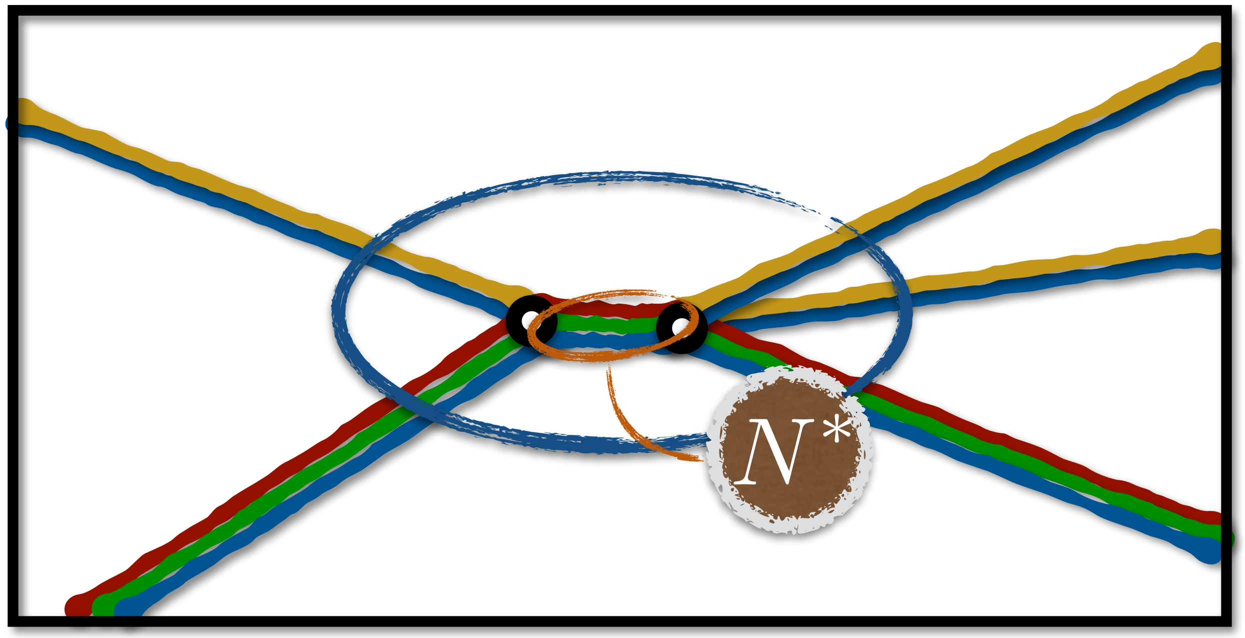
Three-particle scattering from the lattice

Maxwell T. Hansen
Institut für Kernphysik and HIM
Johannes Gutenberg Universität
Mainz, Germany

INT, Seattle
March 30, 2016

Helmholtz-Institut Mainz

Importance of three-particle scattering



Many QCD resonances decay significantly to three particles

$$N\pi \to N^* \to N\pi\pi$$

Ideally we would like to describe resonances by…

(1). Determining QCD scattering amplitudes in a rigorous and model-independent way

(2). Analytically continuing these to the resonance poles

Amplitudes from the path-integral

If we were strong enough, we would proceed as follows

(1). Evaluate the path-integral to obtain the relevant correlator

$$\langle N(x')\pi(y')\pi(z')\; N(x)\pi(y)\rangle$$
$$= \int \mathcal{D}A\mathcal{D}q\mathcal{D}\bar{q}\; \exp[iS_{QCD}]\; N(x')\pi(y')\pi(z')\; N(x)\pi(y)$$

(2). Fourier transform and apply LSZ reduction

$$\langle \tilde{N}(p')\tilde{\pi}(k')\tilde{\pi}(q')\; \tilde{N}(p)\tilde{\pi}(k)\rangle \longrightarrow \frac{iZ_N^{1/2}}{p'^2 - m_N^2} \cdots \frac{iZ_\pi^{1/2}}{k^2 - M_\pi^2} \langle N\pi\pi, \mathrm{out}|N\pi, \mathrm{in}\rangle$$

This approach requires...

Infinite volume to define asymptotic states $\dfrac{1}{k^2 - M_\pi^2}$

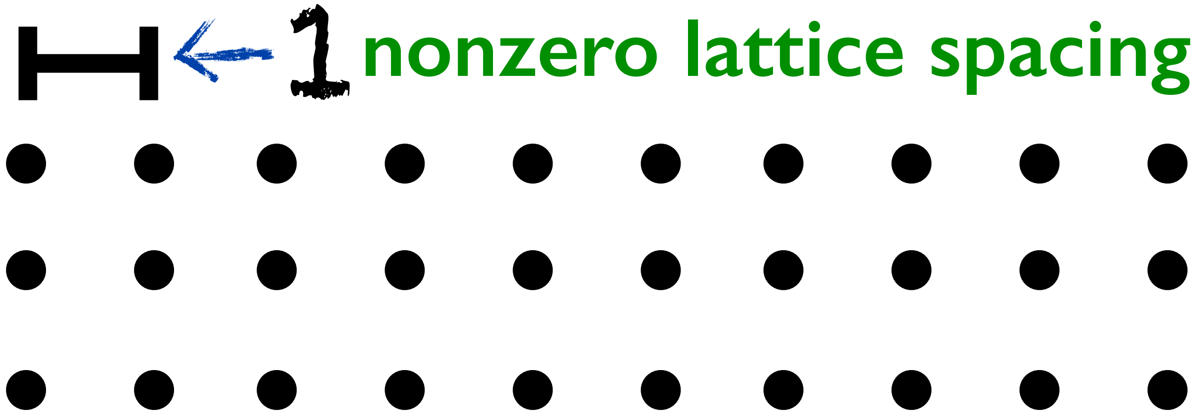
Minkowski momenta to approach the poles **("go on shell")**

$$k^2 \to M_\pi^2$$

In Lattice QCD we are evaluating the path integral numerically…

To do so we have to make four compromises

1 nonzero lattice spacing



Must ensure this is smaller than all relevant length scales

2 Unphysical pion masses

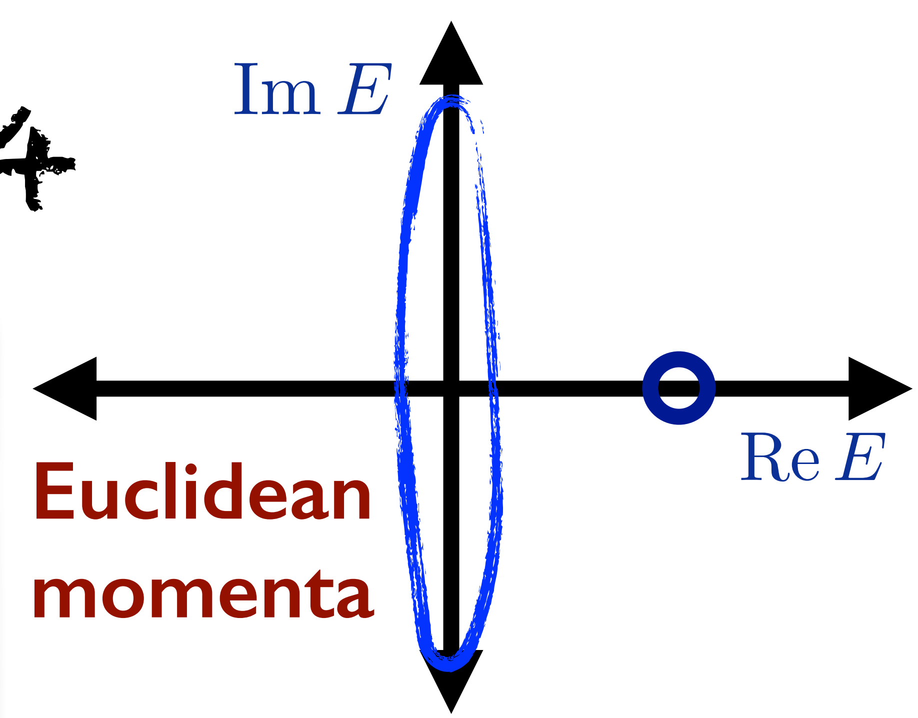
$$M_{\pi,\mathrm{lattice}} > M_{\pi,\mathrm{our\ universe}}$$

But calculations at the physical pion mass do now exist…

and exploring pion mass dependence is interesting

3 finite volume, L

4

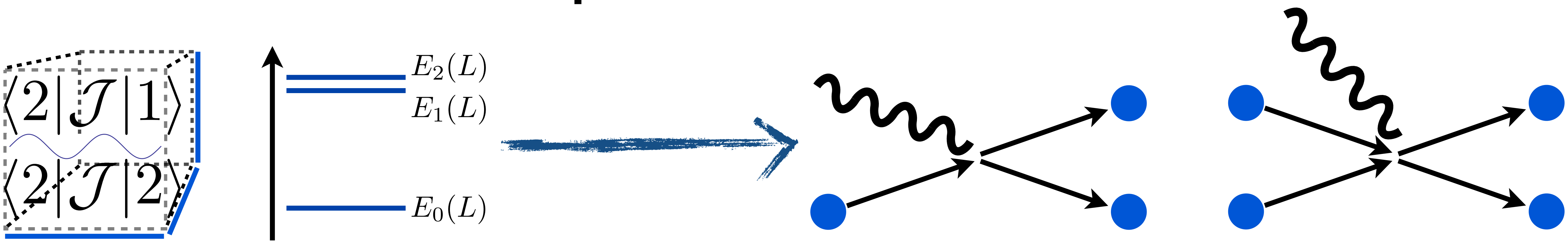


LQCD cannot directly access scattering amplitudes… but it can give finite-volume energies

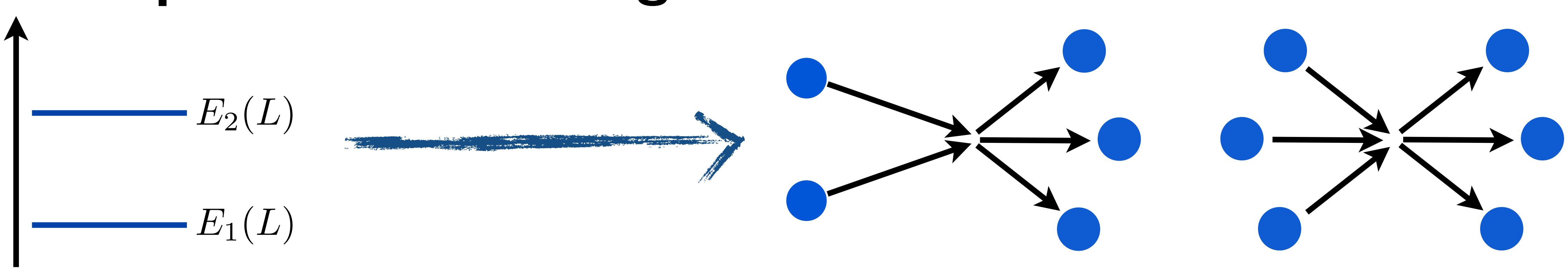
It is possible to derive relations between finite- and infinite-volume physics

Two-particle scattering

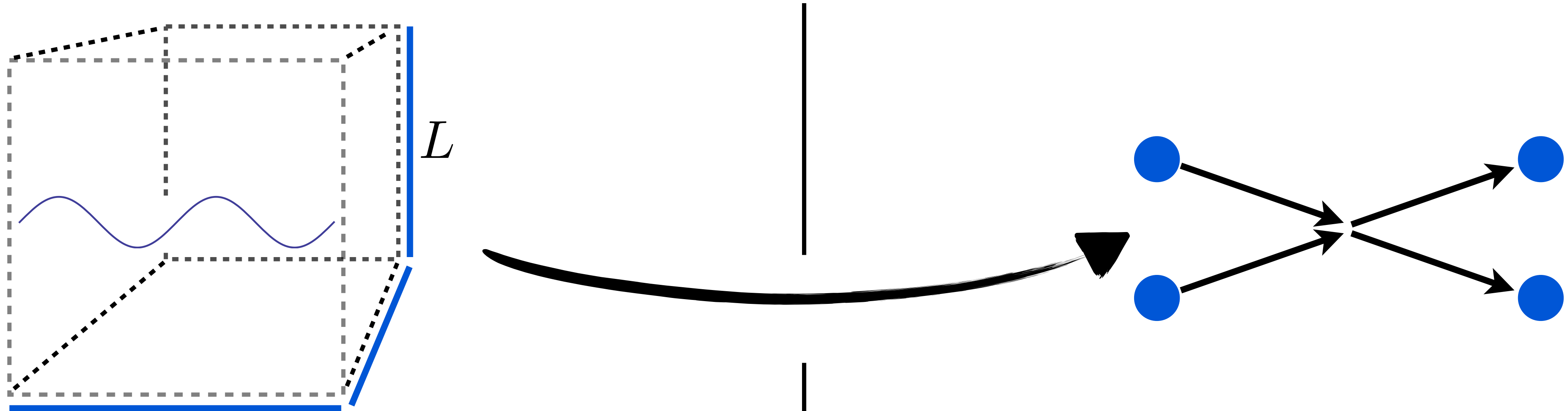
Photo- and electroproduction



Three-particle scattering

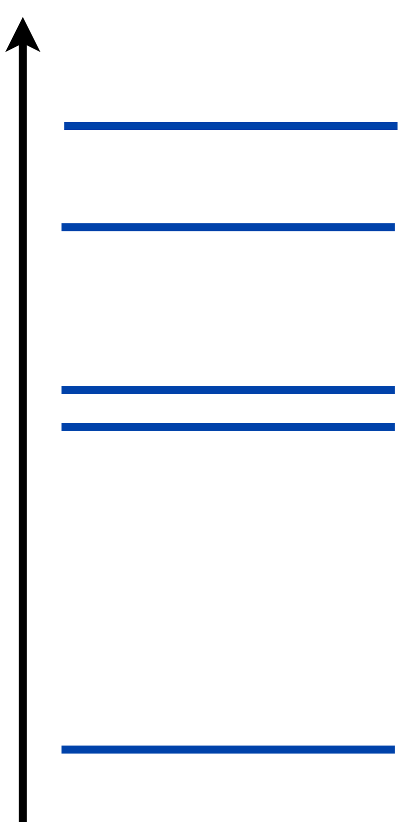


From energy levels to amplitudes



Finite volume

Discrete tower of energy levels



$$E_n(L, \vec{P})$$

depends on
finite-volume size
total momentum

Infinite volume

Decompose scattering amplitude in partial waves

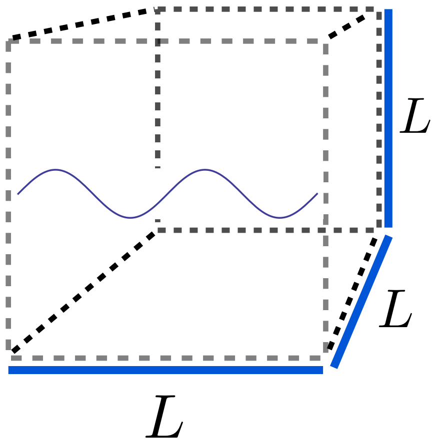
One real observable…

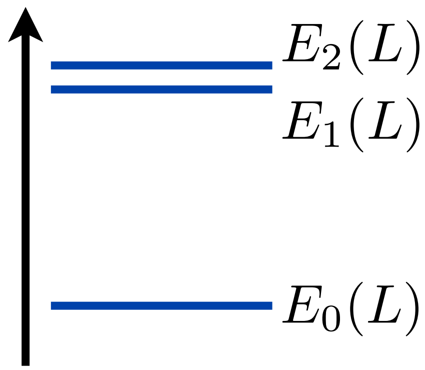
$$\delta_\ell(E^*)$$

in each partial wave

at each CM energy

Finite volume quantum field theory





cubic, spatial volume (extent L)

periodic boundary conditions

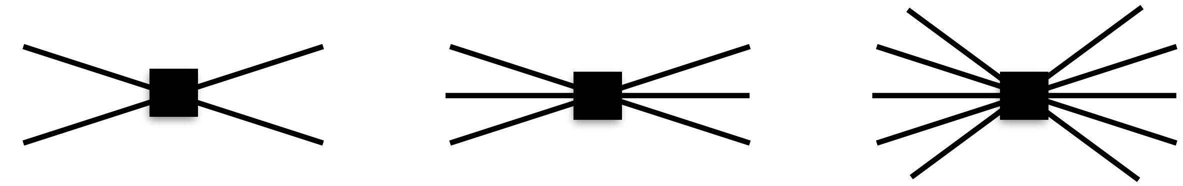
$$\vec{p} \in (2\pi/L)\mathbb{Z}^3$$

time direction **infinite**

L large enough to ignore e^{-mL}

generic relativistic QFT

1. Include all interactions



2. no power-counting scheme

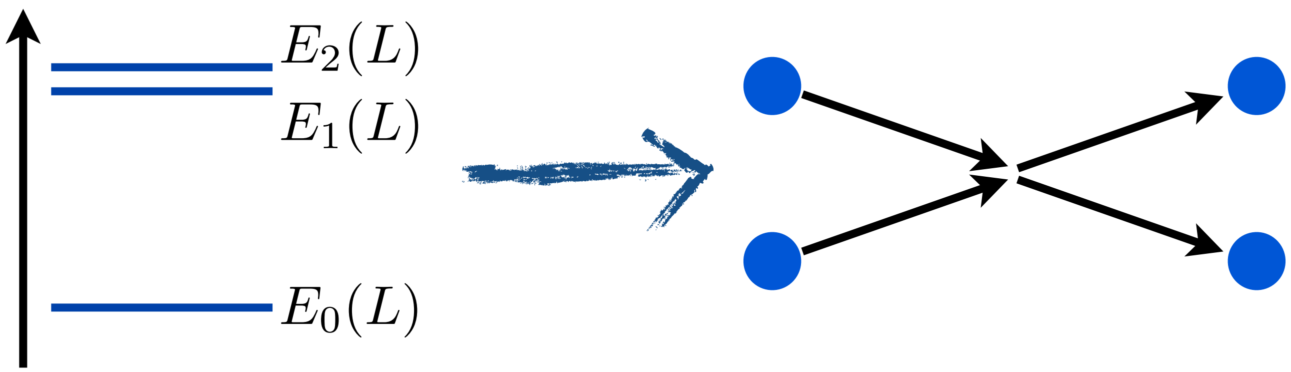
Not possible to directly calculate scattering observables to all orders

But it is possible to derive *general, all-orders relations* to finite-volume quantities

Assume lattice effects are small and accommodated elsewhere

Work in continuum field theory throughout

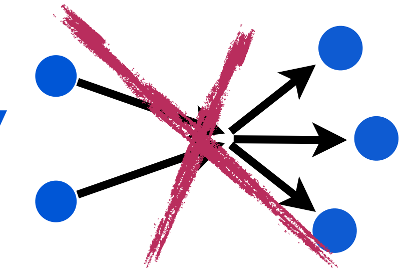
Two-to-two scattering



For now assume…

identical scalars, mass m

$\mathbb{Z}_2$ symmetry



$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

Euclidean convention

two-particle interpolator

$$P = (P_4, \vec{P}) = (P_4, 2\pi \vec{n}/L)$$

but allow P_4 to be real or imaginary

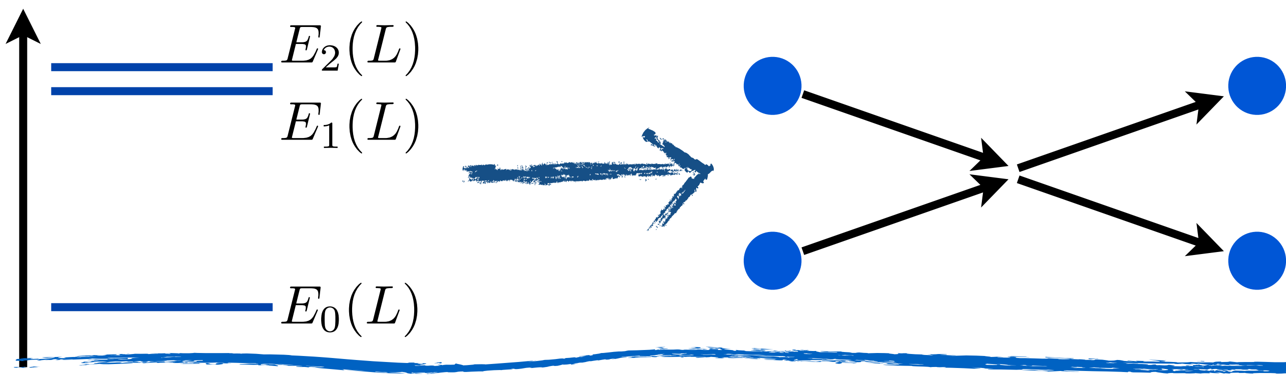
CM frame energy is then $E^{*2} = -P_4^2 - \vec{P}^2$

Require $E^* < 4m$ to isolate two-to-two scattering

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

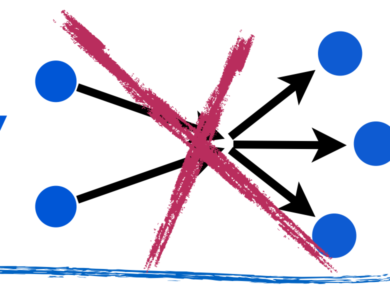
Two-to-two scattering



For now assume…

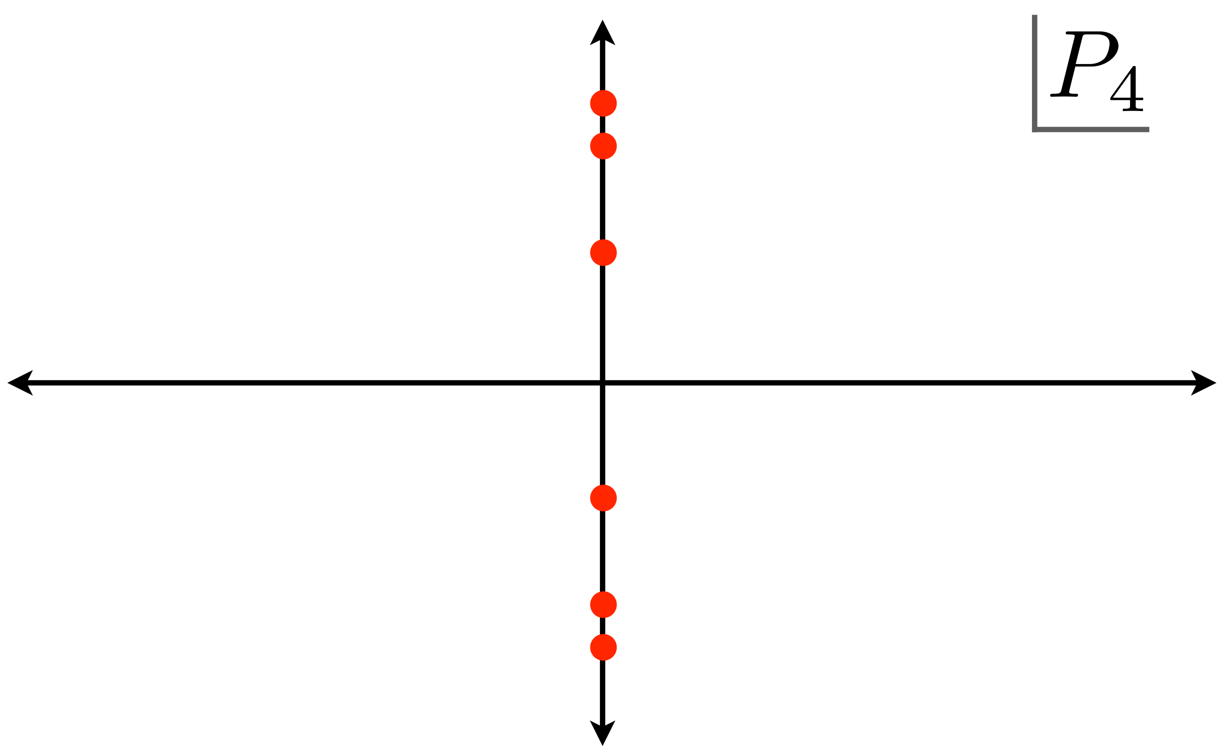
identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**

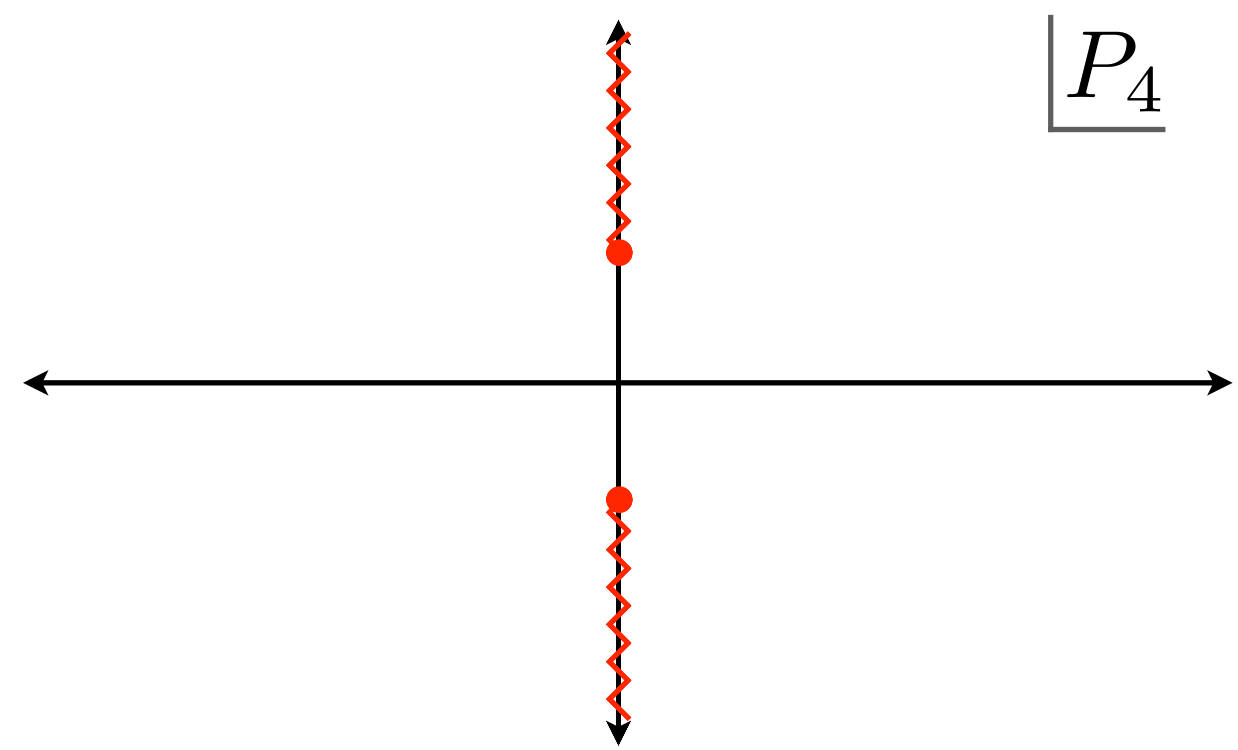


$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

At fixed $L, \vec{P}$, **poles in** C_L **give finite-volume spectrum**

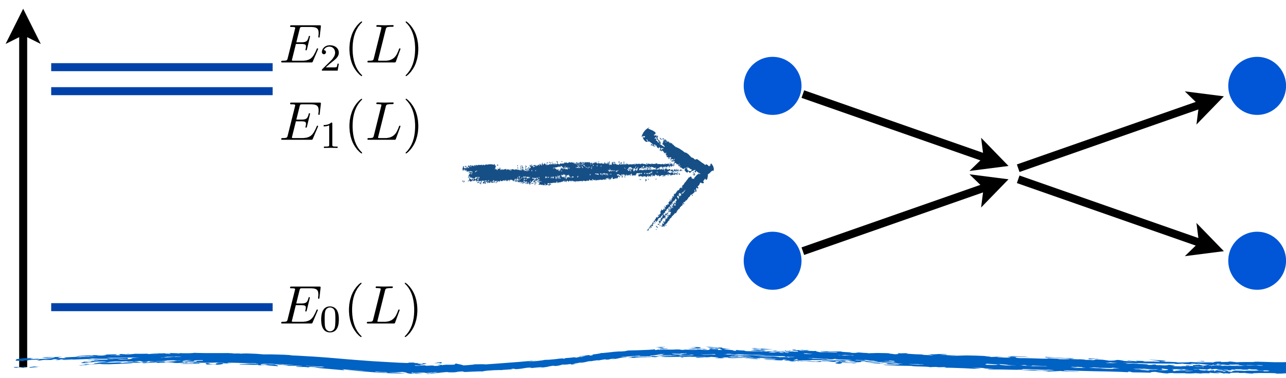


C_L **analytic structure**



C_∞ **analytic structure**

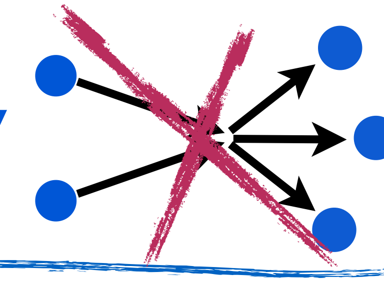
Two-to-two scattering



For now assume…

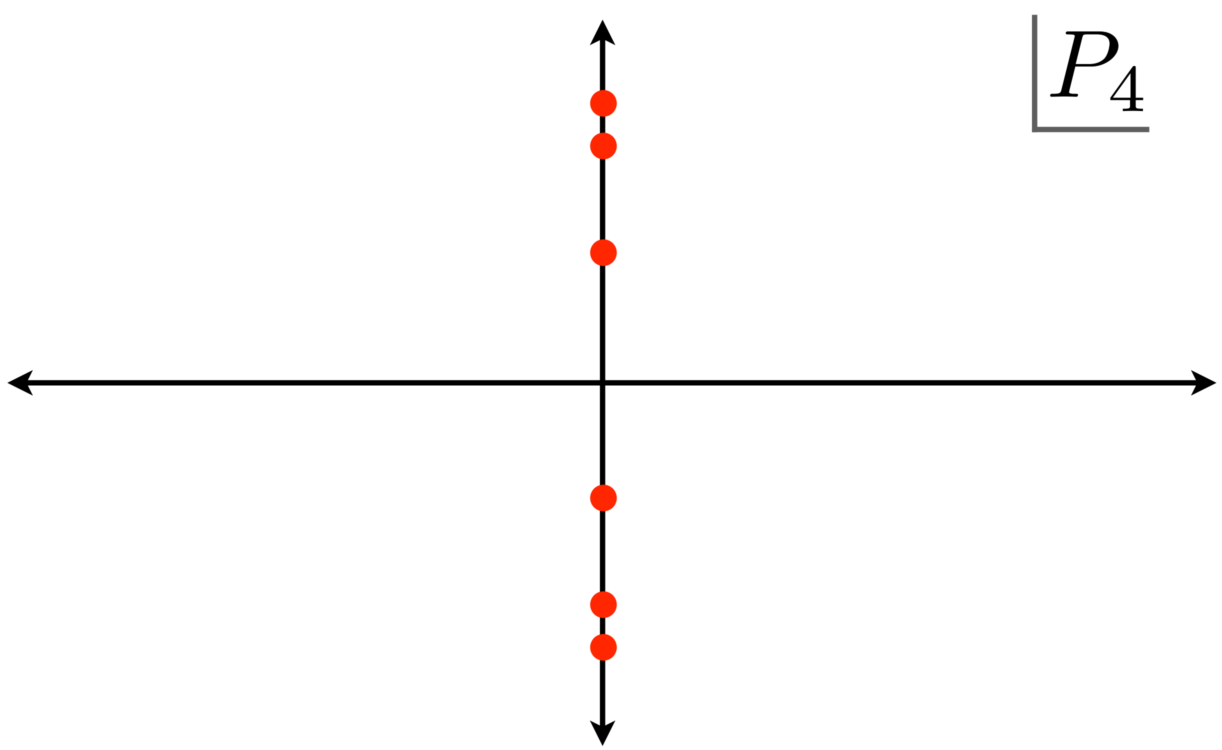
identical scalars, mass m

$\mathbb{Z}_2$ **symmetry**



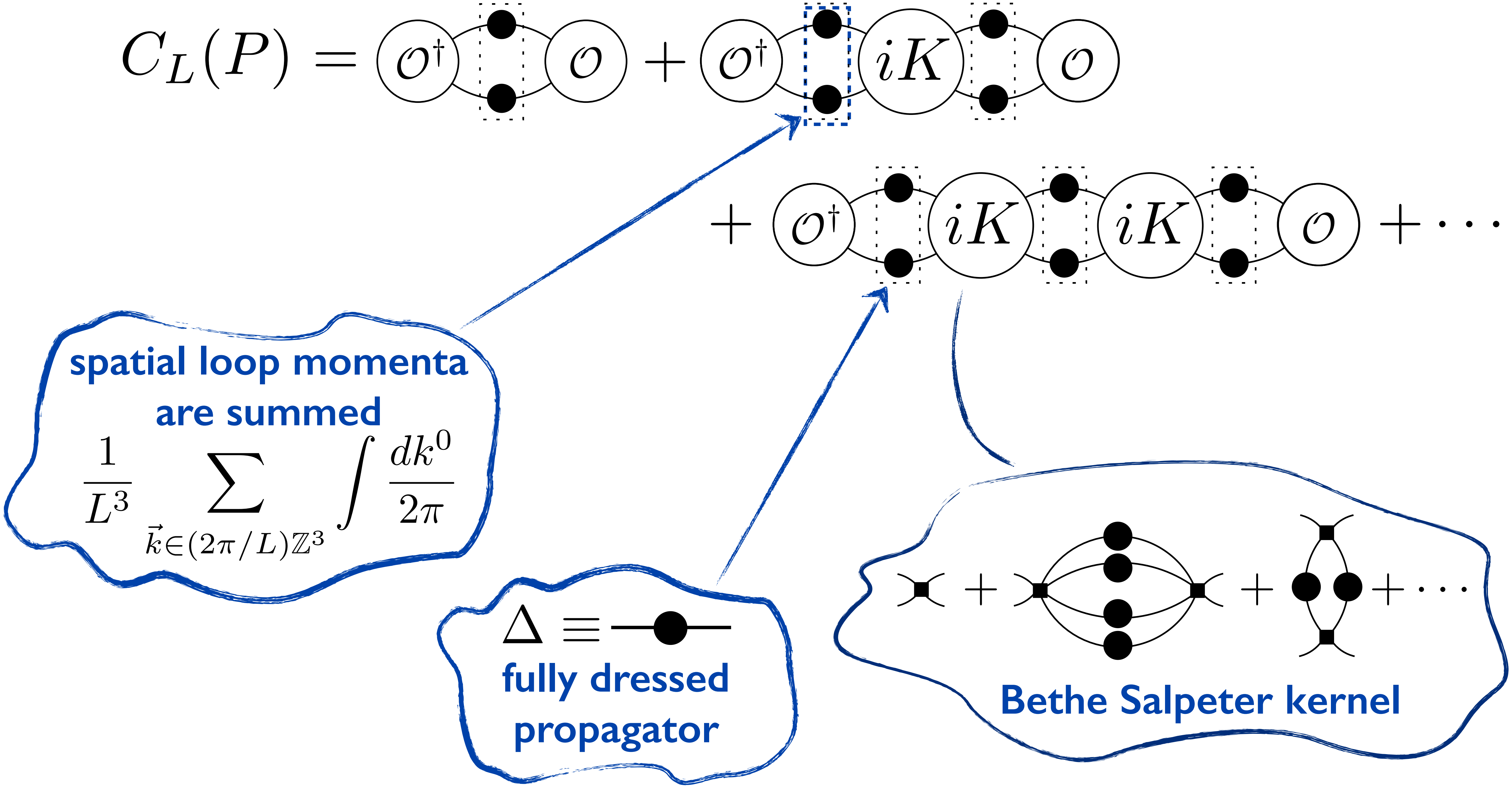
$$C_L(P) \equiv \int_L d^4x \; e^{-iPx} \langle 0 | T \mathcal{O}(x) \mathcal{O}^\dagger(0) | 0 \rangle$$

At fixed $L, \vec{P}$, **poles in** C_L **give finite-volume spectrum**



C_L **analytic structure**

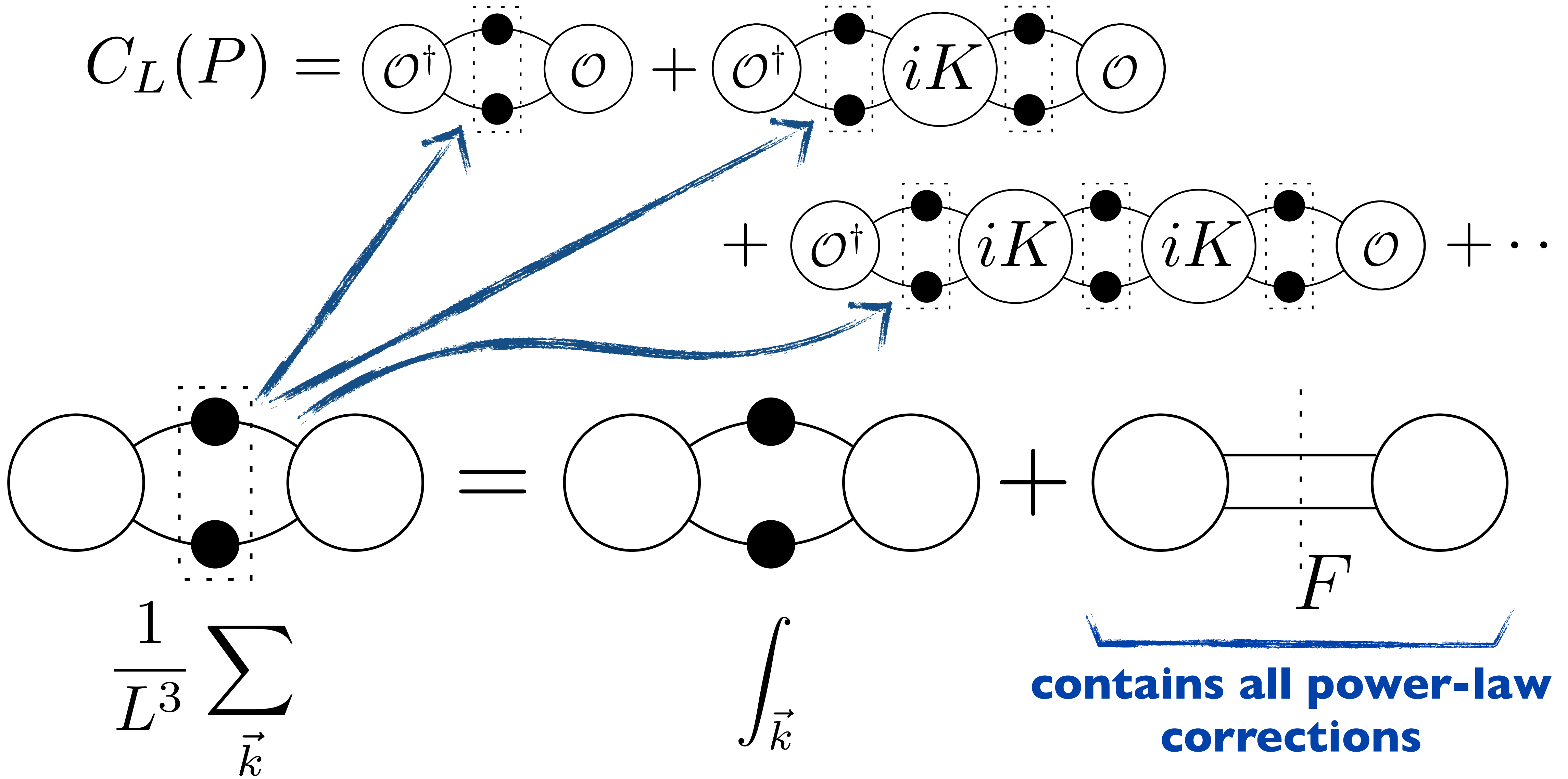
Calculate $C_L(P)$ to all orders in perturbation theory and determine locations of poles.



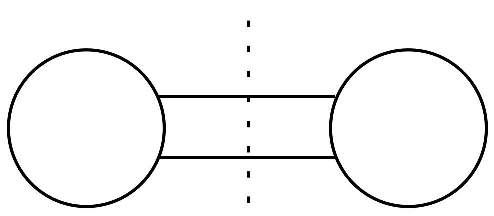
If $E^* < 4m$ **then**
$$K_L = K_\infty + \mathcal{O}(e^{-mL})$$
$$\Delta_L = \Delta_\infty + \mathcal{O}(e^{-mL})$$

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)

Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

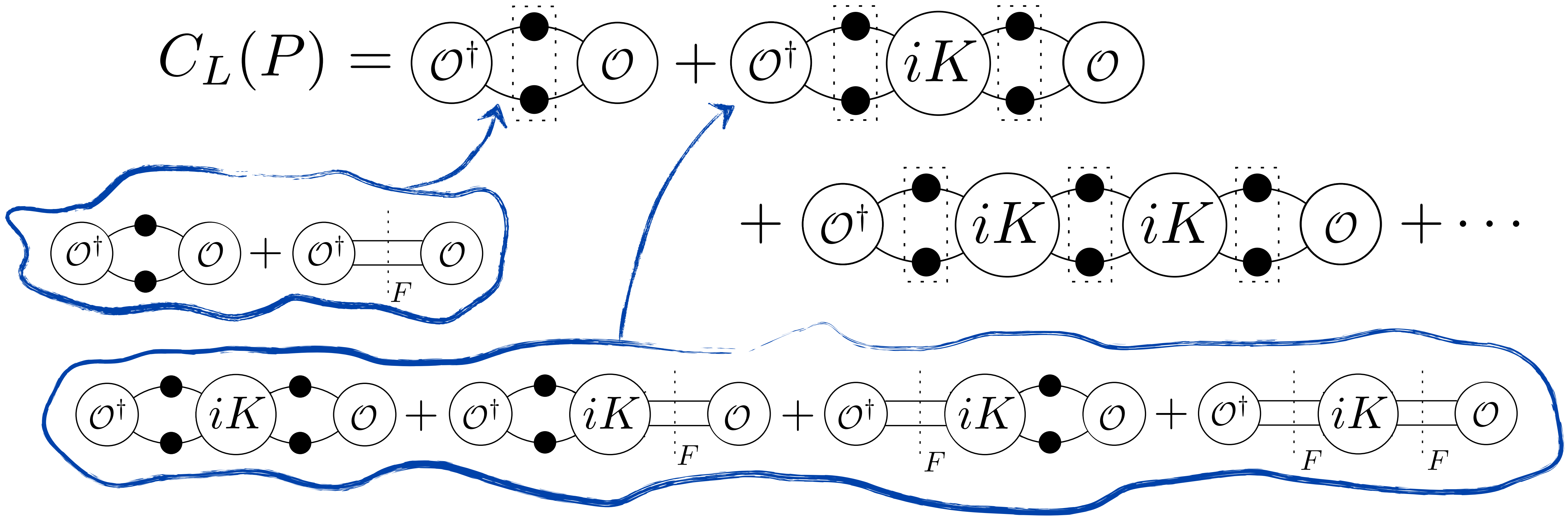


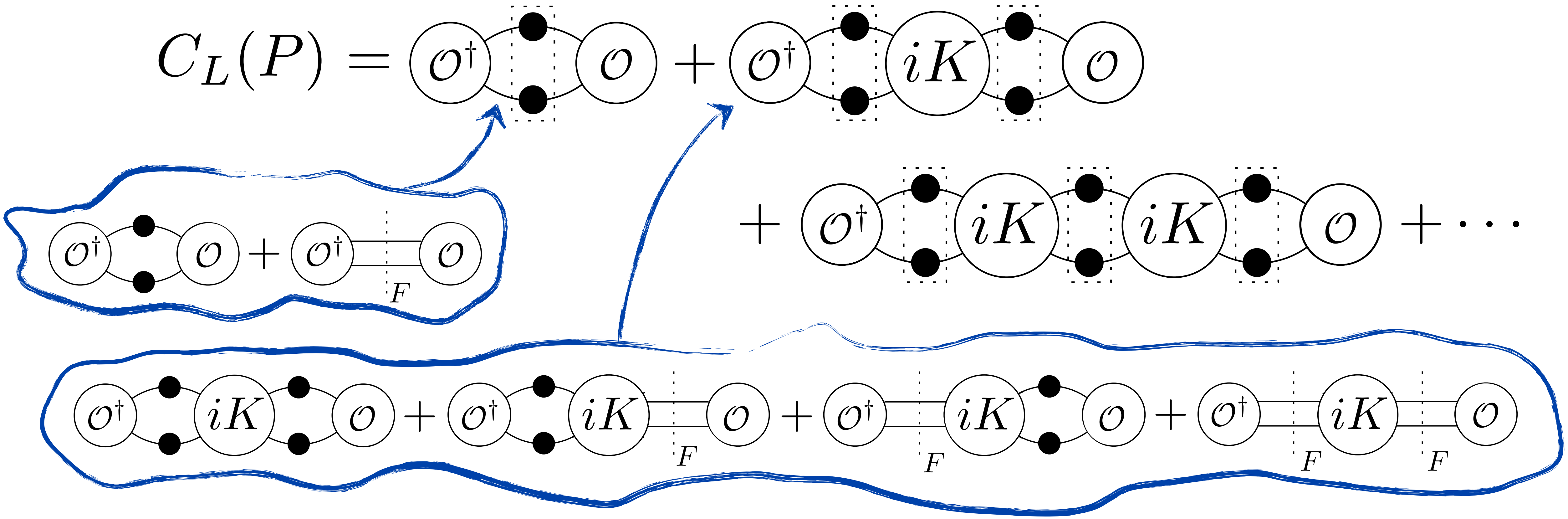
Now we introduce an important identity.

In  all four-momenta are projected **on shell.**

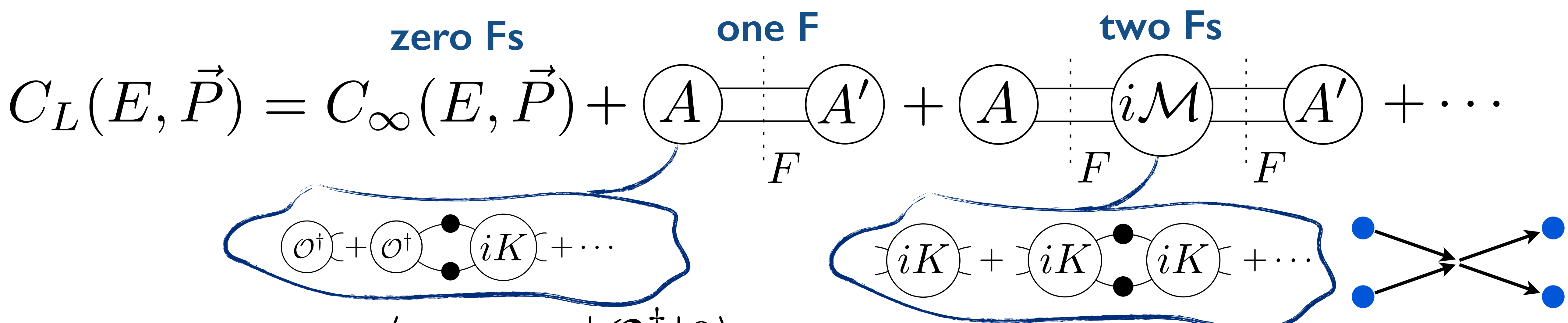
Physical, propagating states give dominate finite-volume effects.

Lüscher, M. *Nucl. Phys* B354, 531-578 (1991)
Derivation from Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)


$C_L(P) =$
$\mathcal{O}^\dagger$
iK
$\mathcal{O}$
F
$+ \cdots$



Now regroup by number of Fs

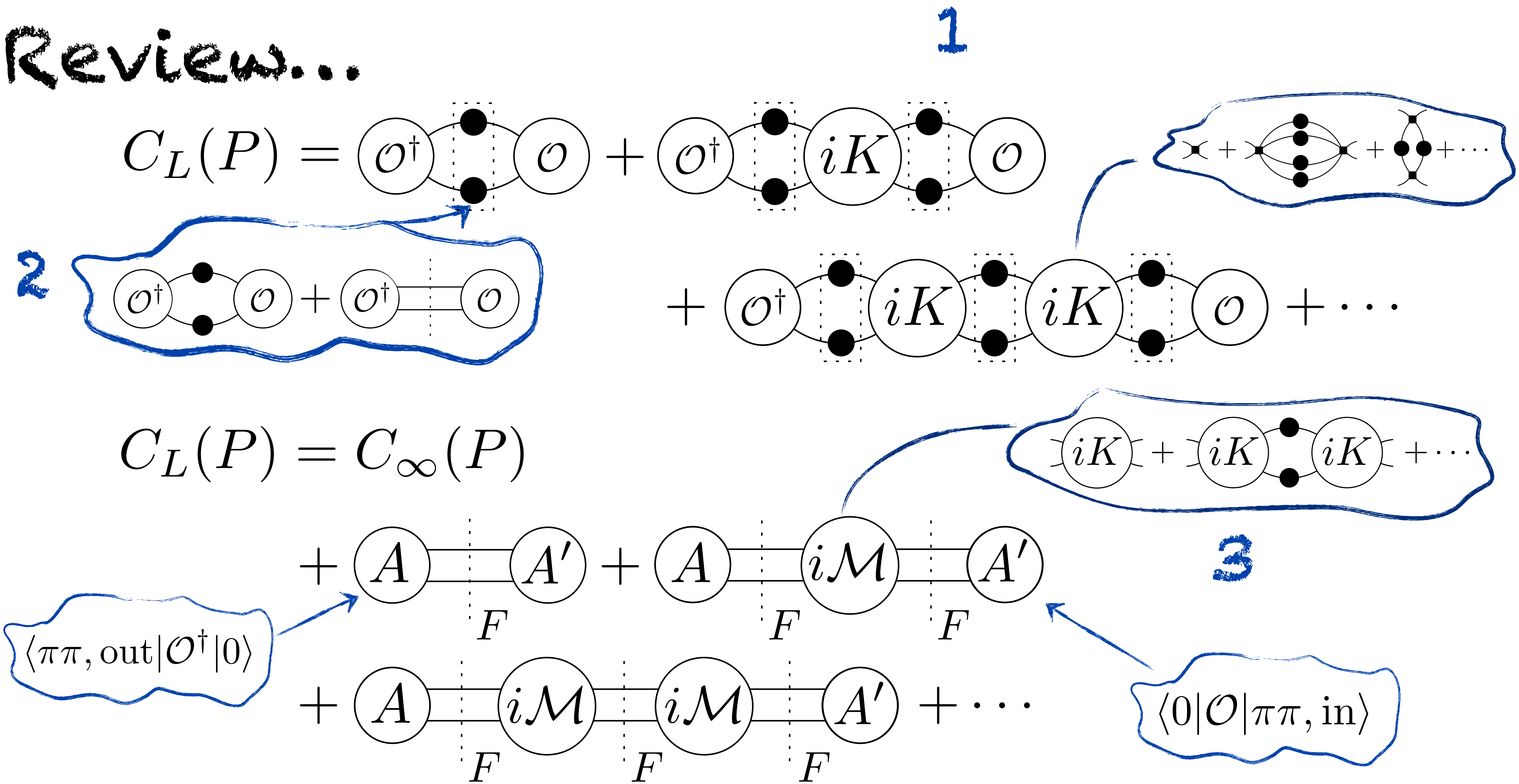


$$= \langle \pi\pi, \mathrm{out}|\mathcal{O}^\dagger|0\rangle$$

When we factorize diagrams and group infinite-volume parts…

physical observables emerge!

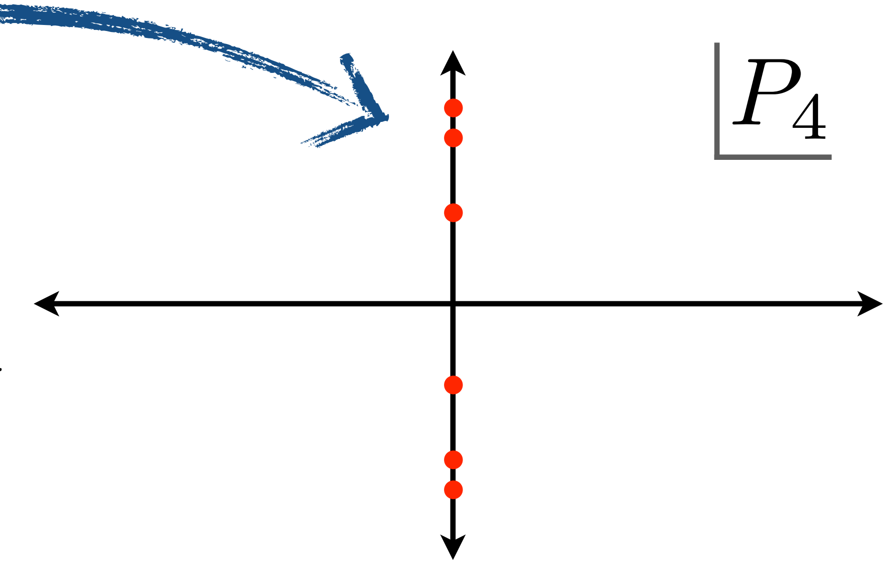
Review...



We deduce...

poles are in here

$$C_L(P) = C_\infty(P) - A' F \frac{1}{1 + \mathcal{M}_{2\to 2} F} A$$

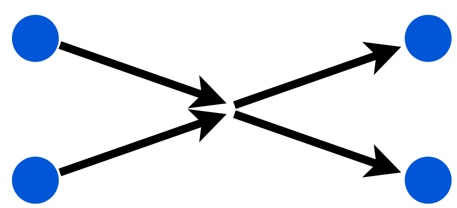


Two-particle result

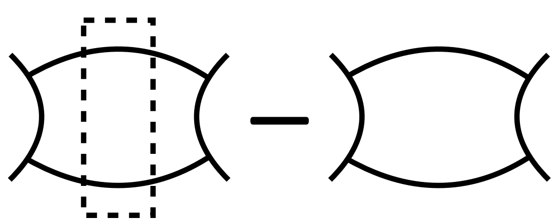
At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det[\mathcal{M}_{2\to2}^{-1} + F] = 0$

Rummukainen and Gottlieb, *Nucl. Phys.* B450, 397 (1995)
Kim, Sachrajda and Sharpe. *Nucl. Phys.* B727, 218-243 (2005)

Matrices defined using angular-momentum states

$\mathcal{M}_{2\to2} \equiv$  **diagonal matrix, parametrized by** $\delta_\ell(E^*)$

$F \equiv$ **non-diagonal matrix of known geometric functions**

$\equiv$ 

difference of two-particle loops in finite and infinite volume

depends on $L, E, \vec{P}$

At low energies, lowest partial waves dominate $\mathcal{M}_{2\to2}$

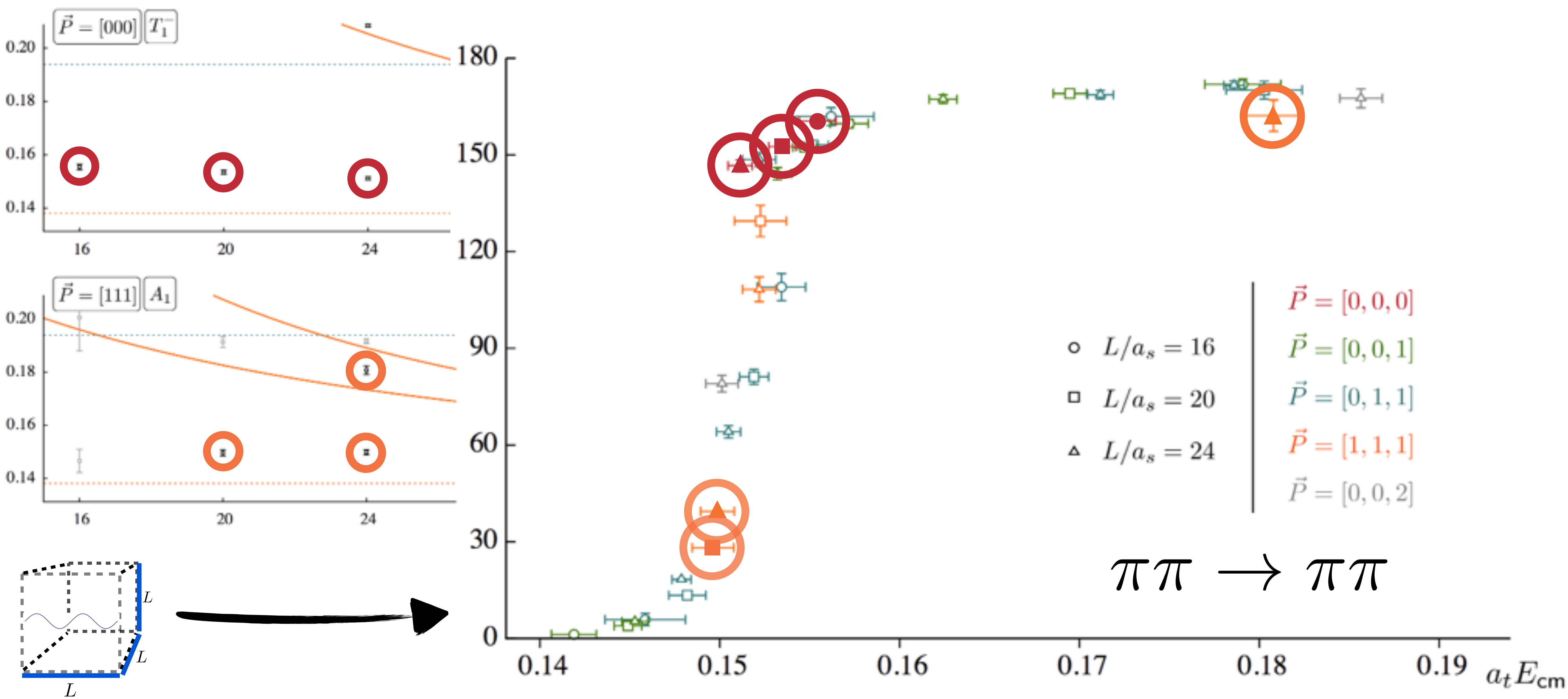
e.g. s-wave only with some rearranging

$$\longrightarrow \cot\delta(E_n^*) + \cot\phi(E_n, \vec{P}, L) = 0$$

scattering phase — known function

Using the result (p-wave)

$$\cot \delta_{\ell=1}(E_n^*) + \cot \phi(E_n, \vec{P}, L) = 0$$



from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

Two-particle scattering

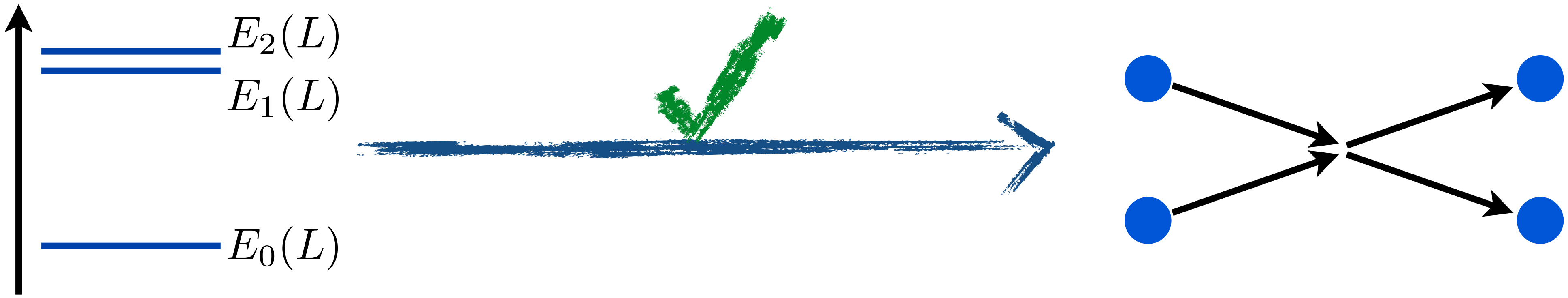
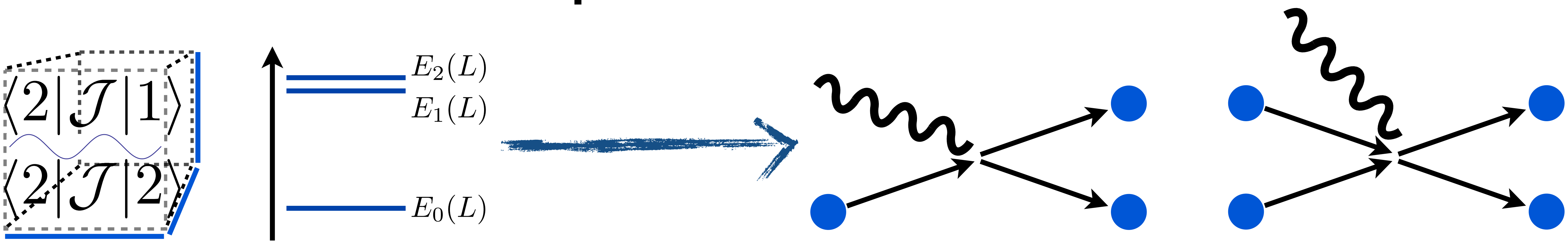
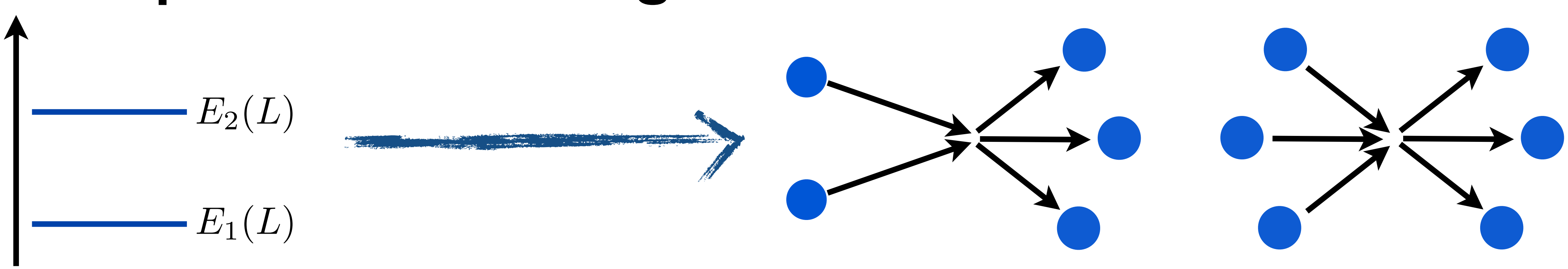


Photo- and electroproduction



Three-particle scattering



Two-particle scattering

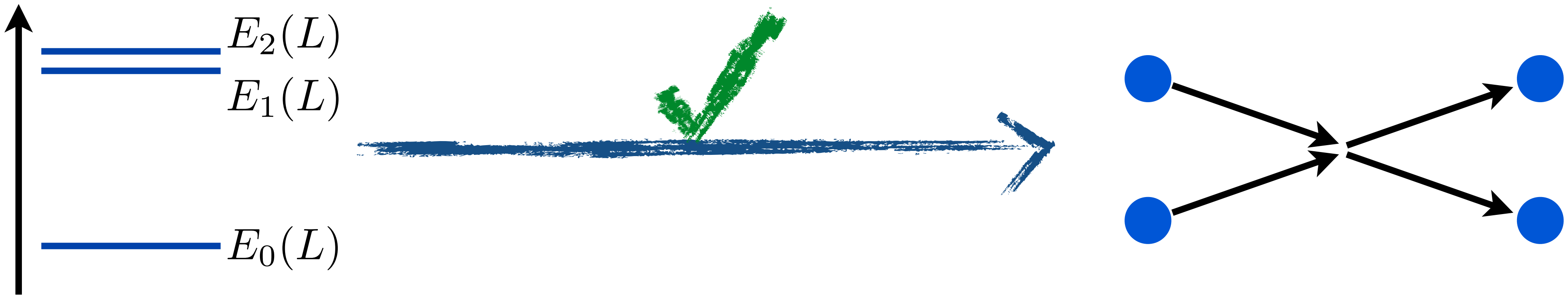
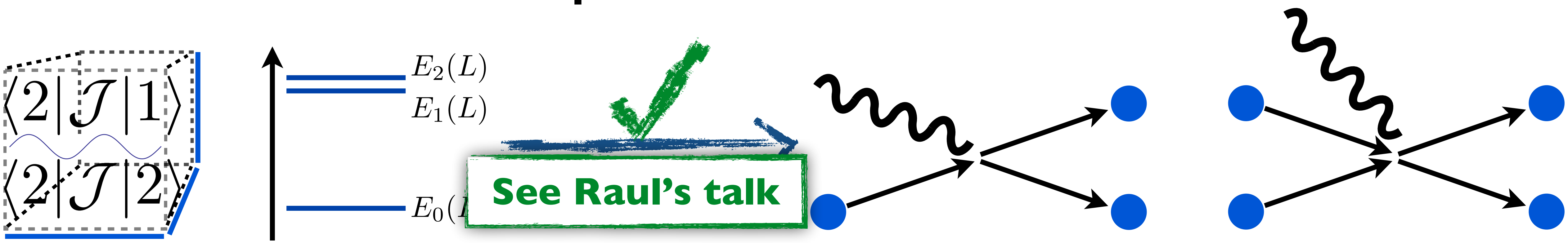
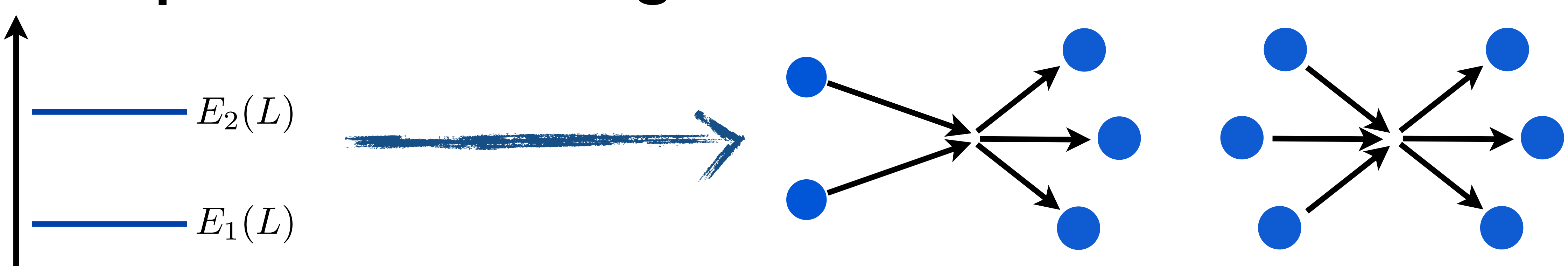


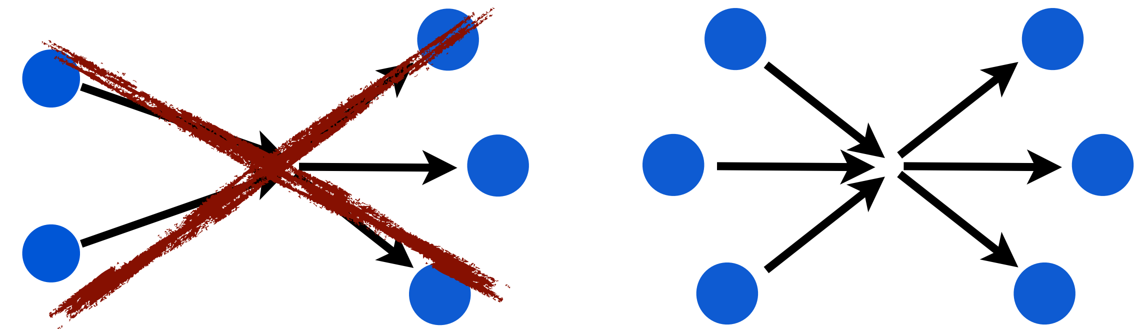
Photo- and electroproduction



Three-particle scattering



Begin by considering the infinite-volume observables

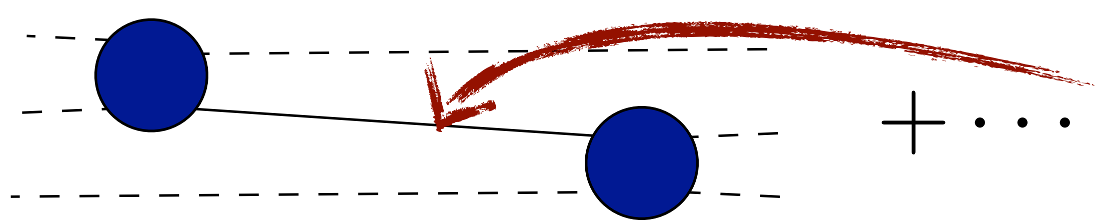


Because of "finite-volume rescattering" it is not possible to access two-to-three without also accessing three-to-three

For now we turn off two-to-three scattering using a symmetry

Three-to-three amplitude has kinematic singularities

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator with six external legs amputated and projected on shell

$=$ 

Certain external momenta put this on-shell!

Three-to-three amplitude has more degrees of freedom

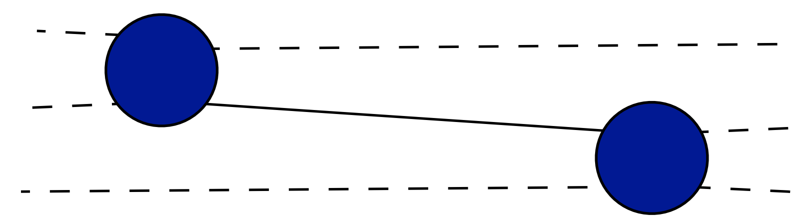
8 degrees of freedom including total energy

Compared with 2 for the two-to-two amplitude

How can we possibly hope to extract a singular, eight-coordinate function using finite-volume energies?

Short answer…

(1). We found that the spectrum depends on a modified quantity with singularities removed

$\mathcal{K}_{\mathrm{df},3} \not\supset$ 

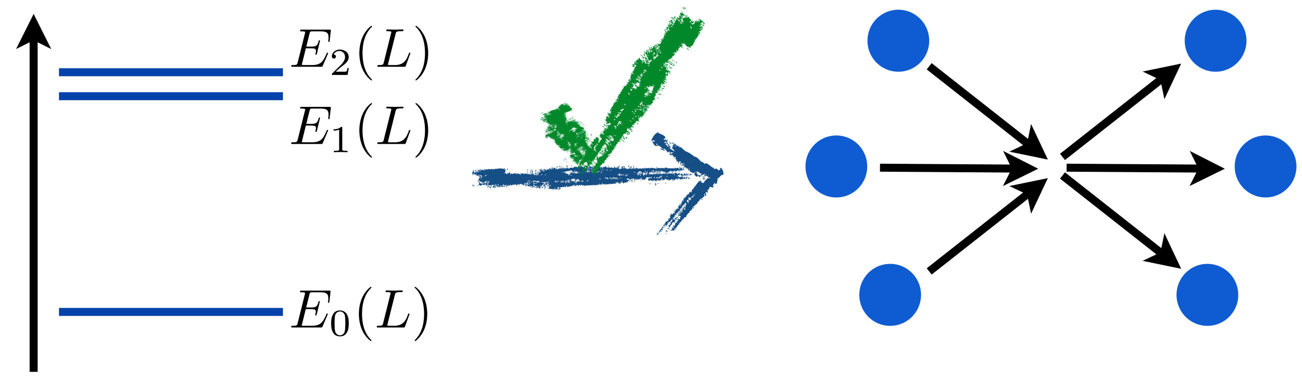
(a) Same degrees of freedom as $\mathcal{M}_{3\to3}$.
(b) Relation to $\mathcal{M}_{3\to3}$ is known (depends only on on-shell $\mathcal{M}_{2\to2}$)
(c) Smooth function (allows harmonic decomposition)

(2). Degrees of freedom encoded in an extended matrix space

($\vec{k}$ is restricted to finite-volume momenta)

$$\vec{k}, \ell, m$$

Three-to-three scattering



Current status:

Formalism is complete for the simplest three-scalar system

General, model-independent relation between finite-volume energies and three-to-three scattering amplitude

Derived using a generic relativistic field theory

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)
MTH and Sharpe, *Phys. Rev.* D92, 114509 (2015)

Important caveats:

Identical particles with no two-to-three transitions

$$\pi\pi\pi \longrightarrow \pi\pi\pi$$

Requires that two-particle scattering phase is bounded

$$|\delta_\ell(E)| < \pi/2$$

Three-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

$F_3 \equiv$ matrix that depends on known geometric functions as well as $\mathcal{M}_{2\to2}$.

---

(1). Use two-particle quantization condition to constrain $\mathcal{M}_{2\to2}$ and thus determine $F_3(E, \vec{P}, L)$

(2). Use harmonic decomposition + various parametrizations to express $\mathcal{K}_{\mathrm{df},3}(E^*)$ in terms of N unknown parameters

(3). Use quantization condition with lattice (or otherwise) determined energies to determine all parameters

(4). Use known relation to recover $\mathcal{M}_{3\to3}$

MTH and Sharpe, *Phys. Rev.* D92, 114509 (2015)

Three-particle result

At fixed $(L, \vec{P})$, finite-volume energies are solutions to $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)

Some nice features...

Matrices automatically truncated in the $\vec{k}$ index

truncate angular momentum space → **solvable system**

Expanding about weak interactions gives an important check

$$E = 3m + \frac{a_3}{L^3} + \frac{a_4}{L^4} + \frac{a_5}{L^5} + \frac{a_6}{L^6} + \mathcal{O}(1/L^7)$$

Our result agrees with existing results for $a_{3\to5}$ and gives a prediction for a_6

K. Huang and C. Yang, *Phys. Rev.* 105 (1957) 767-775

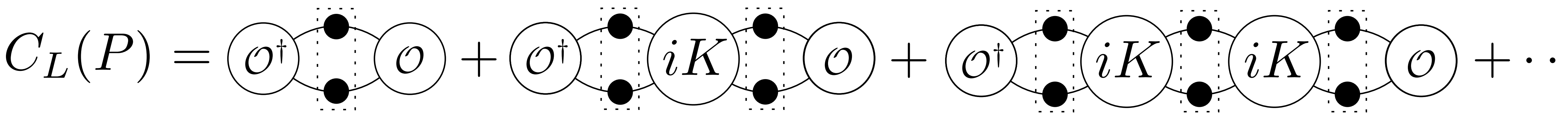
Beane, Detmold, Savage, *Phys. Rev.* D76 (2007) 074507

MTH and Sharpe, arXiv:1602.00324

Three-particle result $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

Sketch of the derivation…

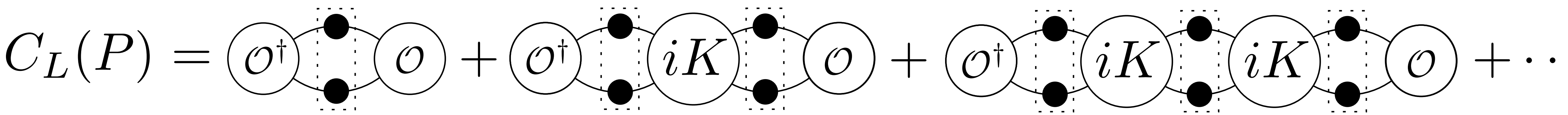
Recall for two particles we started with a "skeleton expansion"



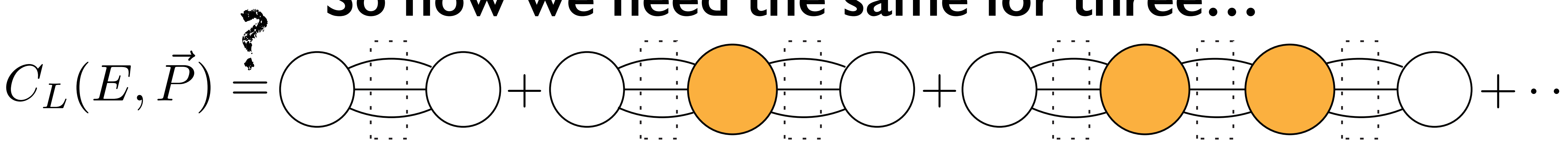
Three-particle result $\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$

Sketch of the derivation…

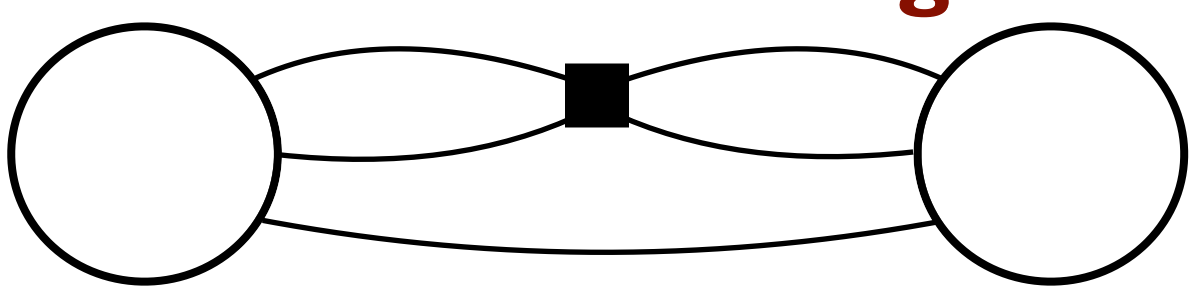
Recall for two particles we started with a "skeleton expansion"

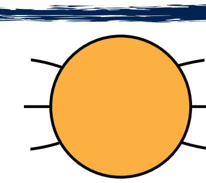


So now we need the same for three…

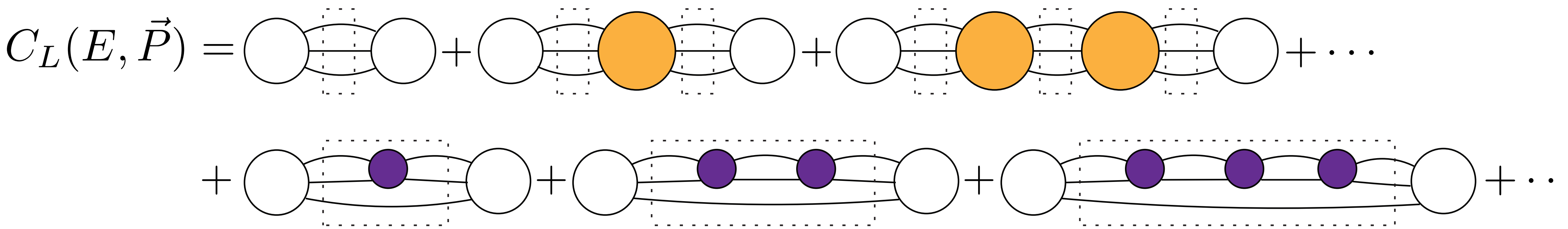


No! We also need diagrams like

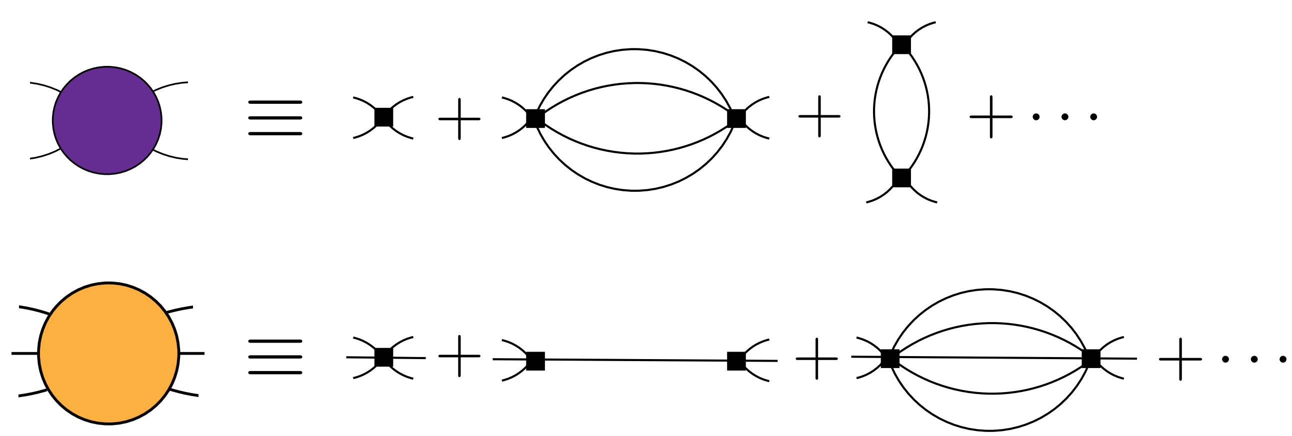


Disconnected diagrams in  **lead to singularities that invalidate the derivation**

New skeleton expansion

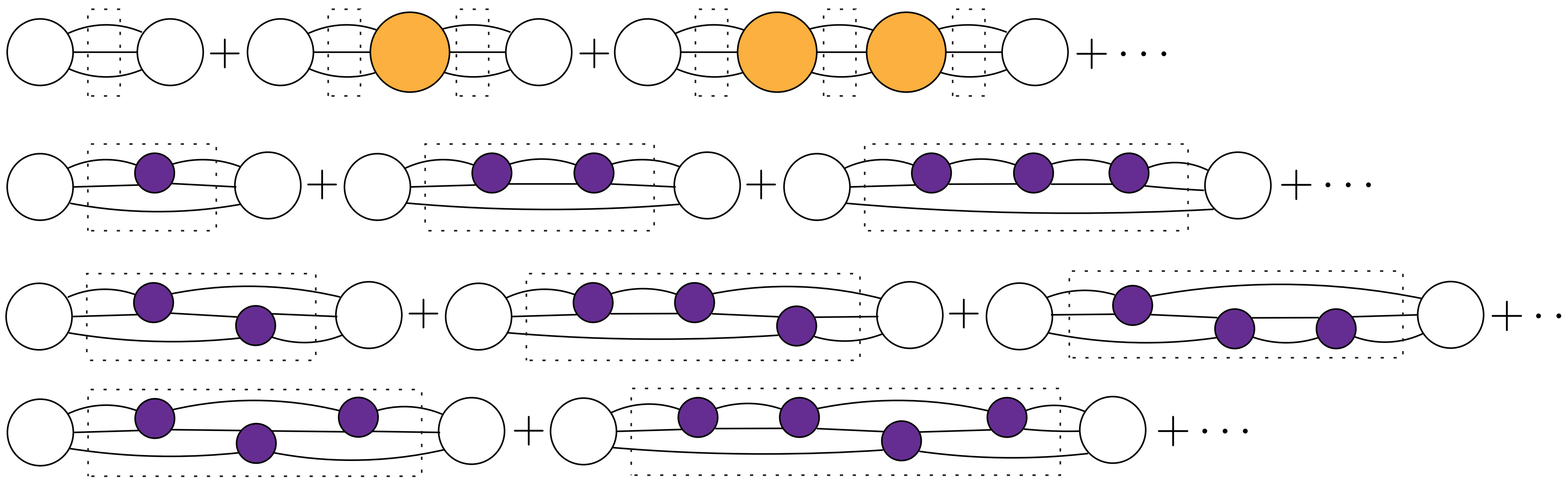


Kernel definitions:

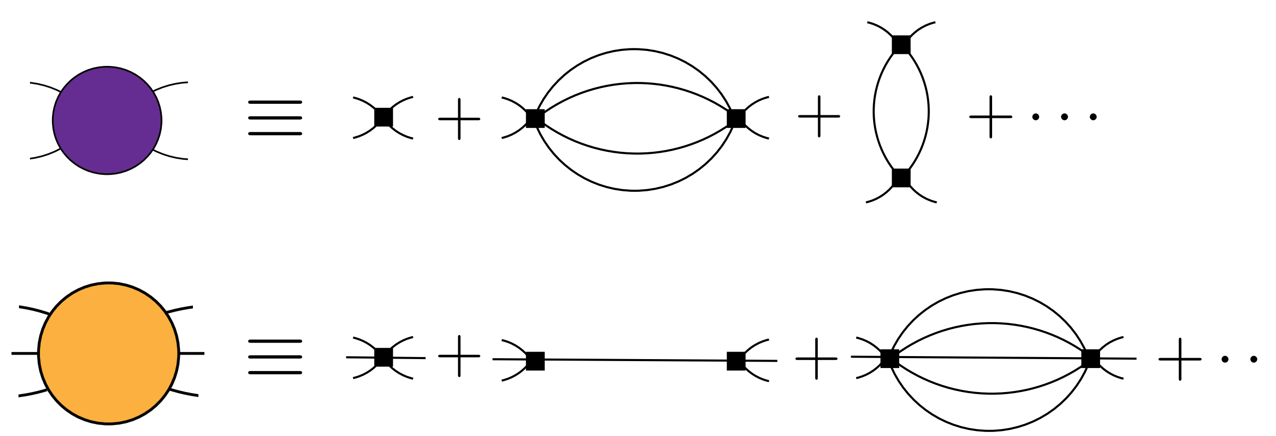


New skeleton expansion

$C_L(E,\vec{P}) =$

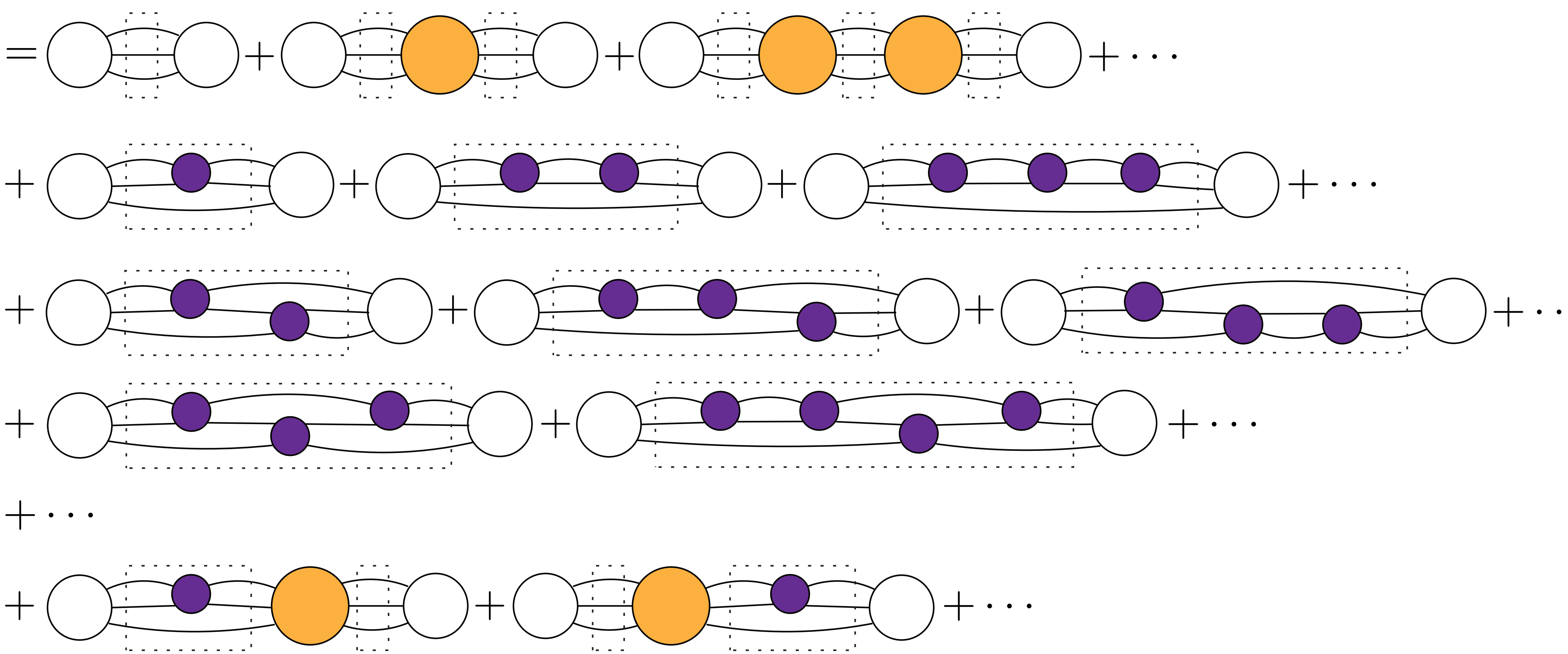


Kernel definitions:

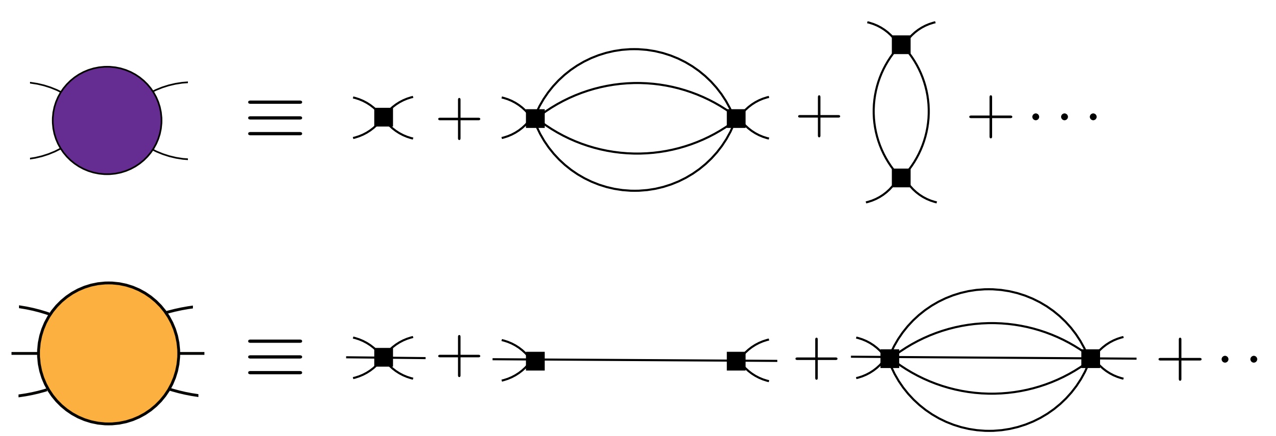


New skeleton expansion

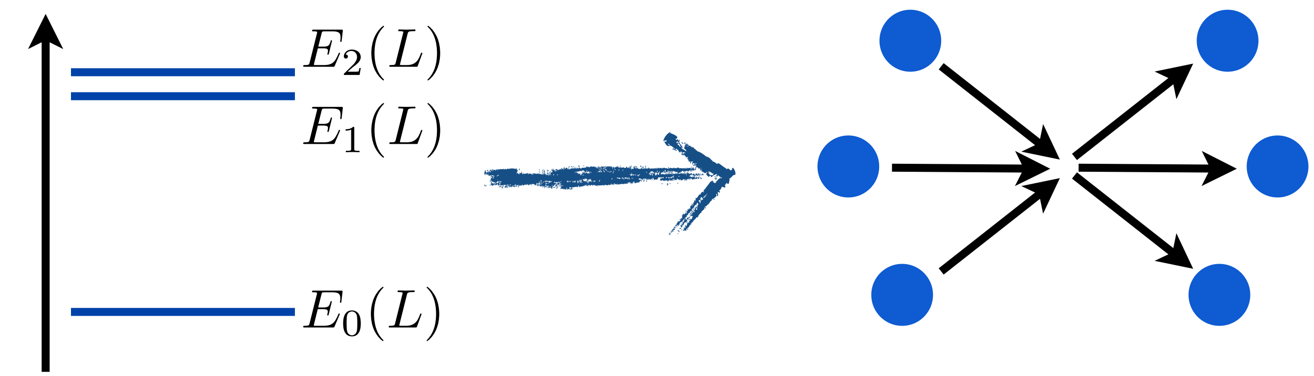
$C_L(E, \vec{P}) =$



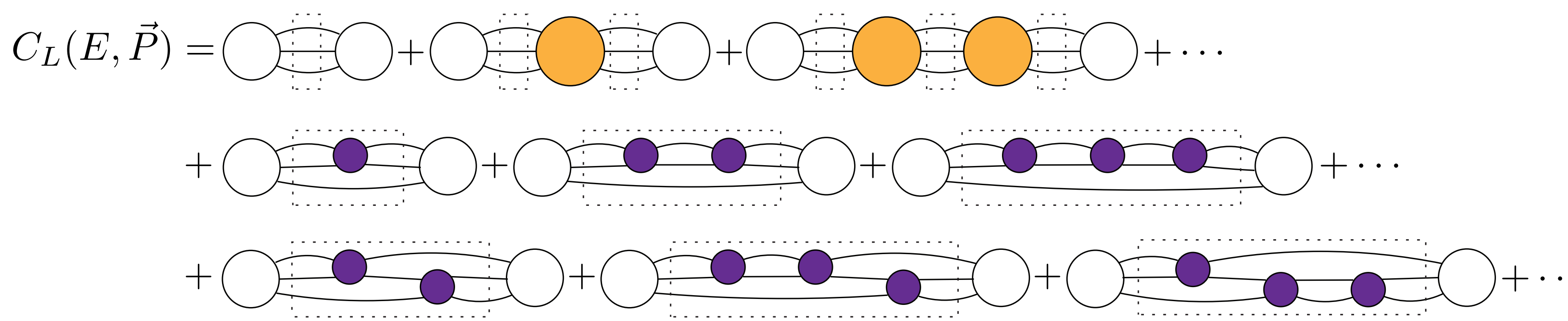
Kernel definitions:



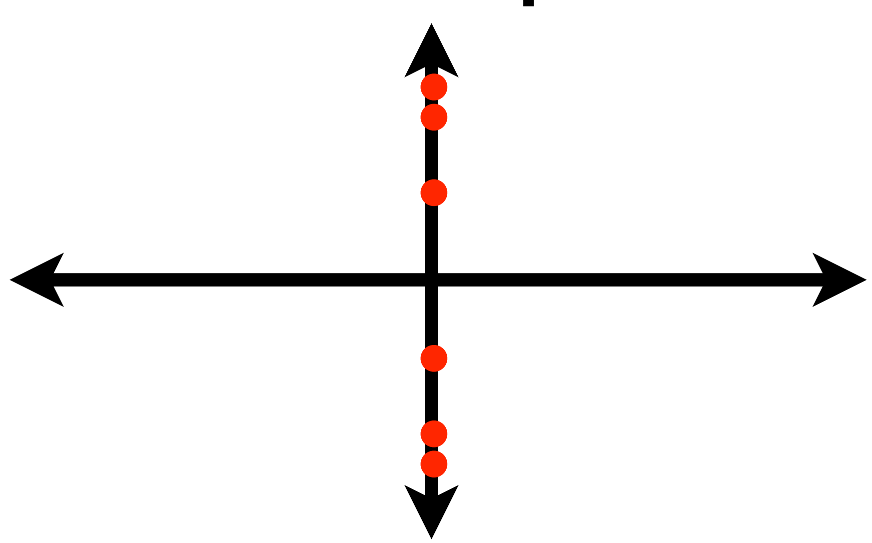
Three-to-three scattering



1. **Work out the three particle skeleton expansion**

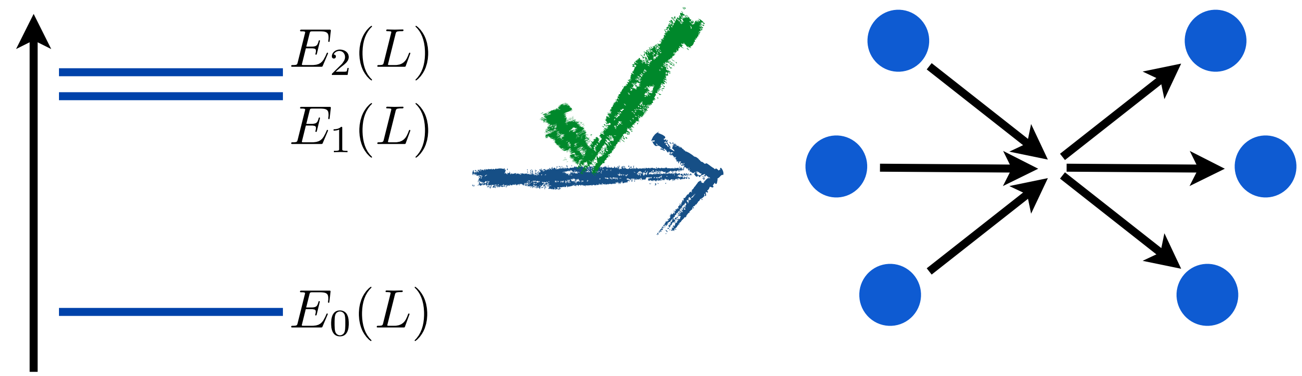


2. **Break diagrams into finite- and infinite-volume parts**
3. **Sum subsets of terms to identify** *infinite-volume quantities*
4. **Relate these to poles in the finite-volume correlator**



$$\det_{k,\ell,m}\left[\mathcal{K}_{\mathrm{df},3}^{-1} + F_3\right] = 0$$

Three-to-three scattering



Current status:

Formalism is complete for the simplest three-scalar system

General, model-independent relation between finite-volume energies and three-to-three scattering amplitude

Derived using a generic relativistic field theory

MTH and Sharpe, *Phys. Rev.* D90, 116003 (2014)
MTH and Sharpe, *Phys. Rev.* D92, 114509 (2015)

Important caveats:

Identical particles with no two-to-three transitions

$$\pi\pi\pi \longrightarrow \pi\pi\pi$$

Requires that two-particle scattering phase is bounded

$$|\delta_\ell(E)| < \pi/2$$

Currently underway:

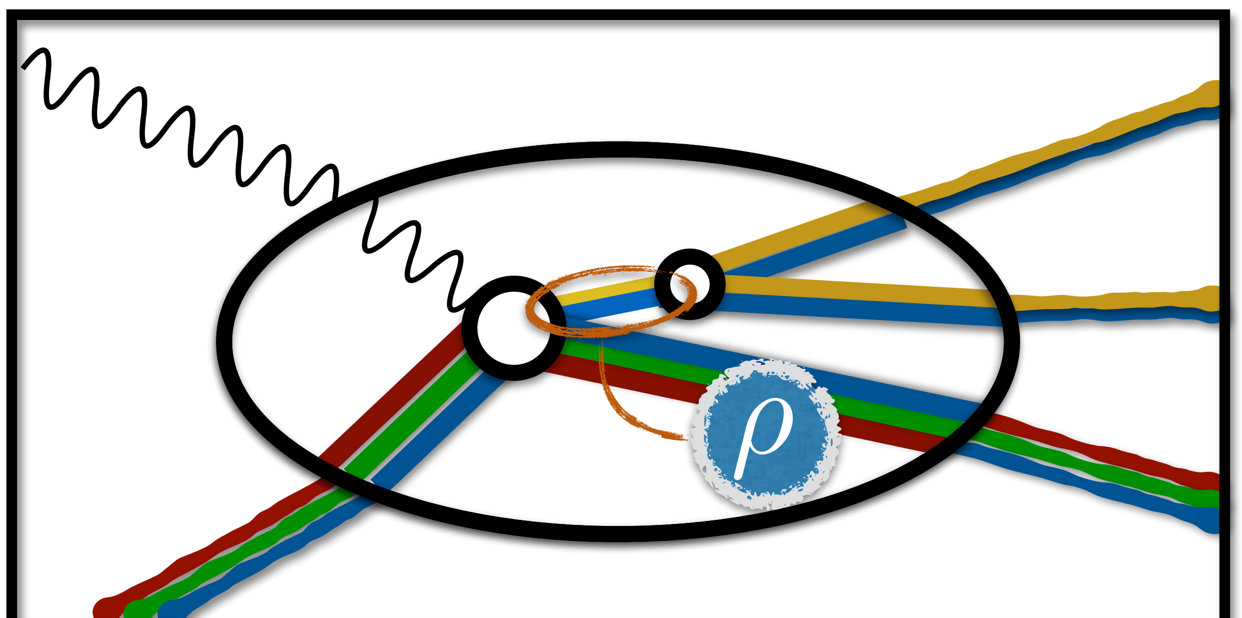
Relax all simplifying assumptions:

Allow all particle types, allow two-to-three couplings, remove bound on phase shift

$K\pi \to K\pi\pi$ $\qquad$ $N\pi \to N\pi\pi$ $\qquad$ $NNN \to NNN$

Briceño, MTH, Sharpe, *in development*

Derive formalism for three-particle transition amplitudes



$$p\gamma \to N\rho \to N\pi\pi$$

Also want to make connections to other work...

Polejaeva and Rusetsky, *Eur. Phys. J.* A48, 67 (2012)

Briceño and Davoudi, *Phys. Rev.* D87, 094507 (2013)

Meißner, Rios and Rusektsky. *Phys. Rev. Lett.* 114, 091602 (2015)