}^{(f)}(t+1)}$$

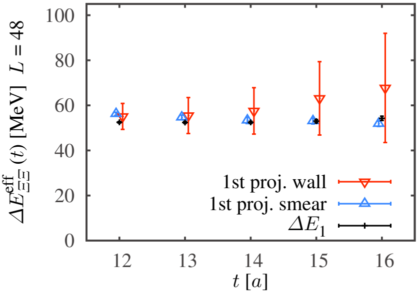
- ground state

$$f(\vec{r}) = \Psi^{\dagger}_{\mathrm{gs}}(\vec{r})$$



- 1st excited state

$$f(\vec{r}) = \Psi^{\dagger}_{1}(\vec{r})$$



Next Leading Order of Derivative Expansion

Derivative expansion: $U(r, r') = \{V_0(r) + V_1(r)\nabla^2\}\delta(r - r')$ (for $^1S_0$)

$$\left[\frac{1}{4m}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right] R(r,t) = \int d^3r' U(r,r') R(r',t)$$

$$\therefore \frac{1}{4m}\frac{(\partial^2/\partial t^2)R}{R} - \frac{(\partial/\partial t)R}{R} - \frac{H_0 R}{R} = V_0(r) + V_1(r)\frac{\nabla^2 R(r,t)}{R(r,t)} \equiv \tilde{V}_{\text{total}}(r,t)$$

- Now, we have R^{smear} and R^{wall}

$$\begin{cases} V_0(r) + V_1(r)\nabla^2 R^{\text{smear}}/R^{\text{smear}} = \tilde{V}^{\text{smear}}_{\text{total}}(r, t_{\text{smear}}) \\ V_0(r) + V_1(r)\nabla^2 R^{\text{wall}}/R^{\text{wall}} = \tilde{V}^{\text{wall}}_{\text{total}}(r, t_{\text{wall}}), \end{cases}$$

- LO $V_0(r)$ and NLO $V_1(r)$ potentials are given by

$$V_1(r) = \frac{\tilde{V}^{\text{smear}}_{\text{total}}(r, t_{\text{smear}}) - \tilde{V}^{\text{wall}}_{\text{total}}(r, t_{\text{wall}})}{\nabla^2 R^{\text{smear}}/R^{\text{smear}} - \nabla^2 R^{\text{wall}}/R^{\text{wall}}}$$

$$V_0(r) = \tilde{V}^{\text{smear}}_{\text{total}}(r, t_{\text{smear}}) - V_1(r)\frac{\nabla^2 R^{\text{smear}}}{R^{\text{smear}}}.$$

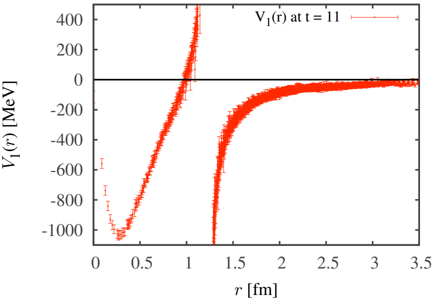
- $\nabla^2 R/R - \nabla^2 R/R$ in denominator

Results: NLO Potential

■ results of HAL QCD method

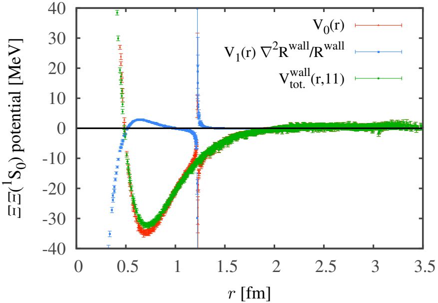
▶ source independent

$$V_{\text{total}}(r,t) = V_0(r) + V_1(r)\frac{\nabla^2 R(r,t)}{R(r,t)}$$



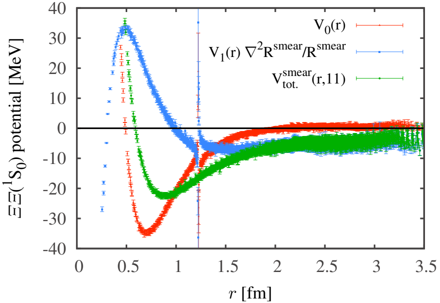
▶ **wall source**

- Good convergence of **LO pot.**



▶ **smeared source**

- **with NLO pot. correction**



Results:NLO Potential (1) $V_0(r)$ and $V_1(r)$

$$V_0(r) = \tilde{V}^{\text{exp.}}_{\text{total}} - V_1(r)\frac{\nabla^2 R^{\text{exp.}}}{R^{\text{exp.}}}, \quad V_1(r) = \frac{\tilde{V}^{\text{exp.}}_{\text{total}} - \tilde{V}^{\text{wall}}_{\text{total}}}{\nabla^2 R^{\text{exp.}}/R^{\text{exp.}} - \nabla^2 R^{\text{wall}}/R^{\text{wall}}}$$

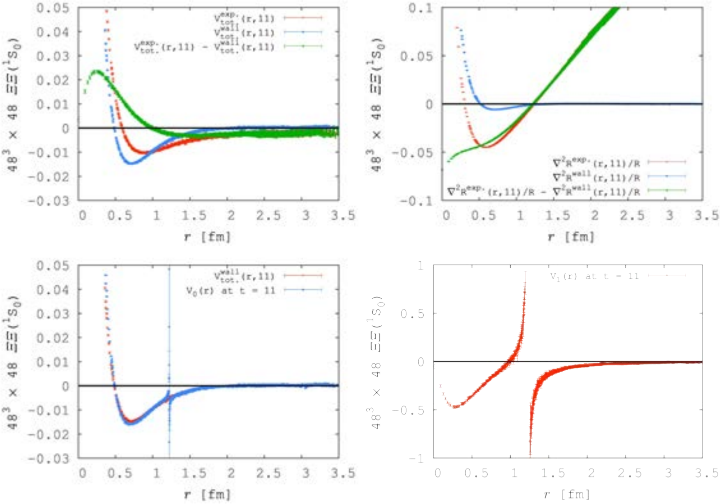


Figure: (a) diff. of $\tilde{V}_{\text{total}}$ (b) diff. of $\nabla^2 R/R$ (c) $V_0(r)$ and $\tilde{V}^{\text{wall}}$ (d) $V_1(r)$

Results:NLO Potential (2) V_0 and V_1 in smeared and wall

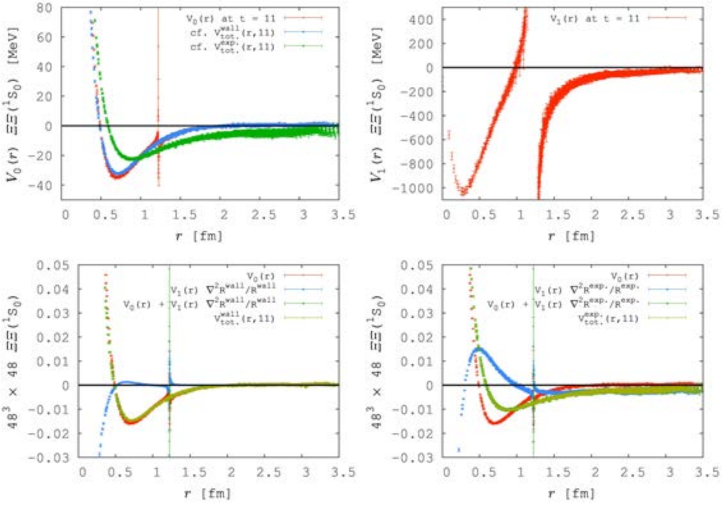


Figure: (a) $V_0(r)$ (b) $V_1(r)$ (c) wall src. summary (d) smeared src. summary

Results:NLO Potential (3) SVD

solving SVD with several t's

$$\begin{pmatrix} 1 & \nabla^2 R^{\text{exp.}}(r,t_1)/R^{\text{exp.}}(r,t_1) \\ 1 & \nabla^2 R^{\text{wall}}(r,t_1)/R^{\text{wall}}(r,t_1) \\ 1 & \nabla^2 R^{\text{exp.}}(r,t_2)/R^{\text{exp.}}(r,t_2) \\ 1 & \nabla^2 R^{\text{wall}}(r,t_2)/R^{\text{wall}}(r,t_2) \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = \begin{pmatrix} \tilde{V}^{\text{exp.}}_{\text{total}}(r,t_1) \\ \tilde{V}^{\text{wall}}_{\text{total}}(r,t_1) \\ \tilde{V}^{\text{exp.}}_{\text{total}}(r,t_2) \\ \tilde{V}^{\text{wall}}_{\text{total}}(r,t_2) \\ \vdots \end{pmatrix}$$

$$A \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = B \longrightarrow U\Sigma V^t \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = B,$$

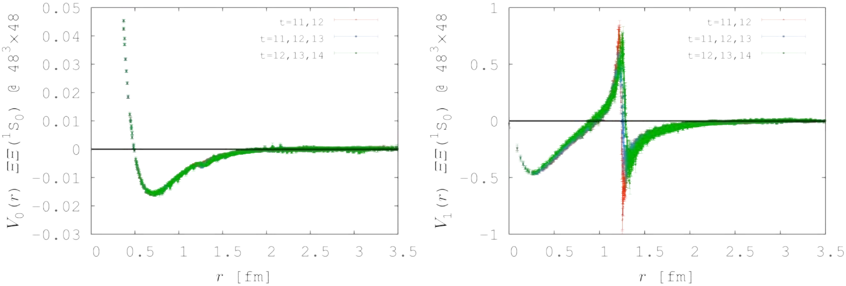


Figure: (a) $V_0(r)$ (b) $V_1(r)$ using SVD

Results:NLO Potential (4) Energy Eigenvalues

$\tilde{V}^{\rm wall}_{\rm tot}$, $V_0(r)$ and $V_0(r)+V_1(r)\nabla^2$
energy eigenvalues are consistent with each other
$\Rightarrow$ wall source potential $\simeq$ LO

volume at $48^3 \times 48$	t	g.s. [MeV]	1st [MeV]
$\tilde{V}^{\rm wall}_{\rm total}(r)$	11	-2.30(20)	53.08(29)
	12	-2.58(22)	52.87(33)
	13	-2.70(30)	52.80(41)
$\tilde{V}^{\rm exp.}_{\rm total}(r)$	13	-5.33(59)	49.86(58)
	14	-4.76(55)	50.46(54)
$V_0(r)$	11,12,13	-2.47(23)	52.91(32)
	12,13,14	-2.62(28)	52.86(37)
$V_0(r)+V_1(r)\nabla^2$	11,12,13	-2.56(1.56)	53.07(2.73)
	12,13,14	-2.74(2.52)	51.58(3.77)

Table: The energy eigenvalues from "potential method".