

Three-particle scattering from the lattice

Maxwell T. Hansen

Institut für Kernphysik and HIM

Johannes Gutenberg Universität

Mainz, Germany

INT, Seattle

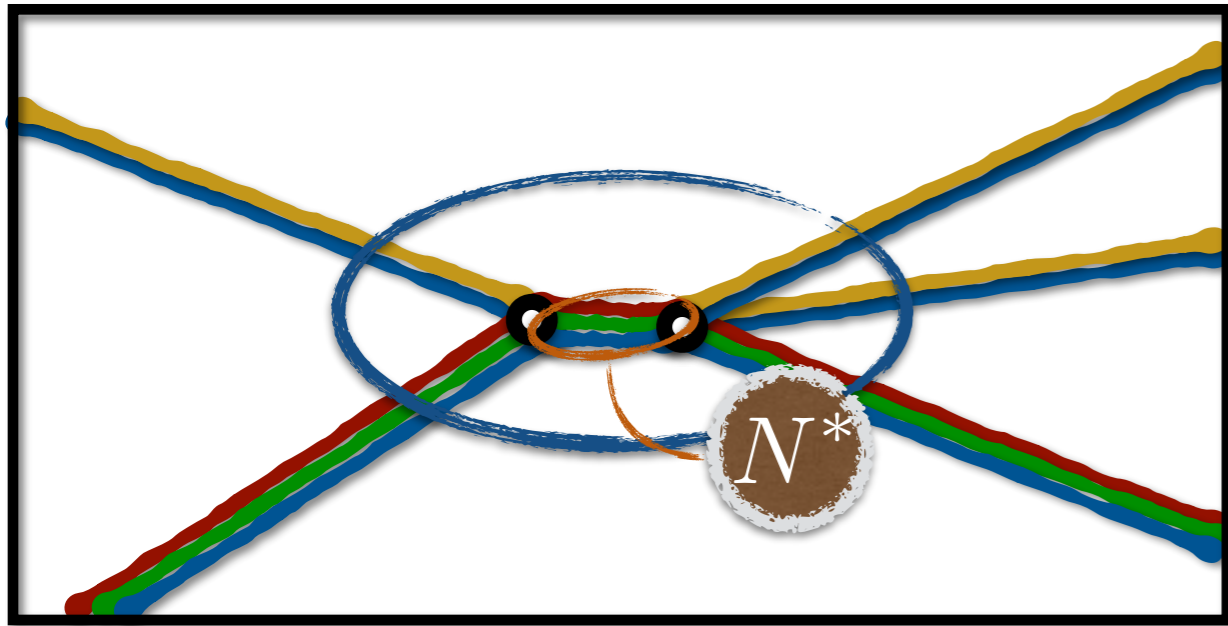
March 30, 2016



Helmholtz-Institut Mainz



Importance of three-particle scattering



Many QCD resonances decay significantly to three particles

$$N\pi \rightarrow N^* \rightarrow N\pi\pi$$

Ideally we would like to describe resonances by...

- (1). **Determining QCD scattering amplitudes** in a rigorous and model-independent way
- (2). **Analytically continuing** these to the resonance poles

Amplitudes from the path-integral

If we were strong enough, we would proceed as follows

(1). Evaluate the path-integral to obtain the relevant correlator

$$\begin{aligned} & \langle N(x')\pi(y')\pi(z') N(x)\pi(y) \rangle \\ &= \int \mathcal{D}A\mathcal{D}q\mathcal{D}\bar{q} \exp[iS_{QCD}] N(x')\pi(y')\pi(z') N(x)\pi(y) \end{aligned}$$

(2). Fourier transform and apply LSZ reduction

$$\langle \tilde{N}(p')\tilde{\pi}(k')\tilde{\pi}(q') \tilde{N}(p)\tilde{\pi}(k) \rangle \longrightarrow \frac{iZ_N^{1/2}}{p'^2 - m_N^2} \cdots \frac{iZ_\pi^{1/2}}{k^2 - M_\pi^2} \langle N\pi\pi, \text{out} | N\pi, \text{in} \rangle$$

This approach requires...

Infinite volume to define asymptotic states $\frac{1}{k^2 - M_\pi^2}$

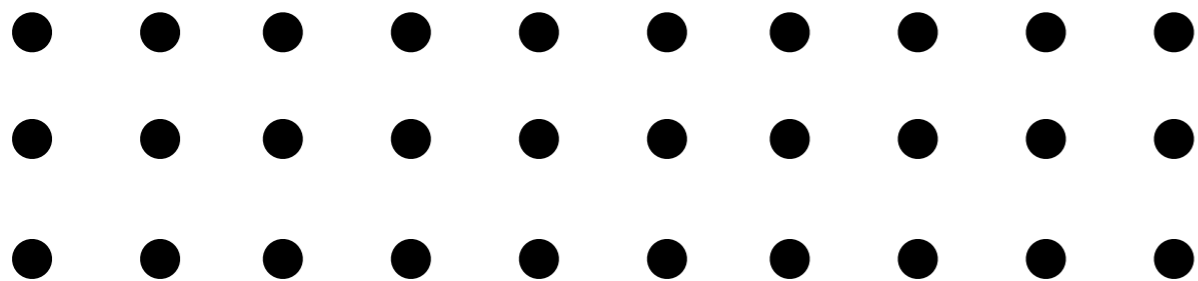
Minkowski momenta to approach the poles (“go on shell”)

$$k^2 \rightarrow M_\pi^2$$

In Lattice QCD we are evaluating the path integral numerically...

To do so we have to make four compromises

1 nonzero lattice spacing



Must ensure this is smaller than all relevant length scales

2 Unphysical pion masses

$$M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$$

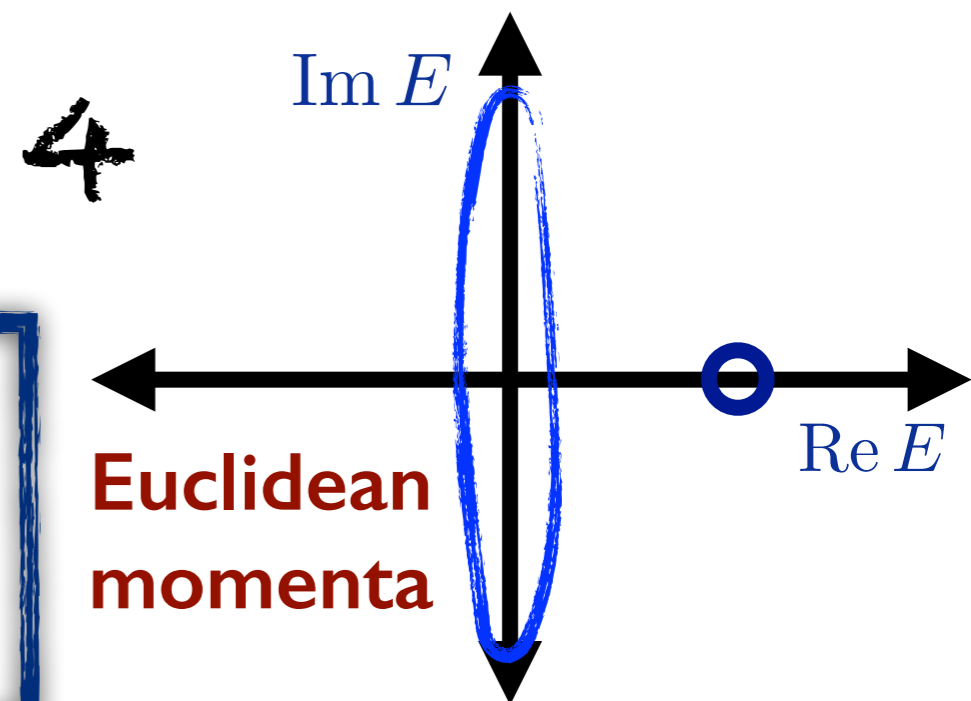
But calculations at the physical pion mass do now exist...

and exploring pion mass dependence is interesting

3 finite volume, L



LQCD cannot directly access scattering amplitudes... but it can give finite-volume energies

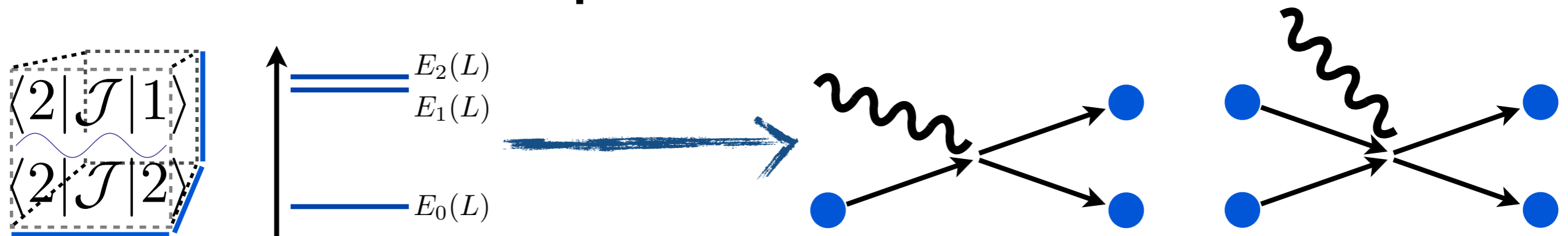


It is possible to derive relations between finite- and infinite-volume physics

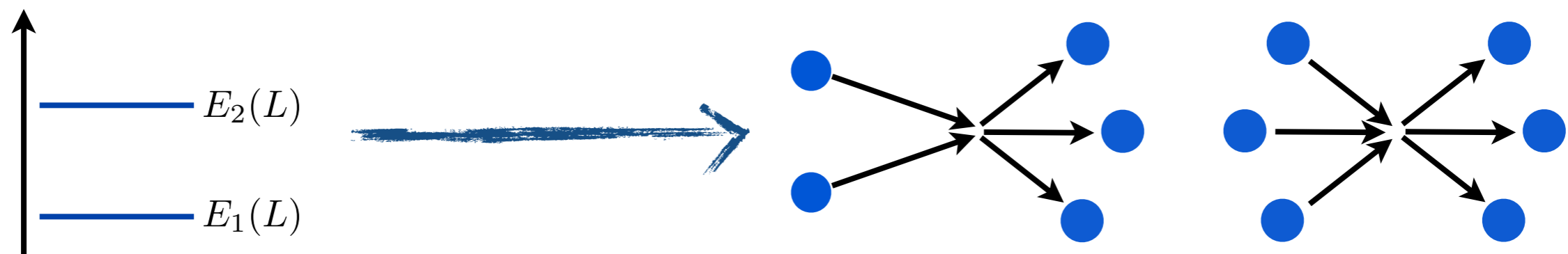
Two-particle scattering



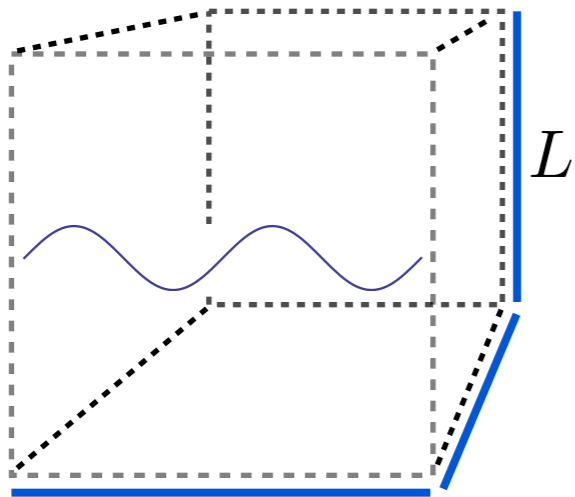
Photo- and electroproduction



Three-particle scattering

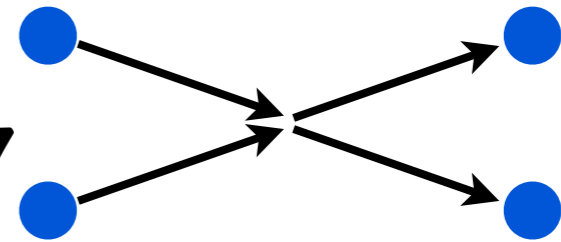
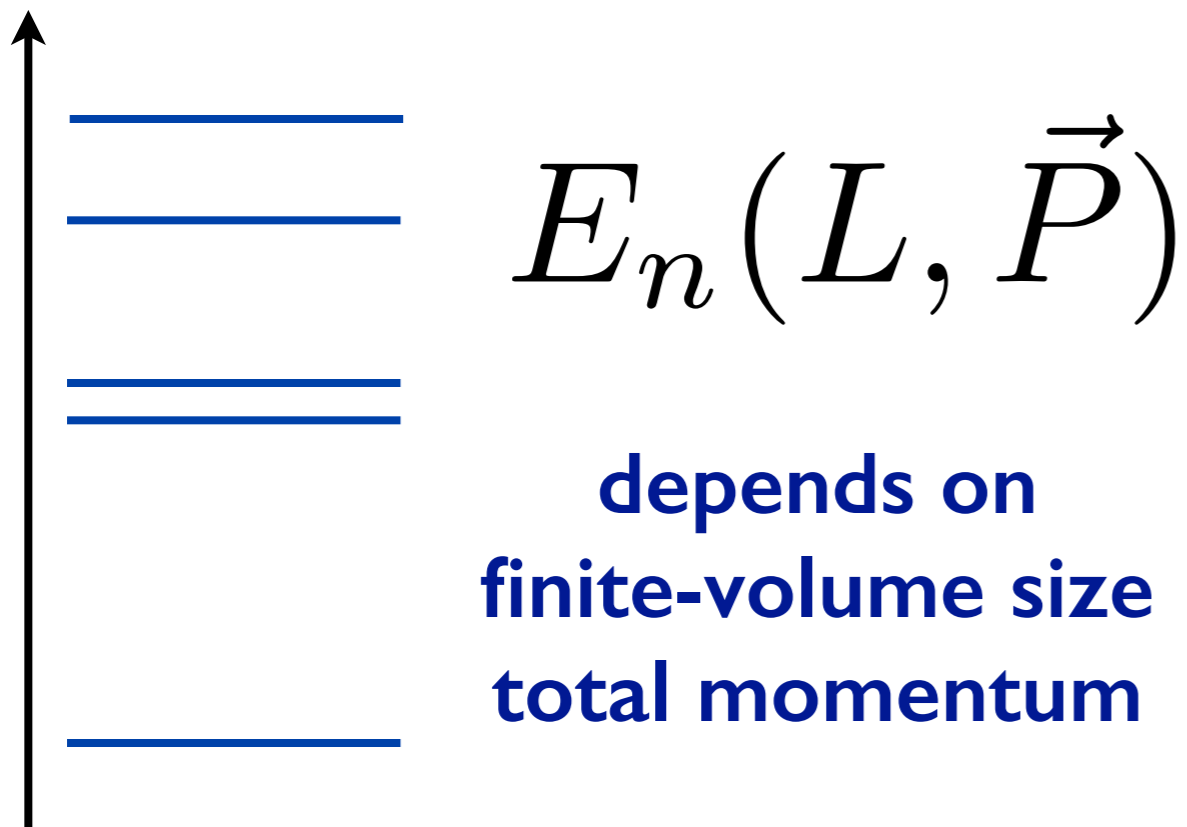


From energy levels to amplitudes



Finite volume

Discrete tower of energy levels



Infinite volume

Decompose scattering amplitude in partial waves

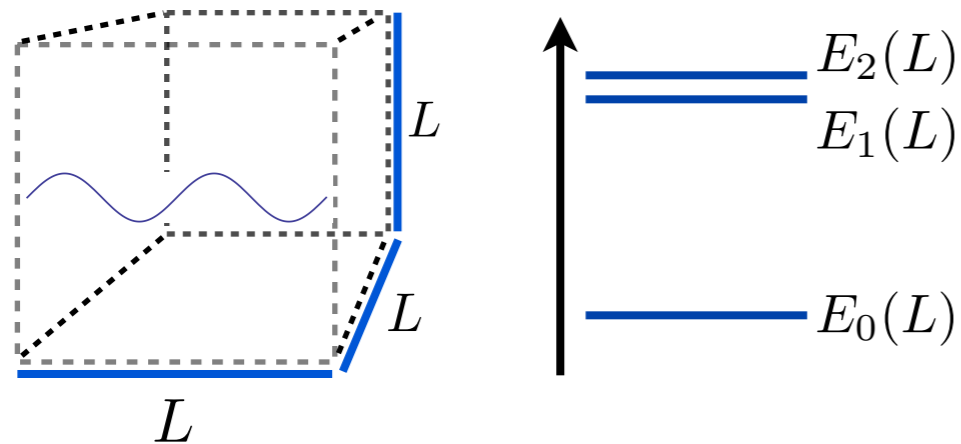
One real observable...

$$\delta_\ell(E^*)$$

in each partial wave

at each CM energy

Finite volume



cubic, spatial volume (extent L)

periodic boundary conditions

$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite**

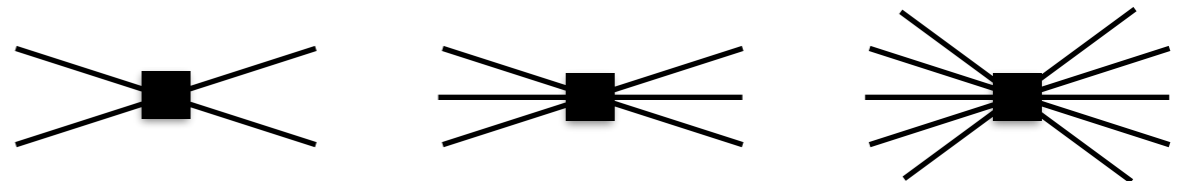


L large enough to ignore e^{-mL}

quantum field theory

generic relativistic QFT

1. Include all interactions



2. no power-counting scheme

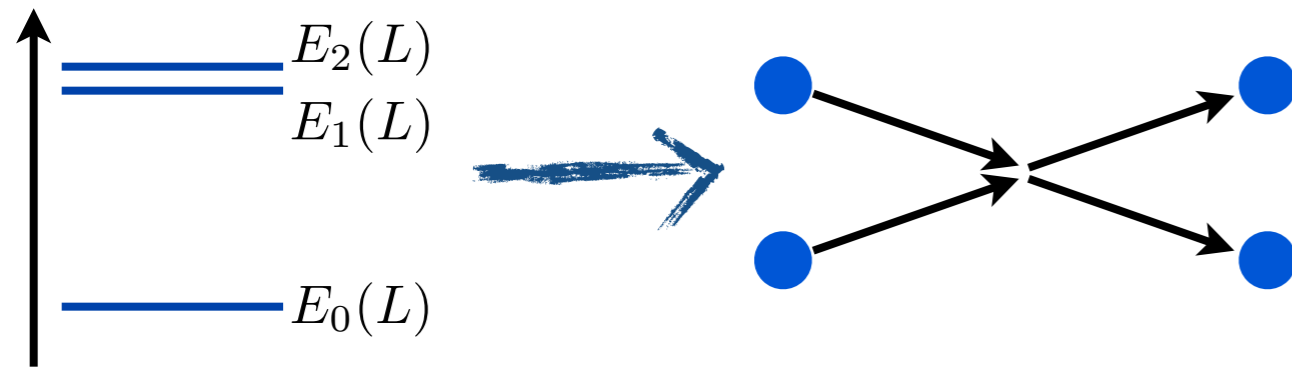
Not possible to directly calculate scattering observables to all orders

But it is possible to derive general, all-orders relations to finite-volume quantities

Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

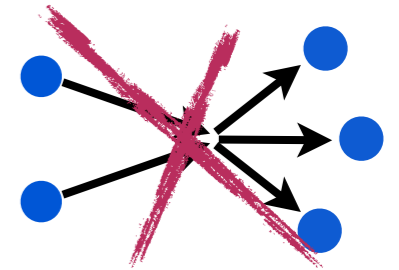
Two-to-two scattering



For now assume...

identical scalars, mass m

\mathbb{Z}_2 symmetry



$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Euclidean convention

two-particle interpolator

$$P = (P_4, \vec{P}) = (P_4, 2\pi\vec{n}/L)$$

but allow P_4 to be real or imaginary

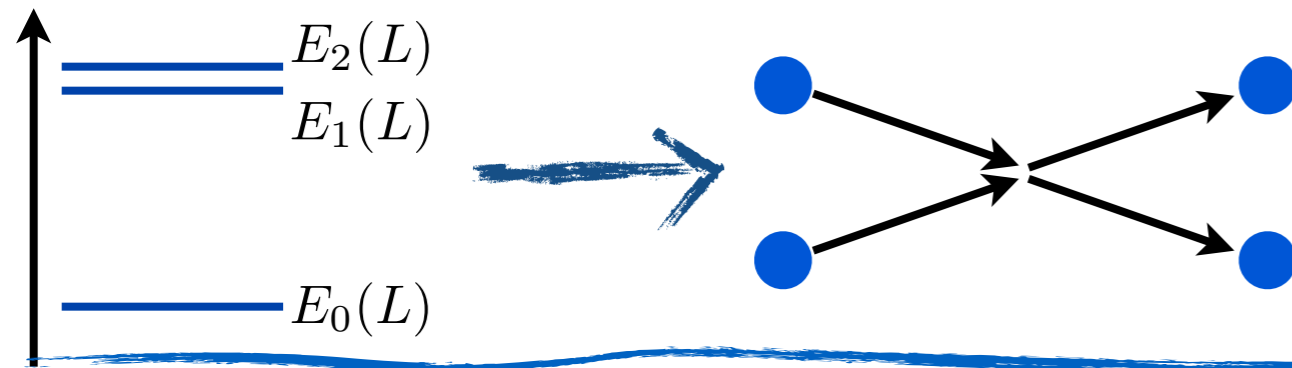
CM frame energy is then $E^{*2} = -P_4^2 - \vec{P}^2$

Require $E^* < 4m$ to isolate two-to-two scattering

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

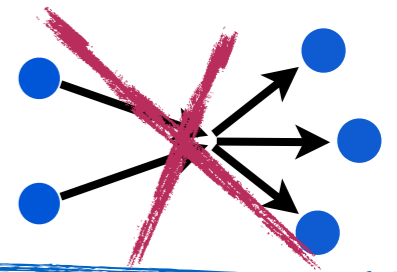
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Two-to-two scattering



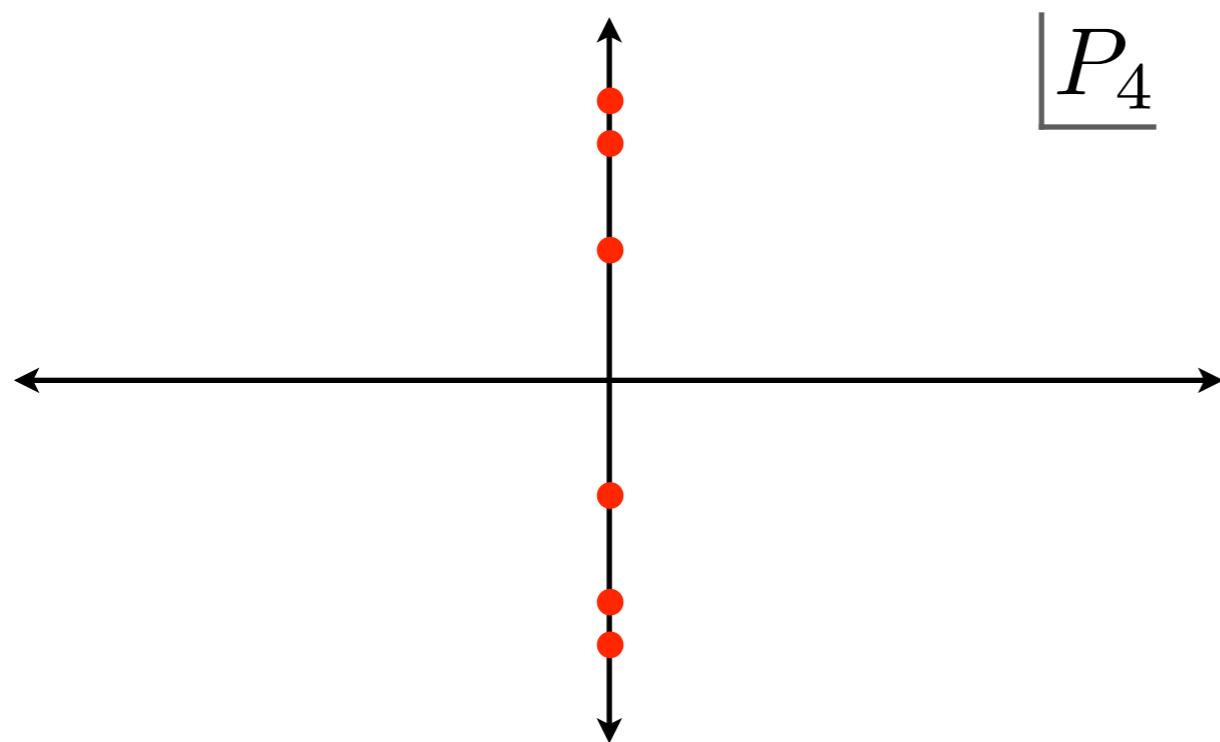
For now assume...
identical scalars, mass m

\mathbb{Z}_2 symmetry

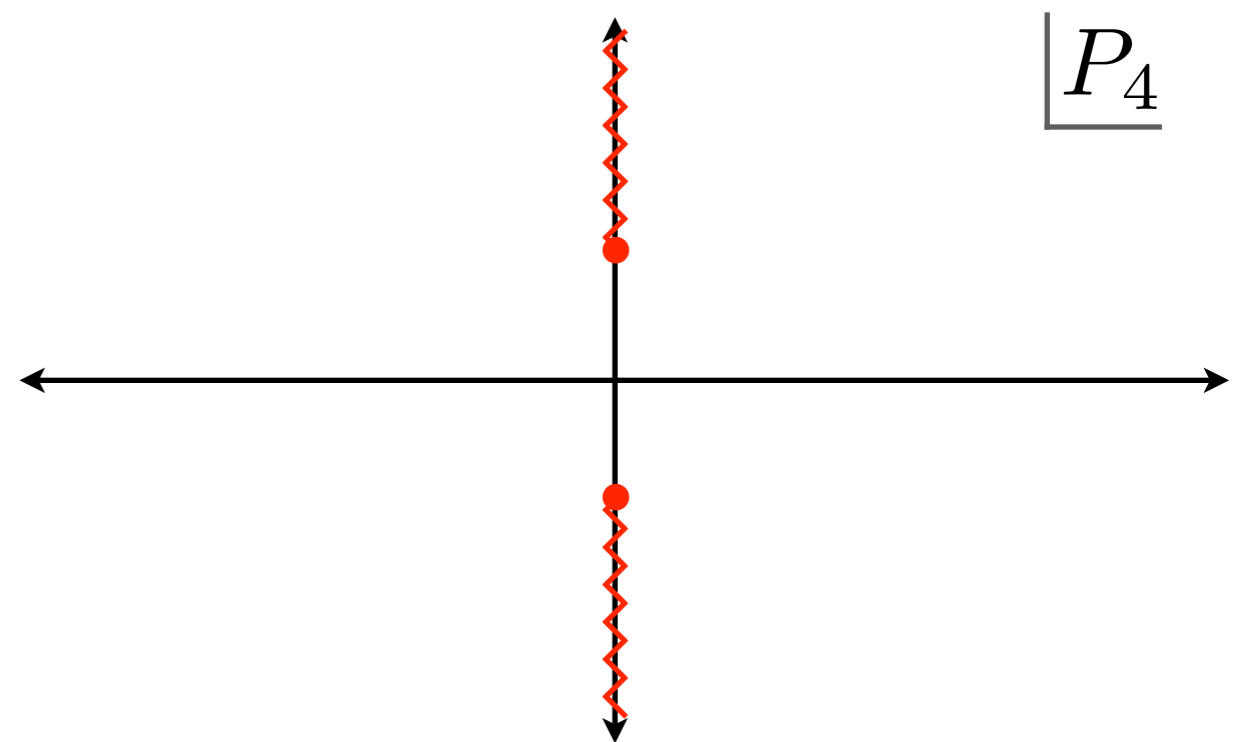


$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

At fixed L, \vec{P} , poles in C_L give finite-volume spectrum

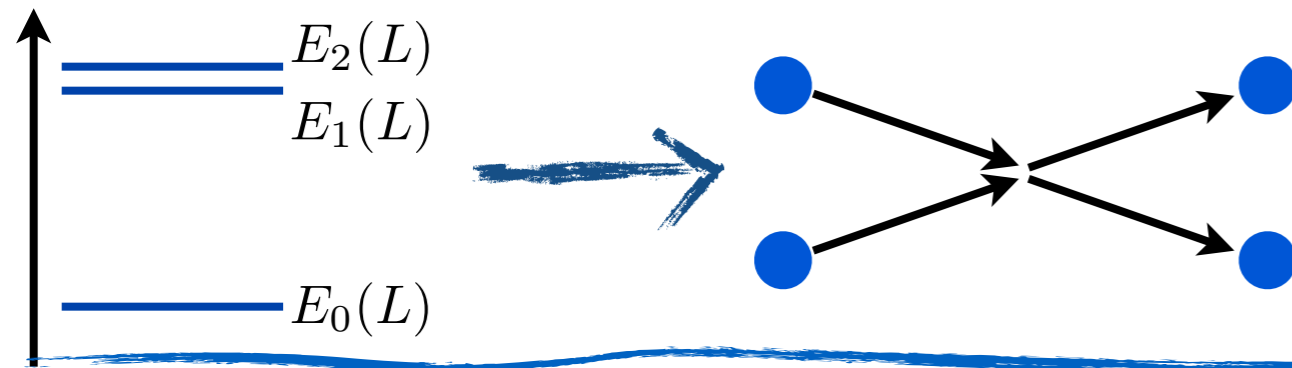


C_L analytic structure



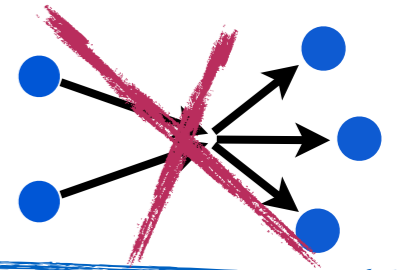
C_∞ analytic structure

Two-to-two scattering



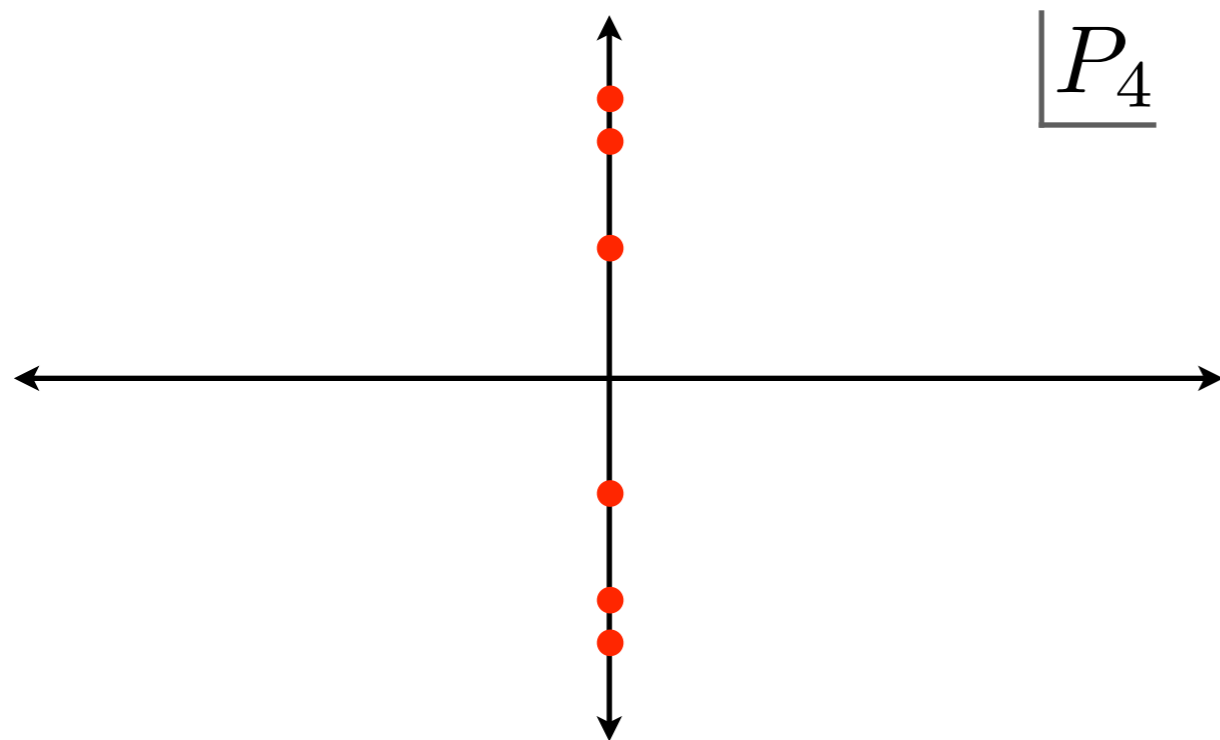
For now assume...
identical scalars, mass m

\mathbb{Z}_2 symmetry



$$C_L(P) \equiv \int_L d^4x e^{-iPx} \langle 0|T\mathcal{O}(x)\mathcal{O}^\dagger(0)|0\rangle$$

At fixed L, \vec{P} , poles in C_L give finite-volume spectrum



C_L analytic structure

Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

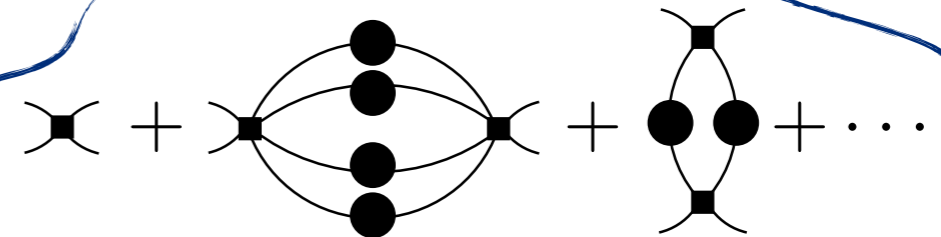
The first diagram shows a circle labeled \mathcal{O}^\dagger connected to a circle labeled \mathcal{O} via two vertices, each with two external legs. The second diagram is similar but includes a circle labeled iK between the two vertices. The third diagram includes two iK circles in series. Blue dashed boxes highlight the vertices in the first two diagrams, and a blue arrow points from the first diagram to the second.

spatial loop momenta
are summed

$$\frac{1}{L^3} \sum_{\vec{k} \in (2\pi/L)\mathbb{Z}^3} \int \frac{dk^0}{2\pi}$$

$\Delta \equiv \text{diagram}$
fully dressed
propagator

The diagram shows a horizontal line with a central black dot, representing a propagator.



if $E^* < 4m$ **then**

$$K_L = K_\infty + \mathcal{O}(e^{-mL})$$

$$\Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL})$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The equation shows a series of diagrams representing terms in a sum. Each diagram consists of two large circles (representing source and sink operators) connected by two smaller circles (representing propagators). The first diagram has two black dots on the left propagator. The second diagram has two black dots on the left propagator and two black dots on the right propagator. The third diagram has two black dots on the left propagator, two on the right propagator, and two on a third propagator in the middle. Blue arrows point from the first two diagrams to the corresponding terms in the equation above.

$$\frac{1}{L^3} \sum_{\vec{k}} \text{diagram}_1 = \text{diagram}_2 + \underbrace{\text{diagram}_3}_F$$

The diagram on the left shows two large circles connected by two smaller circles, with two black dots on the left propagator. This is equal to the sum of two diagrams. The first diagram is identical to the left one. The second diagram shows two large circles connected by a single horizontal line, with a vertical dashed line through the center labeled 'F'. A blue bracket under the 'F' diagram is labeled "contains all power-law corrections".

Now we introduce an important identity.

In  all four-momenta are projected on shell.

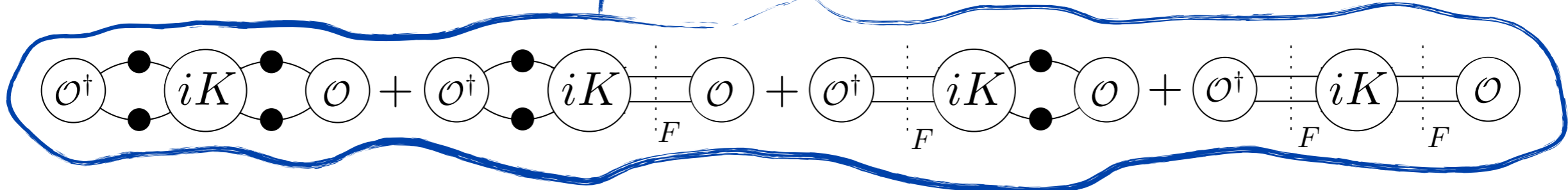
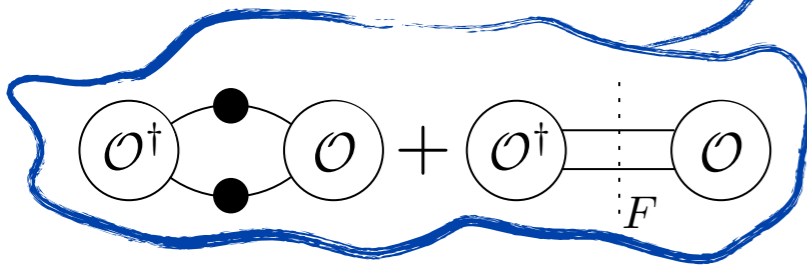
Physical, propagating states give dominate finite-volume effects.

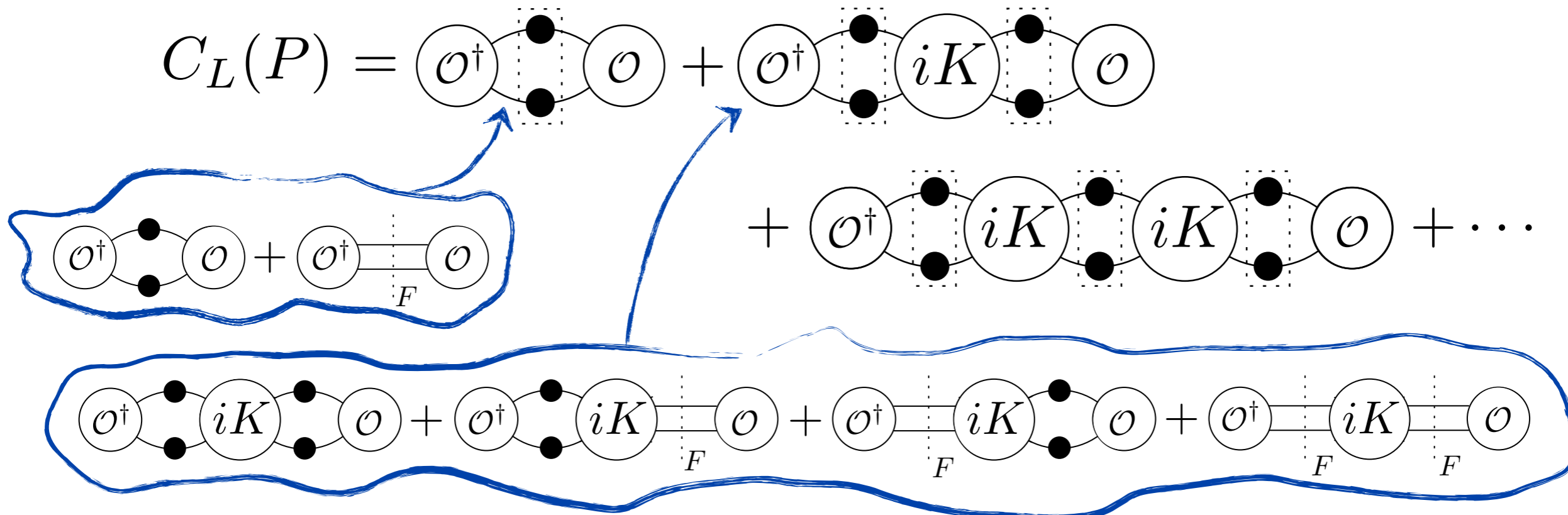
Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

$$C_L(P) = \begin{array}{c} \circlearrowleft \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright \mathcal{O} \\ + \circlearrowleft \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright iK \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright \mathcal{O} \end{array}$$

$$+ \begin{array}{c} \circlearrowleft \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright iK \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright iK \begin{array}{c} \bullet \\ \bullet \end{array} \circlearrowright \mathcal{O} \\ + \dots \end{array}$$





Now regroup by number of Fs

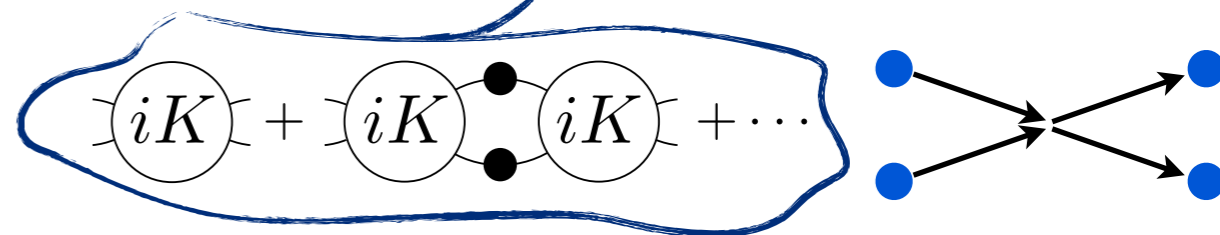
$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \text{diagram}_1 + \text{diagram}_2 + \dots$$

zero Fs
one F
two Fs

The diagram shows the expansion of $C_L(E, \vec{P})$ regrouped by the number of F vertices. The first term is $C_\infty(E, \vec{P})$. The second term has A and A' connected by a line with a vertical dashed line labeled F below it. The third term has A , iM , and A' connected by lines with vertical dashed lines labeled F below each. Ellipses follow. Labels 'zero Fs', 'one F', and 'two Fs' are placed above the terms.

$$= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$$

The diagram shows a chain of \mathcal{O}^\dagger and \mathcal{O} vertices connected by lines. A blue box encloses the chain. Below the box is the equation $= \langle \pi\pi, \text{out} | \mathcal{O}^\dagger | 0 \rangle$.

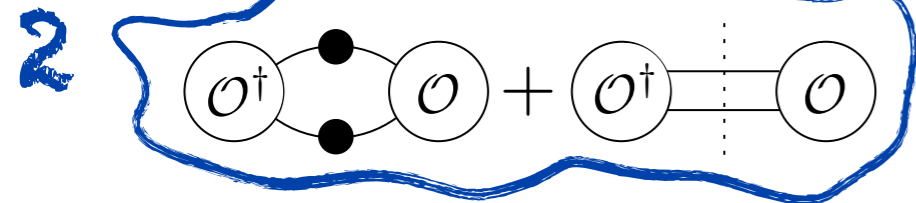


When we factorize diagrams and group infinite-volume parts...

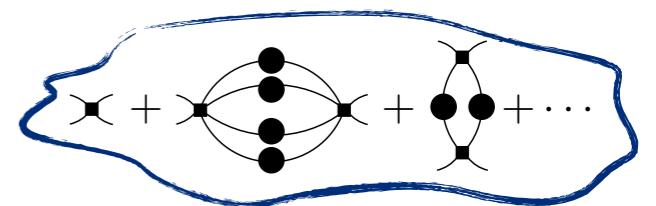
physical observables emerge!

Review...

$$C_L(P) = \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O}$$

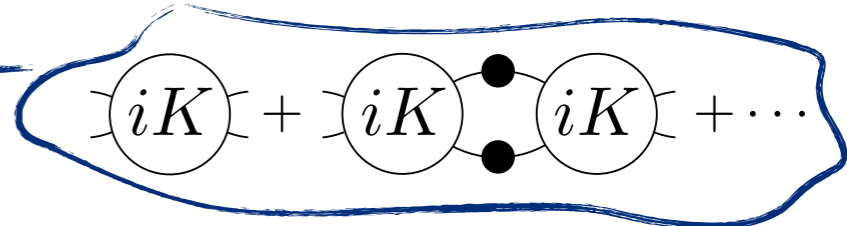
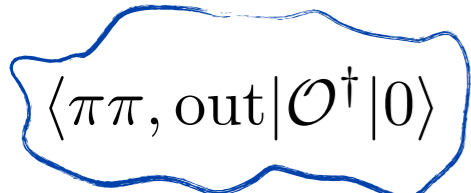


$$+ \mathcal{O}^\dagger \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} iK \begin{array}{c} \bullet \\ \bullet \end{array} \mathcal{O} + \dots$$

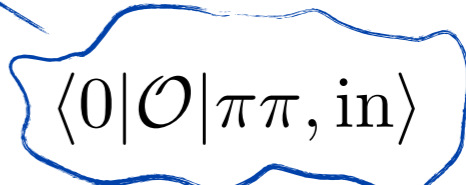


$$C_L(P) = C_\infty(P)$$

$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array} + \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array}$$



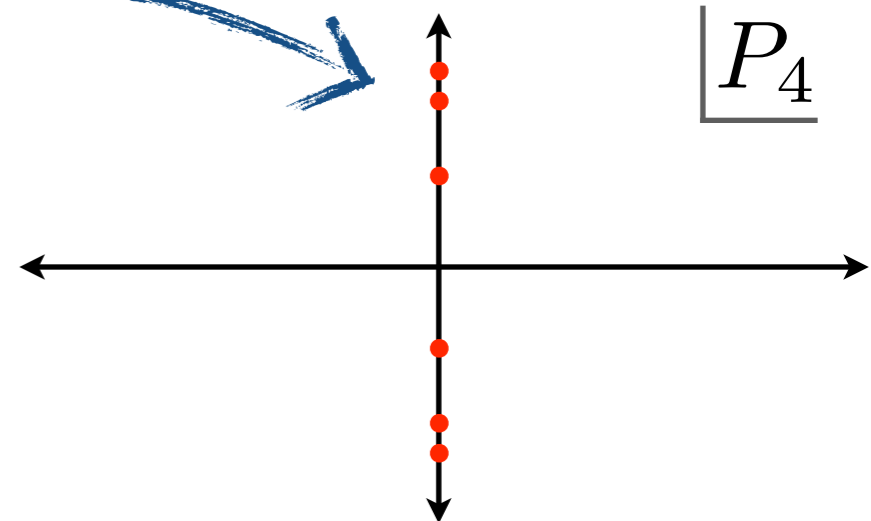
$$+ \begin{array}{c} A \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} i\mathcal{M} \\ \vdots \\ F \end{array} \begin{array}{c} A' \\ \vdots \\ F \end{array} + \dots$$



We deduce...

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2 \rightarrow 2} F} A$$

poles are in here



Two-particle result

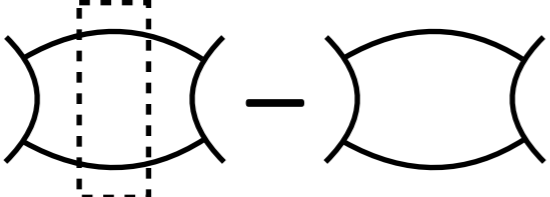
At fixed (L, \vec{P}) , finite-volume energies are solutions to $\det[\mathcal{M}_{2 \rightarrow 2}^{-1} + F] = 0$

Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)
 Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Matrices defined using angular-momentum states

$\mathcal{M}_{2 \rightarrow 2} \equiv$  diagonal matrix, parametrized by $\delta_\ell(E^*)$

$F \equiv$ non-diagonal matrix of known geometric functions

\equiv  difference of two-particle loops in finite and infinite volume depends on L, E, \vec{P}

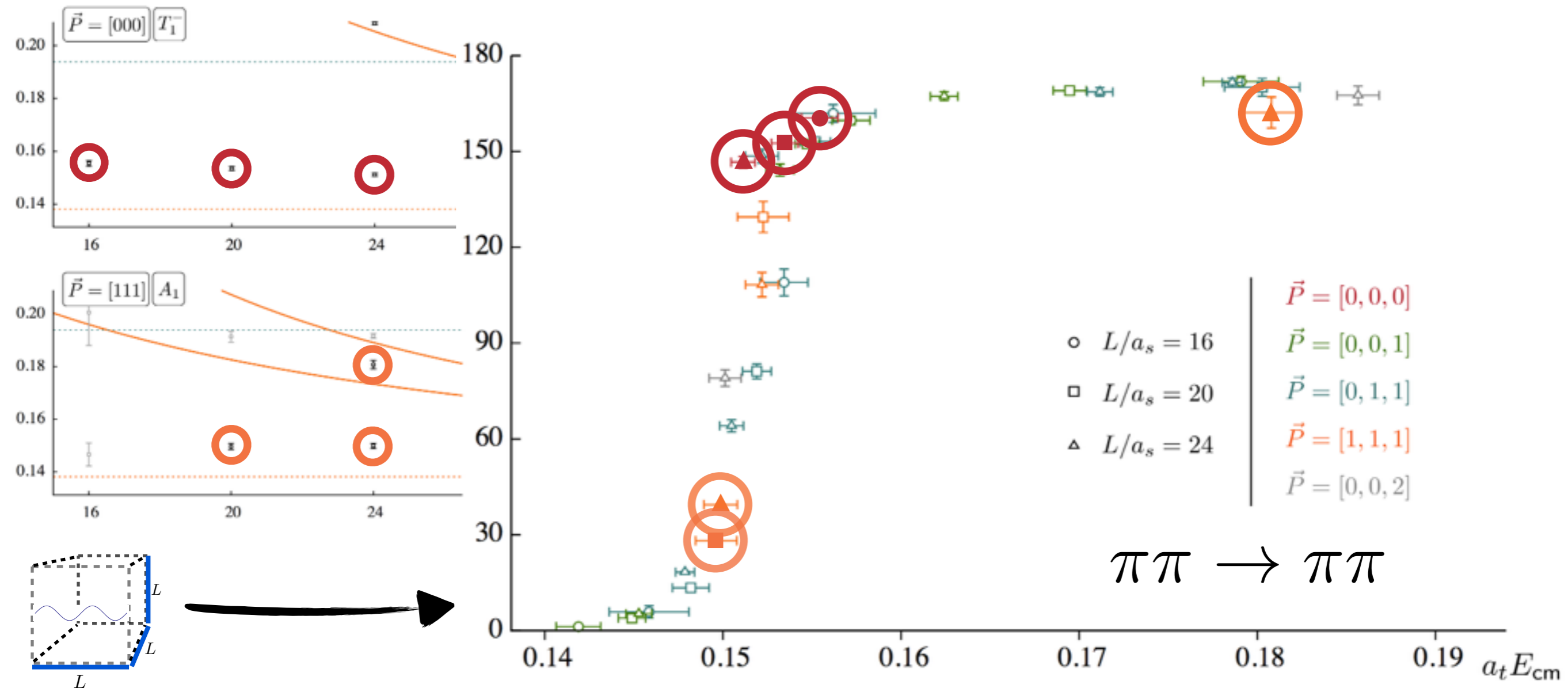
At low energies, lowest partial waves dominate $\mathcal{M}_{2 \rightarrow 2}$

e.g. s-wave only
 with some rearranging \rightarrow $\cot \delta(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$

scattering phase known function

Using the result (p-wave)

$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$

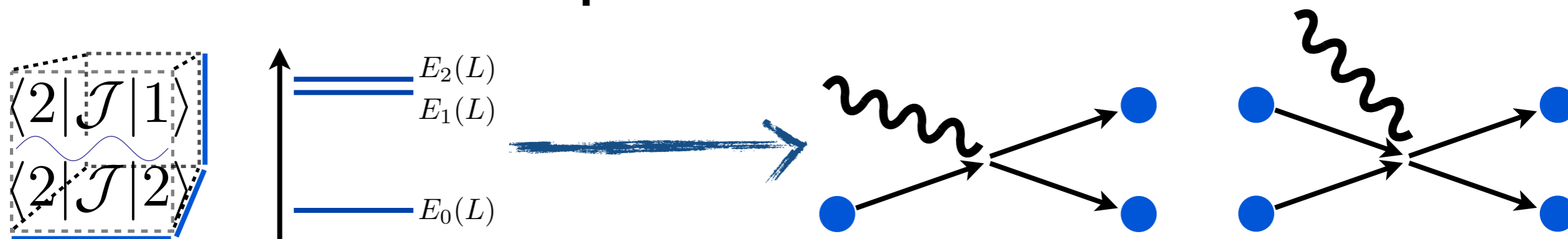


from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

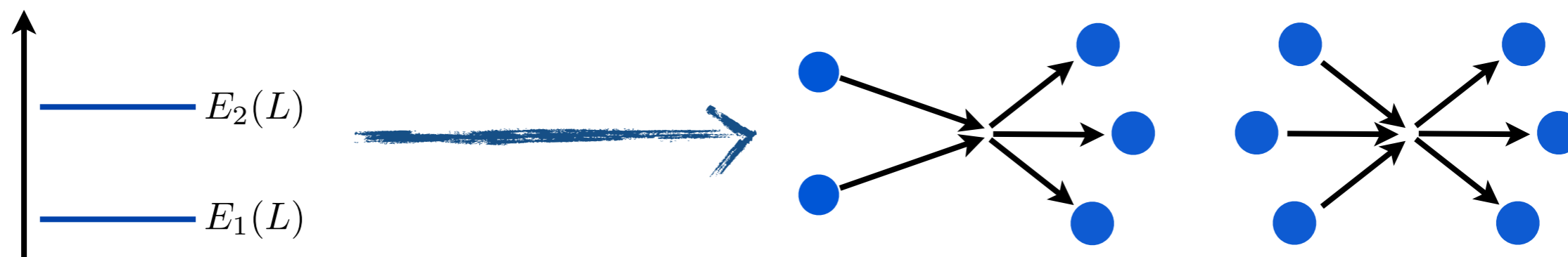
Two-particle scattering



Photo- and electroproduction



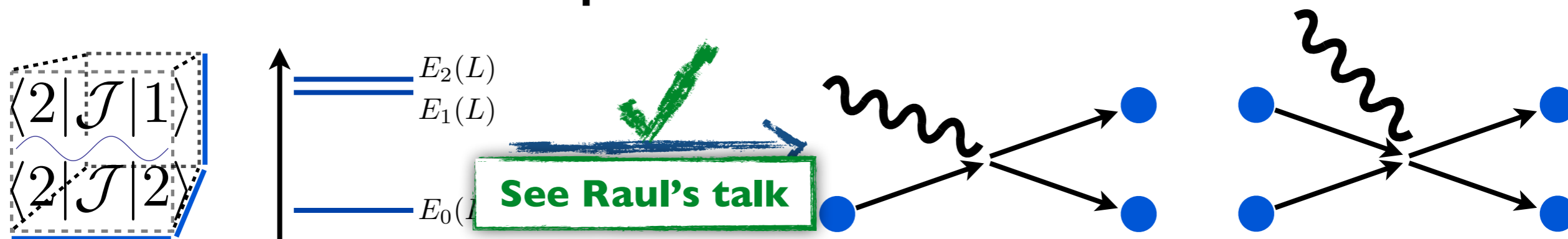
Three-particle scattering



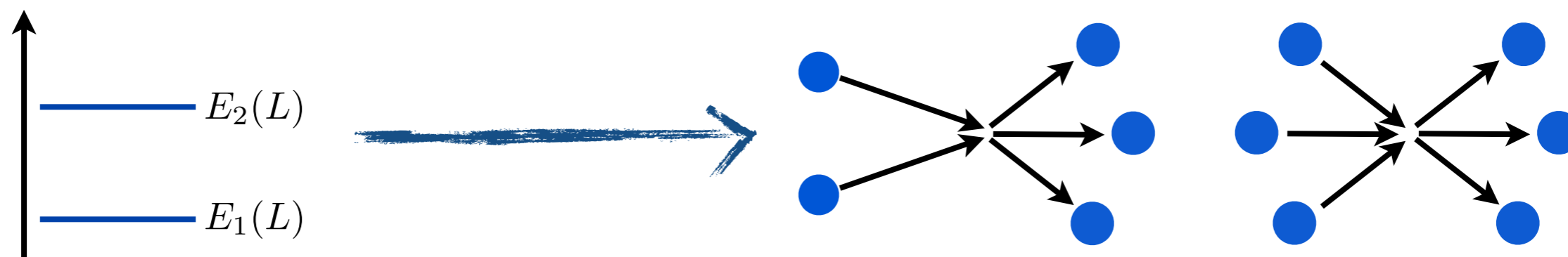
Two-particle scattering



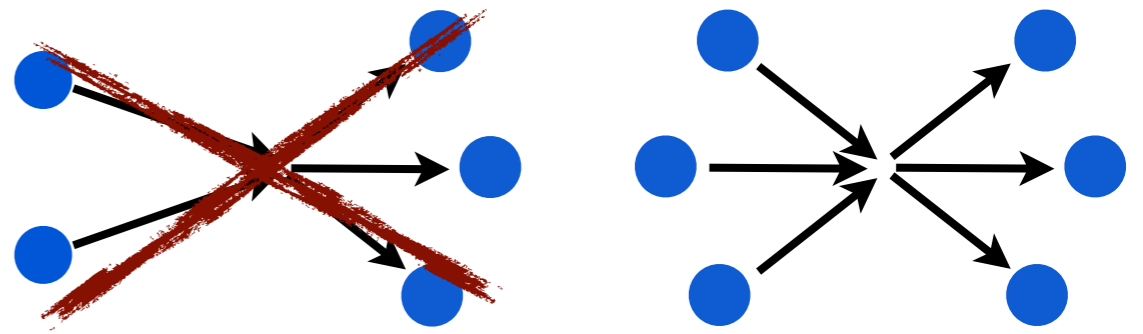
Photo- and electroproduction



Three-particle scattering



Begin by considering the infinite-volume observables

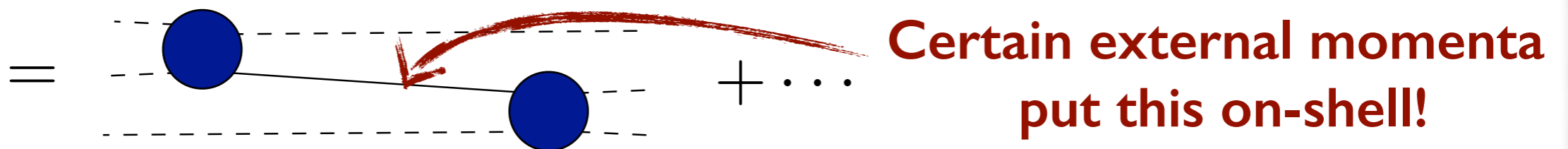


Because of “finite-volume rescattering” it is not possible to access two-to-three without also accessing three-to-three

For now we turn off two-to-three scattering using a symmetry

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\rightarrow 3} \equiv$ fully connected correlator with six external legs amputated and projected on shell



Three-to-three amplitude has more degrees of freedom

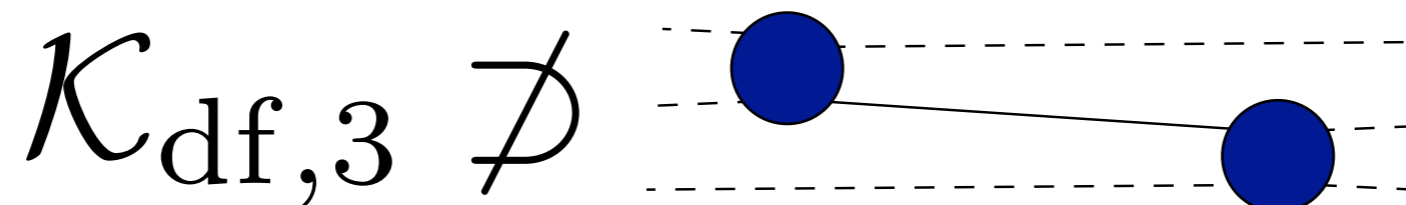
8 degrees of freedom including total energy

Compared with 2 for the two-to-two amplitude

How can we possibly hope to extract a **singular, eight-coordinate function** using finite-volume energies?

Short answer...

(1). We found that the spectrum depends on a modified quantity with singularities removed

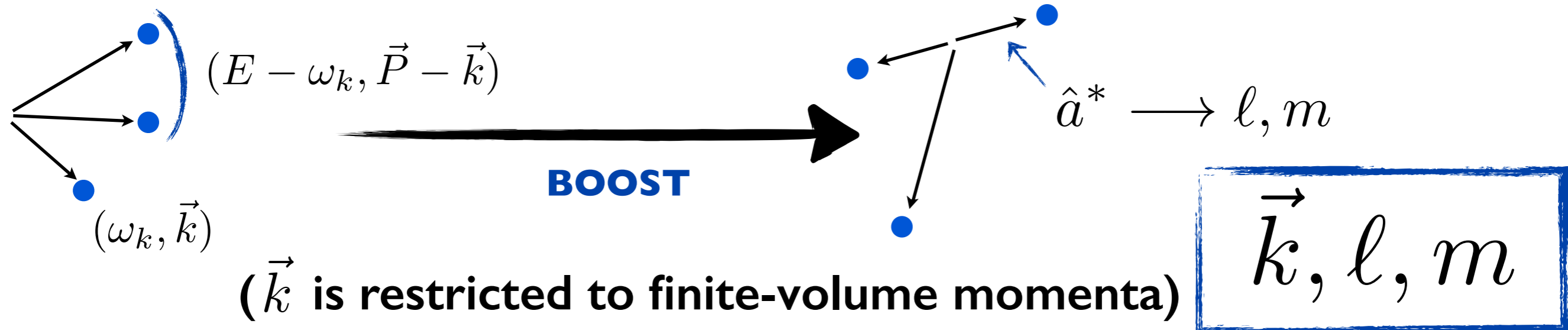


(a) Same degrees of freedom as $\mathcal{M}_{3 \rightarrow 3}$.

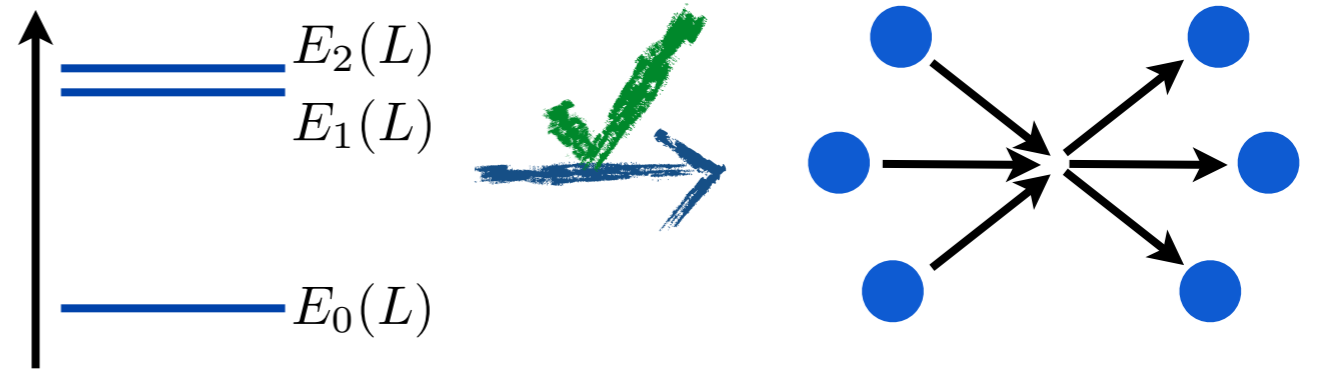
(b) Relation to $\mathcal{M}_{3 \rightarrow 3}$ is known (depends only on on-shell $\mathcal{M}_{2 \rightarrow 2}$)

(c) Smooth function (allows harmonic decomposition)

(2). Degrees of freedom encoded in an extended matrix space



Three-to-three scattering



Current status:

Formalism is complete for the simplest three-scalar system

General, model-independent relation between finite-volume energies and three-to-three scattering amplitude

Derived using a generic relativistic field theory

MTH and Sharpe, *Phys. Rev. D* 90, 116003 (2014)

MTH and Sharpe, *Phys. Rev. D* 92, 114509 (2015)

Important caveats:

Identical particles with no two-to-three transitions

$$\pi\pi\pi \rightarrow \pi\pi\pi$$

Requires that two-particle scattering phase is bounded

$$|\delta_\ell(E)| < \pi/2$$

Three-particle result

At fixed (L, \vec{P}) , finite-volume energies are solutions to $\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$

MTH and Sharpe, *Phys. Rev. D*90, 116003 (2014)

$F_3 \equiv$ matrix that depends on known geometric functions as well as $\mathcal{M}_{2 \rightarrow 2}$.

- (1). Use two-particle quantization condition to constrain $\mathcal{M}_{2 \rightarrow 2}$ and thus determine $F_3(E, \vec{P}, L)$
- (2). Use harmonic decomposition + various parametrizations to express $\mathcal{K}_{\text{df},3}(E^*)$ in terms of N unknown parameters
- (3). Use quantization condition with lattice (or otherwise) determined energies to determine all parameters
- (4). Use known relation to recover $\mathcal{M}_{3 \rightarrow 3}$

MTH and Sharpe, *Phys. Rev. D*92, 114509 (2015)

Three-particle result

At fixed (L, \vec{P}) , finite-volume energies are solutions to $\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$

MTH and Sharpe, *Phys. Rev. D*90, 116003 (2014)

Some nice features...

Matrices automatically truncated in the \vec{k} index

**truncate angular
momentum space**



solvable system

Expanding about weak interactions gives an important check

$$E = 3m + \frac{a_3}{L^3} + \frac{a_4}{L^4} + \frac{a_5}{L^5} + \frac{a_6}{L^6} + \mathcal{O}(1/L^7)$$

Our result agrees with existing results for $a_{3 \rightarrow 5}$ and gives a prediction for a_6

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

Beane, Detmold, Savage, *Phys. Rev. D*76 (2007) 074507

MTH and Sharpe, arXiv:1602.00324

Three-particle result $\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$

Sketch of the derivation...

Recall for two particles we started with a “skeleton expansion”

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagrammatic expansion shows a sequence of terms representing the skeleton expansion for two particles. Each term consists of a chain of circles connected by arcs. The first term has two circles labeled \mathcal{O}^\dagger and \mathcal{O} connected by two arcs, with two black dots in the middle enclosed in a dashed box. The second term adds a circle labeled iK between the two dots. The third term adds two more iK circles in a chain between the dots. Ellipses follow, indicating the series continues.

Three-particle result $\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$

Sketch of the derivation...

Recall for two particles we started with a “skeleton expansion”

$$C_L(P) = \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagrammatic expansion for $C_L(P)$ consists of three terms:

- Term 1: Two circles labeled \mathcal{O}^\dagger and \mathcal{O} connected by two arcs. A dashed box encloses the two vertices.
- Term 2: Two circles labeled \mathcal{O}^\dagger and \mathcal{O} connected by two arcs. A central circle labeled iK is connected to each arc. A dashed box encloses the two vertices and the iK circle.
- Term 3: Two circles labeled \mathcal{O}^\dagger and \mathcal{O} connected by two arcs. Two central circles labeled iK are connected to each arc. A dashed box encloses the two vertices and both iK circles.

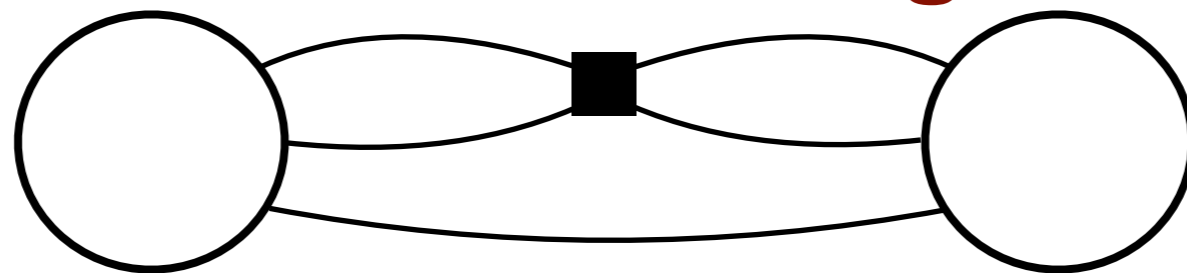
So now we need the same for three...

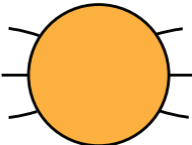
$$C_L(E, \vec{P}) \stackrel{?}{=} \text{diagram}_1 + \text{diagram}_2 + \text{diagram}_3 + \dots$$

The diagrammatic expansion for $C_L(E, \vec{P})$ consists of three terms:

- Term 1: Three circles connected in a chain by two arcs. A dashed box encloses the two outer vertices.
- Term 2: Three circles connected in a chain by two arcs. The middle circle is orange. A dashed box encloses the two outer vertices and the orange circle.
- Term 3: Three circles connected in a chain by two arcs. The two middle circles are orange. A dashed box encloses the two outer vertices and both orange circles.

No! We also need diagrams like



Disconnected diagrams in  lead to singularities that invalidate the derivation

New skeleton expansion

$$C_L(E, \vec{P}) = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

$$+ \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots$$

The diagrams in the expansion are:

- Diagram 1: Two white circles connected by two arcs, enclosed in a dashed box.
- Diagram 2: A white circle, a dashed box containing an orange circle, and another white circle, all connected by two arcs.
- Diagram 3: A white circle, a dashed box containing two orange circles, and another white circle, all connected by two arcs.
- Diagram 4: A white circle, a dashed box containing one purple circle, and another white circle, all connected by two arcs.
- Diagram 5: A white circle, a dashed box containing two purple circles, and another white circle, all connected by two arcs.
- Diagram 6: A white circle, a dashed box containing three purple circles, and another white circle, all connected by two arcs.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram A} + \text{Diagram B} + \text{Diagram C} + \dots$$

The diagrams in the kernel definition for the purple circle are:

- Diagram A: A vertex with two external lines.
- Diagram B: A vertex with two external lines and a loop.
- Diagram C: A vertex with two external lines and a bubble.

$$\text{Orange circle} \equiv \text{Diagram D} + \text{Diagram E} + \text{Diagram F} + \dots$$

The diagrams in the kernel definition for the orange circle are:

- Diagram D: A vertex with two external lines.
- Diagram E: A vertex with two external lines and a horizontal line.
- Diagram F: A vertex with two external lines and a loop.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots
 \end{aligned}$$

The diagrams in the expansion are arranged in four rows. The first row contains three diagrams with two white circles connected by two lines, with one, two, or three orange circles inserted between them. The second row contains three diagrams with two white circles and one, two, or three purple circles inserted between them. The third row contains three diagrams with two white circles and two, three, or four purple circles inserted between them. The fourth row contains two diagrams with two white circles and three purple circles inserted between them. Dashed boxes in each diagram indicate the skeleton structure.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The purple circle kernel is defined as the sum of three diagrams: a single vertex, a vertex with two arcs, and a vertex with two lines.

$$\text{Orange circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The orange circle kernel is defined as the sum of three diagrams: a single vertex, a vertex with two lines, and a vertex with two arcs.

New skeleton expansion

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the expansion represent various Feynman-like diagrams with external lines and internal vertices. The vertices are either white circles, orange circles, or purple circles. Dashed boxes indicate sub-diagrams that are summed over in the expansion.

Kernel definitions:

$$\text{Purple circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The purple circle kernel is defined as the sum of three diagrams: a vertex with four external lines, a vertex with four external lines and two internal lines forming a loop, and a vertex with four external lines and two internal lines forming a figure-eight shape.

$$\text{Orange circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The orange circle kernel is defined as the sum of three diagrams: a vertex with four external lines, a vertex with four external lines and two internal lines forming a straight line, and a vertex with four external lines and two internal lines forming a loop.

Three-to-three scattering



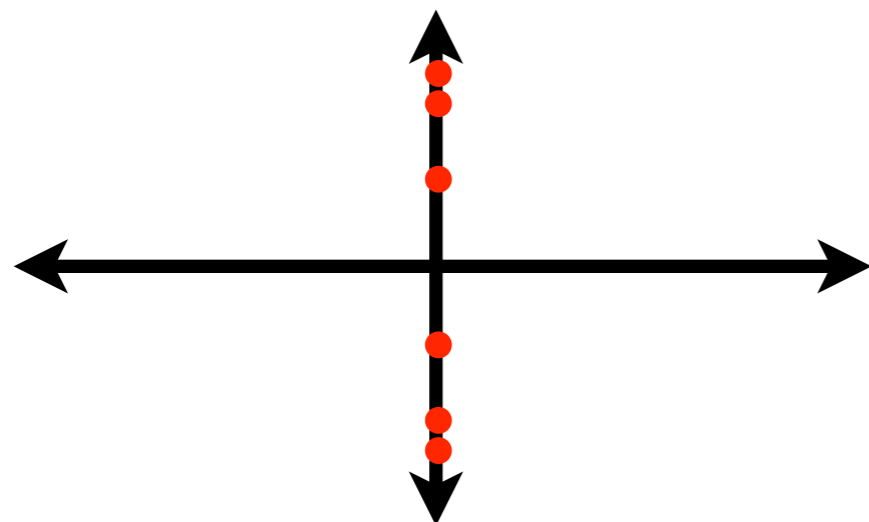
1. Work out the three particle skeleton expansion

$$C_L(E, \vec{P}) = \text{[Diagrammatic expansion of } C_L(E, \vec{P}) \text{ showing skeleton terms with orange and purple particles]} + \dots$$

2. Break diagrams into finite- and infinite-volume parts

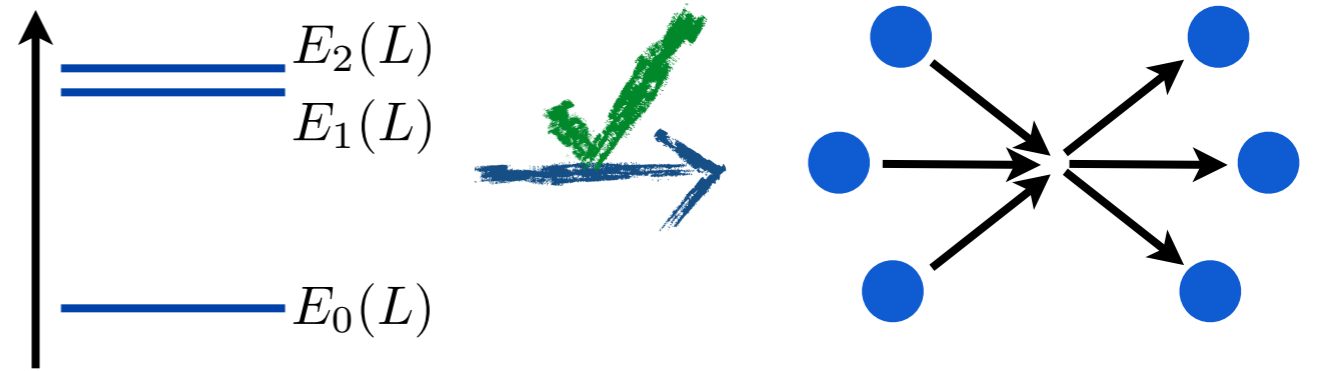
3. Sum subsets of terms to identify *infinite-volume quantities*

4. Relate these to poles in the finite-volume correlator



$$\det_{k,\ell,m} \left[\mathcal{K}_{\text{df},3}^{-1} + F_3 \right] = 0$$

Three-to-three scattering



Current status:

Formalism is complete for the simplest three-scalar system

General, model-independent relation between finite-volume energies and three-to-three scattering amplitude

Derived using a generic relativistic field theory

MTH and Sharpe, *Phys. Rev. D*90, 116003 (2014)

MTH and Sharpe, *Phys. Rev. D*92, 114509 (2015)

Important caveats:

Identical particles with no two-to-three transitions

$$\pi\pi\pi \rightarrow \pi\pi\pi$$

Requires that two-particle scattering phase is bounded

$$|\delta_\ell(E)| < \pi/2$$

Currently underway:

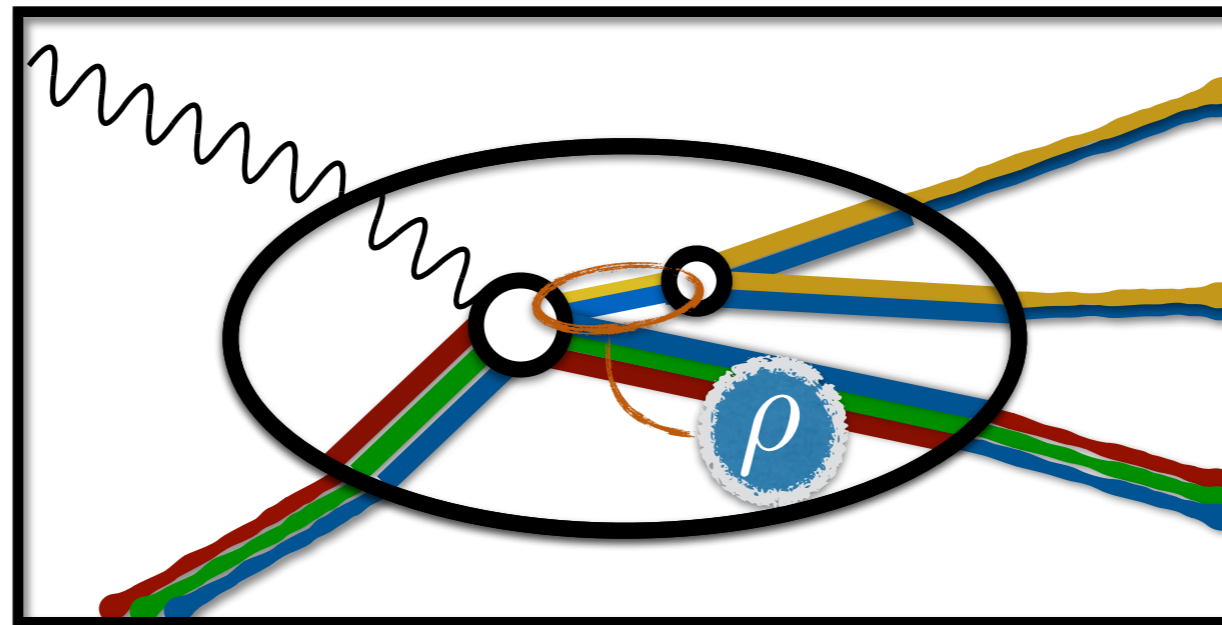
Relax all simplifying assumptions:

**Allow all particle types, allow two-to-three couplings,
remove bound on phase shift**

$$K\pi \rightarrow K\pi\pi \quad N\pi \rightarrow N\pi\pi \quad NNN \rightarrow NNN$$

Briceño, MTH, Sharpe, *in development*

Derive formalism for three-particle transition amplitudes



$$p\gamma \rightarrow N\rho \rightarrow N\pi\pi$$

Also want to make connections to other work...

Polejaeva and Rusetsky, *Eur. Phys. J. A*48, 67 (2012)

Briceño and Davoudi, *Phys. Rev. D*87, 094507 (2013)

Meißner, Rios and Rusektsky. *Phys. Rev. Lett.* 114, 091602 (2015)