Nucleon Polarisabilities in χEFT and Lattice Perspectives – With Uncertainties

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Office of Science U.S. DEPARTMENT OF ENERGY

Office of Nuclear Physics Exploring Nuclear Matter - Quarks to Stars

How do constituents of the nucleon react to external fields?

How to reliably extract proton, neutron, spin polarisabilities?

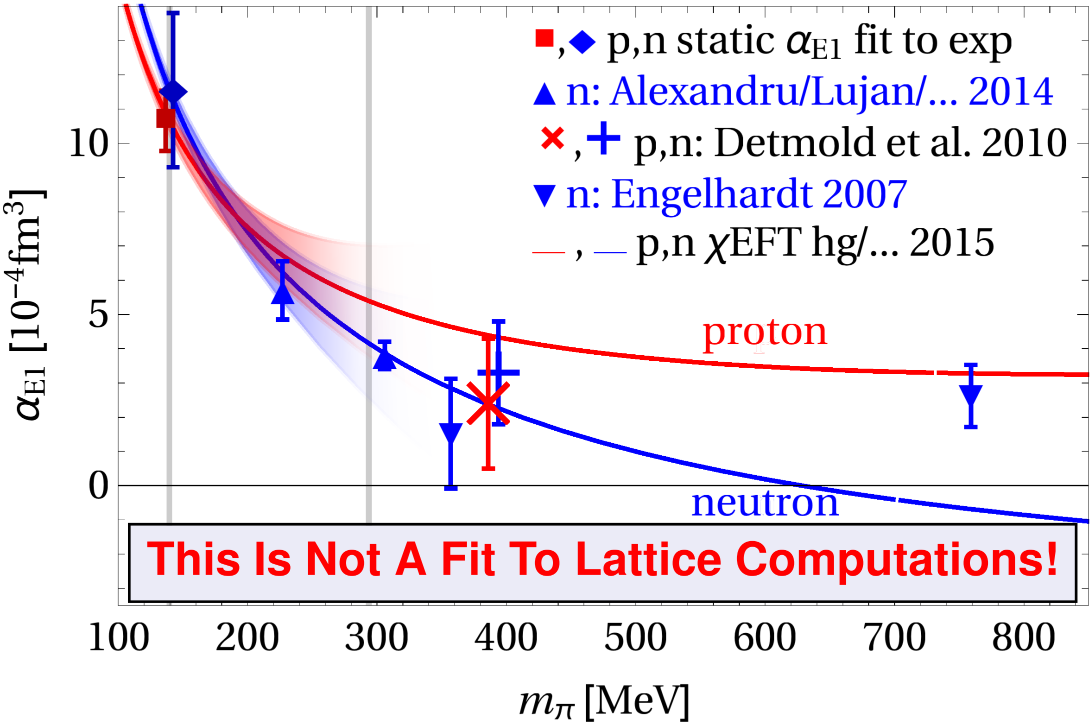
How to bridge between QCD and Nuclear Physics?

Comprehensive Theory Effort:

hg, J. A. McGovern (Manchester), D. R. Phillips (Ohio U): *Eur. Phys. J.* **A49** (2013), 12 (proton)

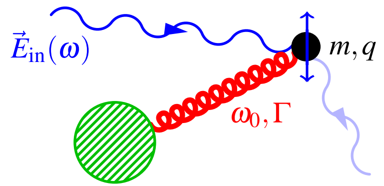
hg/JMcG/DRP/G. Feldman: **Prog. Part. Nucl. Phys. 67 (2012) 841**; as COMPTON@MAX-lab: *Phys. Rev. Lett.* **113** (2014) 262506

Polarisabilities & Bayes in χEFT for lattice-QCD: hg/JMcG/DRP 1511.01952


■,◆ p,n static α_{E1} fit to exp
▲ n: Alexandru/Lujan/... 2014
×,+ p,n: Detmold et al. 2010
▼ n: Engelhardt 2007
—, — p,n χEFT hg/... 2015
proton
neutron
This Is Not A Fit To Lattice Computations!
α_{E1} [10^{-4}fm^3]
10
5
0
100
200
300
400
500
600
700
800
m_π [MeV]

(a) Polarisabilities: Stiffness of Charged Constituents in El.- Mag. Fields

Example: induced electric dipole radiation from harmonically bound charge, damping Γ Lorentz/Drude 1900/1905

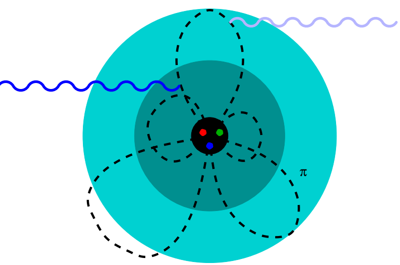


$$\vec{d}_{\text{ind}}(\omega) = \underbrace{\frac{q^2}{m} \frac{1}{\omega_0^2 - \omega^2 - \mathrm{i}\Gamma\omega}}_{=:\, 4\pi\, \alpha_{E1}(\omega)} \vec{E}_{\text{in}}(\omega)$$

$$\mathcal{L}_{\text{pol}} = 2\pi \Big[\underbrace{\alpha_{E1}(\omega)\, \vec{E}^2 + \beta_{M1}(\omega)\, \vec{B}^2}_{} + \dots \Big]$$

electric, magnetic scalar dipole
"displaced volume" $[10^{-3}\ \text{fm}^3]$

$\Longrightarrow$ **Clean, perturbative probe of $\Delta(1232)$ properties, nucleon spin-constituents, χiral symmetry of pion-cloud & its breaking (proton-neutron difference).**



– fundamental hadron property $\Longrightarrow$ **link to emergent lattice-QCD results**
Alexandru/Lee/…2005-, NPLQCD 2006-, LHPC 2007-, Leinweber/…2013

(b) Studying the Two-Photon Response

2015 LRP: ***Great progress has been made in determining the electric and magnetic polarizabilities. Within the next few years, data are expected from [HIγS] that will allow accurate extraction of proton-neutron differences and spin polarizabilities. . . .***

2015 QCD White Paper: "*Synergistic Blend of Theory and Experiment*"

Lattice QCD: relate to fundamental interactions

$\longrightarrow$ *polarQCD* (Alexandru/Lee) 2005-; *NPLQCD* 2006-; *LHPC* (Engelhardt) 2007-; Leinweber/. . . (Adelaide) 2013

Experiment: Significant investments; data taken/scheduled/approved:

HIγS (DOE): a central goal; > 3000 hrs committed at $60 - 100$ MeV

proton doubly & beam pol. (E-06-09/10) deuteron beam pol. (E-18-09, running)

^{3}He unpol & doubly pol. (E-07-10, E-08-16) ^{4}He unpol ^{6}Li unpol. (E-15-11, first!)

A2 @ MAMI (DFG: 5-year SFB): running, data cooking and planned

proton $100 - 400$ MeV: beam & target pol.

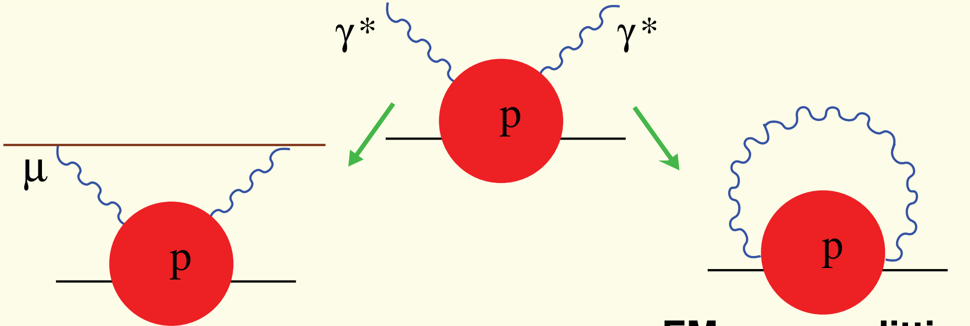
deuteron, ^{3}He, ^{4}He unpol., beam & target pol.

MAXlab: data cooking deuteron $100 - 160$ MeV: unpol.

Chiral EFT: data consistency, binding effects, analysis, extraction

Goal: Unified framework with reliable error bars for proton, deuteron, ^{3}He (elastic & inelastic) into $\Delta(1232)$ region.

(c) Why the Magnetic Polarisability β_{M1} Matters

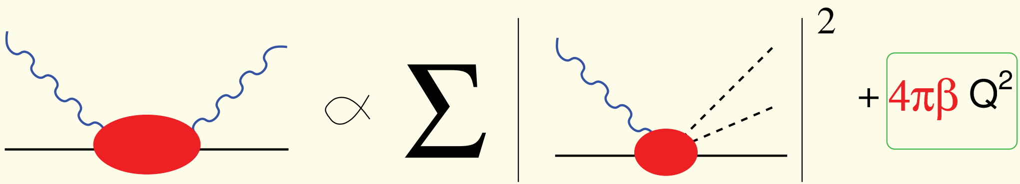


Muonic H

2γ in Lamb shift: proton radius

EM mass splitting

$M_\gamma^{\mathrm{p}} - M_\gamma^{\mathrm{n}} \approx [1.1 \pm 0.5]$ MeV

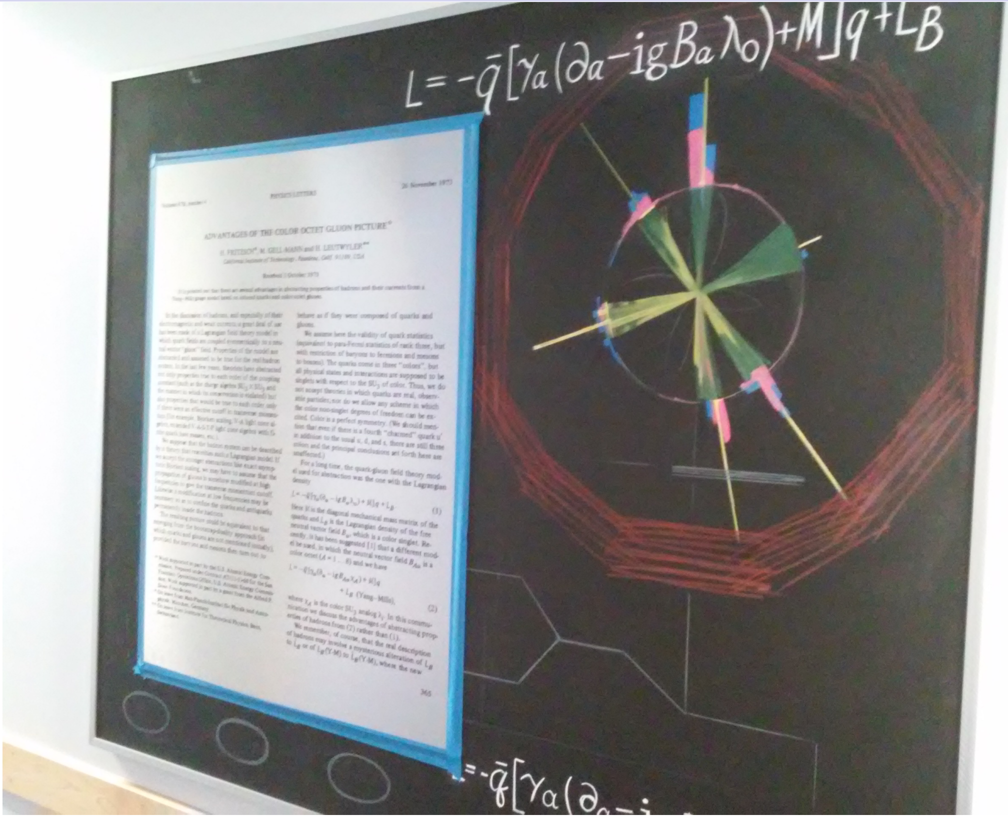


Subtracted DR

Cottingham Sum Rule and VVCS

$$\overline{T}_1(\nu, Q^2) = -\nu^2 \int_{\nu_{th}^2}^{\infty} \frac{d\nu'^2}{\nu'^2} \frac{W_1(\nu', Q^2)}{\nu'^2 - \nu^2} + 4\pi\beta Q^2 + O(Q^4)$$

(d) He Who Controls the Past, Controls the Future. George Orwell: 1984



4 April 2016

(d) He Who Controls the Past, Controls the Future. George Orwell: 1984

5 April 2016

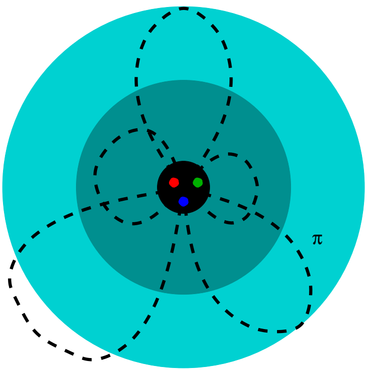
(a) The Method: Chiral Effective Field Theory

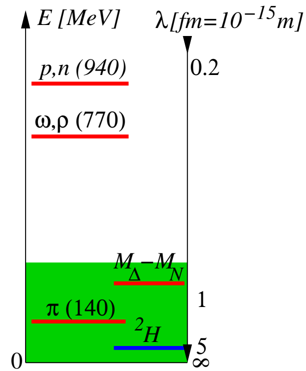
Degrees of freedom $\pi, N, \Delta(1232)$ + **all interactions allowed by symmetries:** Chiral SSB, gauge, iso-spin,. . .

$\Longrightarrow$ **Chiral Effective Field Theory** χ**EFT** $\equiv$ **low-energy QCD**

$$\mathcal{L}_{\chi\text{EFT}} = (D_\mu \pi^a)(D^\mu \pi^a) - m_\pi^2\, \pi^a \pi^a + \dots + N^\dagger [\mathrm{i}\, D_0 + \frac{\vec{D}^2}{2M} + \frac{g_A}{2f_\pi}\, \vec{\sigma}\cdot\vec{D}\pi + \dots] N + C_0 \left(N^\dagger N\right)^2 + \dots$$

Controlled approximation $\Longrightarrow$ ***Model-independent, error-estimate*** **(they say. . .)**





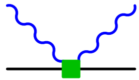
Expand in $\delta = \frac{M_\Delta - M_N}{\Lambda_\chi \approx 1\,\text{GeV}} \approx \sqrt{\frac{m_\pi}{\Lambda_\chi}} = \frac{p_\text{typ}}{\Lambda_\chi} \ll 1$ **(numerical fact)** Pascalutsa/Phillips 2002.

Bernard/Kaiser/Meißner 1992-4, Butler/Savage/Springer 1992-3, Hemmert/... 1998
McGovern 2001, hg/Hemmert/Hildebrandt/Pasquini 2003
McGovern/Phillips/hg 2013

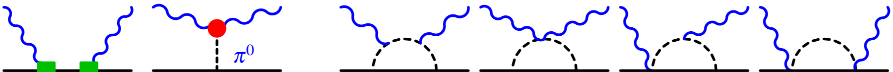
Unified Amplitude: gauge & RG invariant set of all contributions which are

	in low régime	$\omega \lesssim m_\pi$	at least N^{4}LO $(e^2\delta^4)$:	accuracy $\delta^5 \lesssim 2\%$;
or	in high régime	$\omega \sim M_\Delta - M_N$	at least NLO $(e^2\delta^0)$:	accuracy $\delta^2 \lesssim 20\%$.

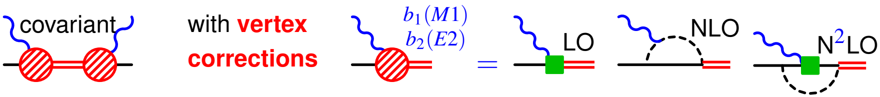
$\omega \lesssim m_\pi$ | $\sim M_\Delta - M_N \approx 300$ MeV



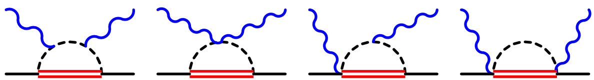
$e^2\delta^0$ ***LO*** | $e^2\delta^0 \searrow$NLO



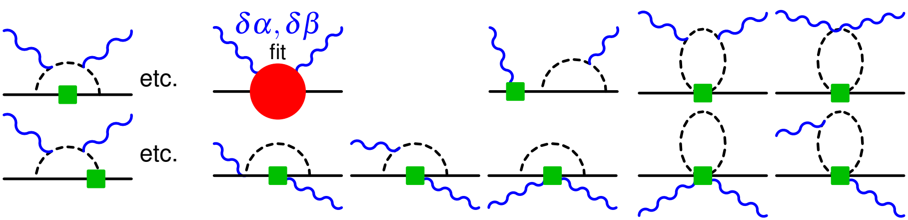
$e^2\delta^2$ N^{2}LO | $e^2\delta^1$ N^{2}LO



$e^2\delta^3$ N^{3}LO | $e^2\delta^{-1} \nearrow$***LO***



$e^2\delta^3$ N^{3}LO | $e^2\delta^1$ N^{2}LO



$e^2\delta^4$ N^{4}LO | $e^2\delta^2$ N^{3}LO

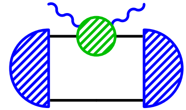
Unknowns: short-distance $\delta\alpha, \delta\beta \Longleftrightarrow$ static α_{E1}, β_{M1}

(c) Neutron Polarisabilities and Nuclear Binding

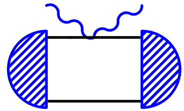
hg/. . . /+Phillips/+McGovern 2004-2014
MECs: Beane/. . . 1999-2005

Need model-independent, systematic subtraction of binding effects. $\Longrightarrow$ χEFT: reliable uncertainties.

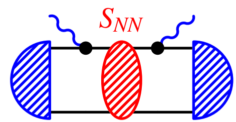
– **Nucleon structure: average of neutron & proton polarisabilities:**
  χ**EFT, Disp. Rel.: p-n difference is small** hg/Pasquini/. . . 2005



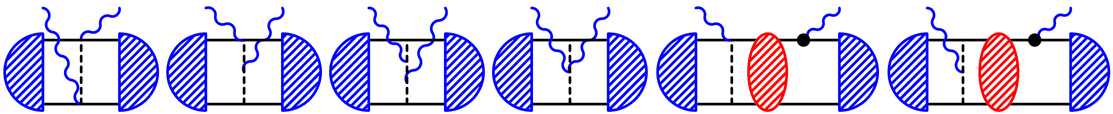
– **Parameter-free one-nucleon contributions:**

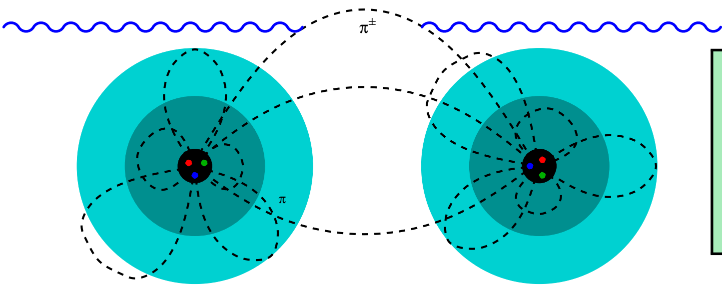


$\sum$ partial waves



– **Parameter-free charged meson-exchange currents** dictated in χEFT by gauge & chiral symmetry:



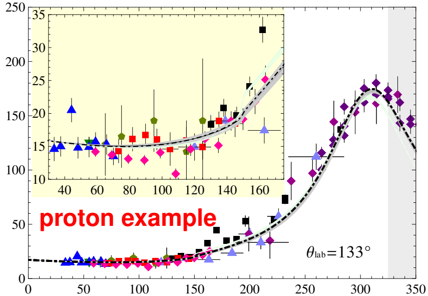


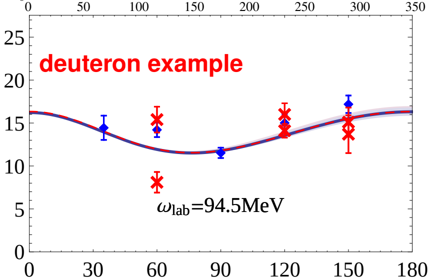
Test charged-pion component of NN force.
Rescattering pivotal for Thomson limit

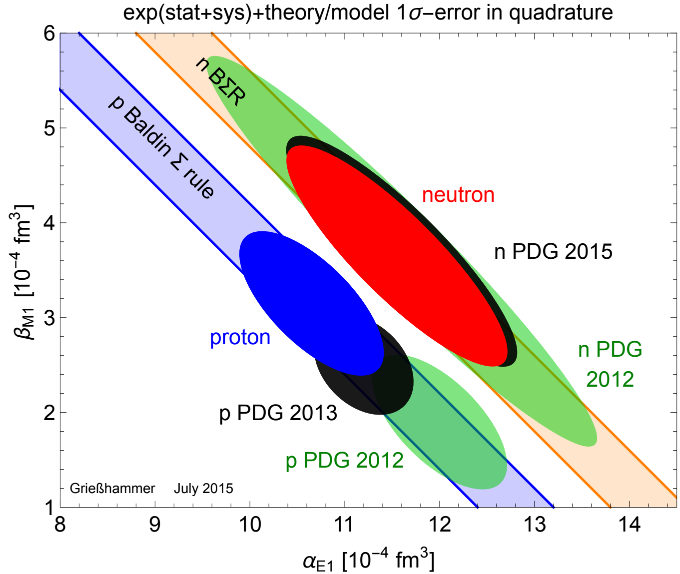
$$\mathcal{A}(\omega=0) = -\frac{e^2}{M_d}\vec{\epsilon}\cdot\vec{\epsilon}'.$$

$\Longrightarrow$ **tiny dep. on d wave fu. & NN pot.**

(d) Scalar Dipole Polarisabilities: Values, Data and Theory Errors in χEFT







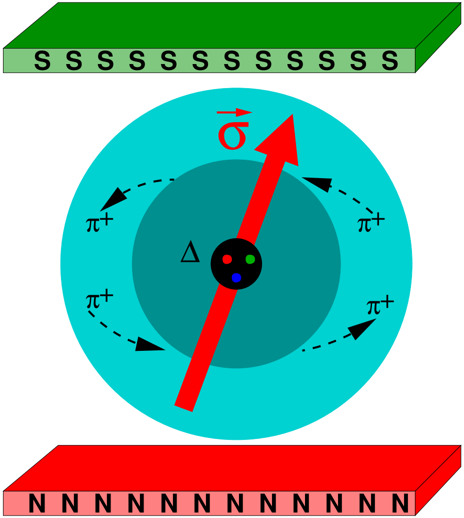
	α_{E1} $[10^{-4}\ \text{fm}^3]$	β_{M1} $[10^{-4}\ \text{fm}^3]$	χ^2/d.o.f.
proton (Baldin, N^{2}LO) McGovern/Phillips/hg EPJA 2013	$10.65 \pm 0.35_{\text{stat}} \pm 0.2_{\Sigma} \pm 0.3_{\text{theory}}$	$3.15 \mp 0.35_{\text{stat}} \pm 0.2_{\Sigma} \mp 0.3_{\text{theory}}$	$113.2/135$
neutron, new data from **Compton@MAXlab** COMPTON@MAX-lab PRL 2014	$11.55 \pm 1.25_{\text{stat}} \pm 0.2_{\Sigma} \pm 0.8_{\text{theory}}$	$3.65 \mp 1.25_{\text{stat}} \pm 0.2_{\Sigma} \mp 0.8_{\text{theory}}$	$45.2/44$

$\Longrightarrow \alpha_{E1}^{p-n} = -0.9 \pm 1.6_{\text{tot}}$: **neutron ≈ proton polarisabilities; exp. error dominates.**

Cottingham ΣR: $M_\gamma^p - M_\gamma^n$ **explained;** $\alpha_{E1}^{p-n} = -1.7 \pm 0.4_{\text{tot}}$. **Gasser/Hoferichter/Leutwyler/Rusetzky 1506.06747**

(e) Spin-Polarisabilities: Nucleonic Bi-Refringence and Faraday Effect

Optical Activity: Response of spin-degrees of freedom, experimental frontier.



$$\mathcal{L}_{\rm pol} = 4\pi\, N^\dagger \times \Big\{ \frac{1}{2}\Big[\alpha_{E1}\,\vec{E}^2 + \beta_{M1}\,\vec{B}^2\Big]$$ **scalar dipole**

$$+\frac{1}{2}\Big[\gamma_{E1E1}\,\vec{\sigma}\cdot(\vec{E}\times\dot{\vec{E}}) + \gamma_{M1M1}\,\vec{\sigma}\cdot(\vec{B}\times\dot{\vec{B}})$$ **"pure" spin-dependent dipole**

$$-2\,\gamma_{M1E2}\,\sigma_i\,B_j\,E_{ij} + 2\,\gamma_{E1M2}\,\sigma_i\,E_j\,B_{ij}\Big] + \dots\Big\}\,N$$

$E_{ij} := \frac{1}{2}(\partial_i E_j + \partial_j E_i)$ etc. **"mixed" spin-dependent dipole**

$+$ quadrupole etc.

$\mathcal{O}(e^2\delta^4)$ χEFT prediction hg/McGovern/Phillips 2014 vs. MAMI extraction Martel/...2014

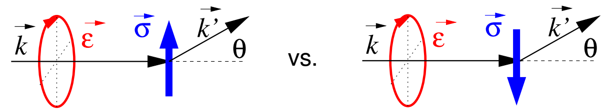
static $[10^{-4}\ \text{fm}^4]$	γ_{E1E1}	γ_{M1M1}	γ_{E1M2}	γ_{M1E2}
MAMI 2014 proton	-3.5 ± 1.2	3.2 ± 0.9	-0.7 ± 1.2	2.0 ± 0.3
χEFT proton	$-1.1\pm1.9_{\rm th}$	$2.2\pm0.5_{\rm stat}\pm0.6_{\rm th}$ **fit to unpol.**	$-0.4\pm0.6_{\rm th}$	$1.9\pm0.5_{\rm th}$
χEFT neutron	$-4.0\pm1.9_{\rm th}$	$1.3\pm0.5_{\rm stat}\pm0.6_{\rm th}$	$-0.1\pm0.6_{\rm th}$	$2.4\pm0.5_{\rm th}$

Spin-Polarisabilities from Polarised Photons

$\mathcal{O}(e^2\delta^3)$: hg/Hildebrandt/...2003
$\mathcal{O}(e^2\delta^4)$: hg/McGovern/Phillips 1511.0952 & in prep.
exp: Martel/...(MAMI) PRL 2014

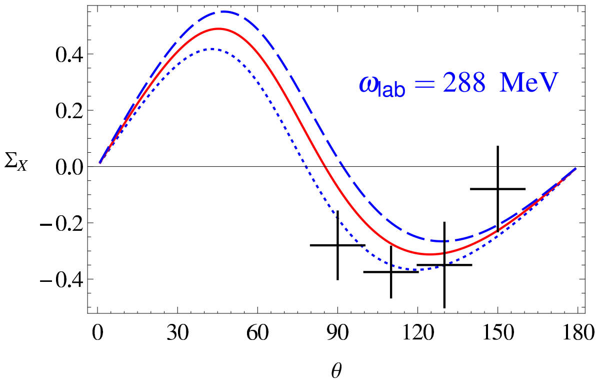
Proton best: Incoming γ circularly polarised, sum over final states. N-spin in $(\vec{k},\vec{k}')$-plane, perpendicular to $\vec{k}$:

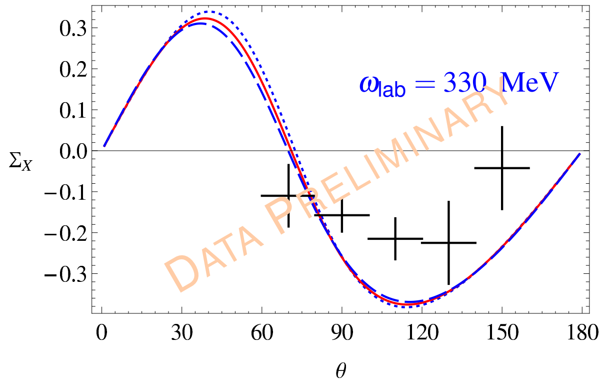
Σ_x :



Compare to Martel/...(MAMI) PRL 2014

$\gamma_{E1E1} =$ (solid) -1.1: χEFT prediction; (dashed) $-1.1+2$; (dotted) $-1.1-2=-3.1 \Longleftrightarrow$ Martel fit: -3.5 ± 1.2





Polarisabilities beyond dipoles negligible – ω-dependence important.
Also good signal for linear polarisations.

(f) (Dis)Agreement Significant Only When All Error Sources Explored

Editorial PRA 83 (2011) 040001

physical effects not included in the calculation from the beginning, such as electron correlation and relativistic corrections. It is of course never possible to state precisely what the error is without in fact doing a larger calculation and obtaining the higher accuracy. However, the same is true for the uncertainties in experimental data. The aim is to estimate the uncertainty, not to state the exact amount of the error or provide a rigorous bound.

There are many cases where it is indeed not practical to give a meaningful error estimate for a theoretical calculation; for example, in scattering processes involving complex systems. The comparison with experiment itself provides a test of our theoretical understanding. However, there is a broad class of papers where estimates of theoretical uncertainties can and should be made. Papers presenting the results of theoretical calculations are expected to include uncertainty estimates for the calculations whenever practicable, and especially under the following circumstances:

1. If the authors claim high accuracy, or improvements on the accuracy of previous work.
2. If the primary motivation for the paper is to make comparisons with present or future high precision experimental measurements.
3. If the primary motivation is to provide interpolations or extrapolations of known experimental measurements.

These guidelines have been used on a case-by-case basis for the past two years. Authors have adapted well to this, resulting in papers of greater interest and significance for our readers.

The Editors

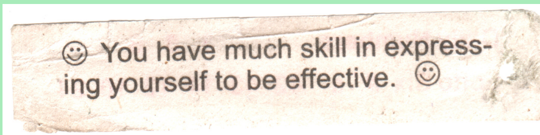
$$\alpha_{E1}^{p} = 10.65 \pm 0.35_{\text{stat}} \pm 0.2_{\Sigma} \pm 0.3_{\text{theory}}$$

Non-Theory Errors: Numerical $\Longrightarrow$ better computers.

Statistical/parameter $\Longrightarrow$ better data.

Theoretical uncertainty: Truncation of Physics

EFT claim: systematic in $Q = \dfrac{\textit{typ. low scale } p_{\textit{typ}}}{\textit{typ. high scale } \overline{\Lambda}_{\textit{EFT}}}$

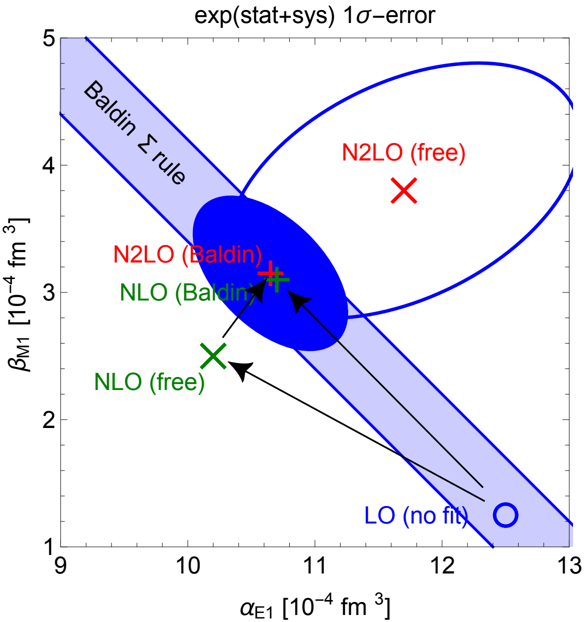


Scientific Method: Quantitative results with corridor of theoretical uncertainties for *falsifiable predictions*.

Need procedure which is established, economical, reproducible: room to argue about "error on the error".

"Double-Blind" Theory Errors: Assess with pretense of no/very limited data.

(g) Fit Discussion: Parameters and Uncertainties



1σ-contours

Consistent with Baldin Σ Rule

$$\alpha_{E1}+\beta_{M1}=\frac{1}{2\pi^2}\int_{\nu_0}^{\infty}\mathrm{d}\nu\,\frac{\sigma(\gamma p\to X)}{\nu^2}$$

$$=13.8\pm0.4$$ Olmos de Leon 2001

need more forward data to constrain.

Residual Theoretical Uncertainty

McGovern/Phillips/hg: EPJ**A49** 12 (2013); many before

Convergence pattern of $\alpha_{E1}-\beta_{M1}$ by

most conservative/worst-case of:

(1) $\delta\approx\frac{2}{5}$ of NLO→N^{2}LO;

(2) $\delta^2\approx\frac{1}{6}$ of LO→NLO;

(3) $\delta^2\approx\frac{1}{6}$ of LO→N^{2}LO. ⟸

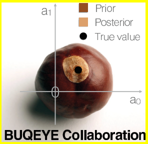
Fit Stability: floating norms within exp. sys. errors; vary dataset, b_1, vertex dressing,. . .

	α^p_{E1} [10^{-4} fm^3]	β^p_{M1} [10^{-4} fm^3]	χ^2/d.o.f.
N^{2}LO Baldin constrained $\alpha^p_{E1}+\beta^p_{M1}=13.8\pm0.4$	$10.65\pm0.4_{\text{stat}}\pm0.2_{\Sigma}\pm0.3_{\text{theory}}$	$3.15\mp0.4_{\text{stat}}\pm0.2_{\Sigma}\mp0.3_{\text{theory}}$	$113.2/135$

(h) Fit Discussion: What Does "Conservative" Error Mean?

Observable/Series $\mathcal{O} = \delta^n \left(c_0 + c_1\delta^1 + c_2\delta^2 + \textbf{unknown} \times \delta^4\right) \Longrightarrow$

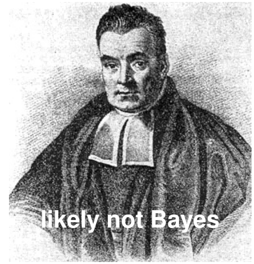
Estimate next term *"conservatively"* as $|\textbf{unknown}\ c_3| \lesssim R := \max\{|c_0|;|c_1|;|c_2|\}$.



No infinite sampling pool; data fixed; more data changes confidence.

$\Longrightarrow$ **Call upon the Reverend Bayes!**

see e.g. **BUQEYE collaboration** **Furnstahl/Phillips/…1506.01343**



Rev. Bayes frequents his local bar. Bartender: "What do you want?"

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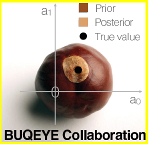
Observable/Series $\mathcal{O} = \delta^n \left(c_0 + c_1\delta^1 + c_2\delta^2 + \textbf{unknown} \times \delta^4\right) \implies$

Estimate next term *“conservatively”* as $|\textbf{unknown } c_3| \lesssim R := \max\{|c_0|;|c_1|;|c_2|\}$.



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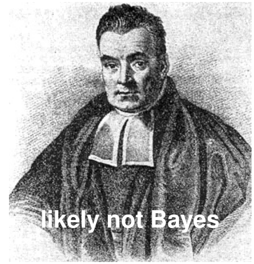


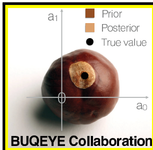
Rev. Bayes frequents his local bar. Bartender: “What do you want?”

– Bayes: “What do you think?”

(h) Fit Discussion: What Does "Conservative" Error Mean?

Observable/Series $\mathcal{O} = \delta^n \left(c_0 + c_1 \delta^1 + c_2 \delta^2 + \textbf{unknown} \times \delta^4 \right) \implies$

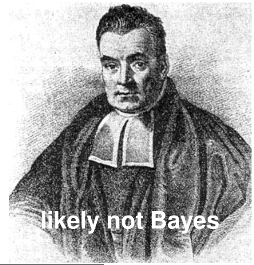
Estimate next term *"conservatively"* as $|\textbf{unknown } c_3| \lesssim R := \max\{|c_0|;|c_1|;|c_2|\}$.



No infinite sampling pool; data fixed; more data changes confidence.

$\implies$ **Call upon the Reverend Bayes!**

see e.g. **BUQEYE collaboration** **Furnstahl/Phillips/...1506.01343**



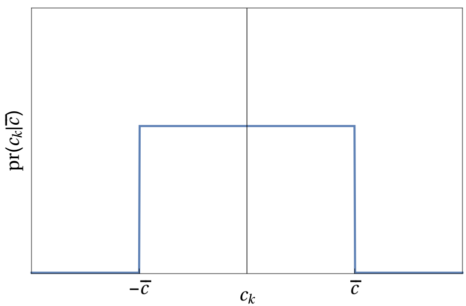
Rev. Bayes frequents his local bar. Bartender: "What do you want?"

– Bayes: "What do you think?"

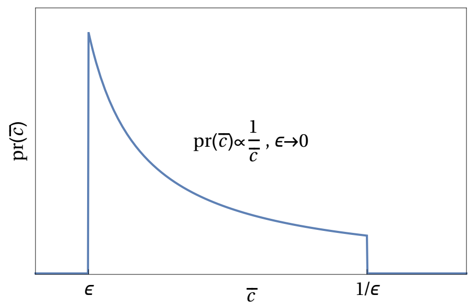
Bayes makes you specify your premises/assumptions about series.

Priors: leading-omitted term dominates ($\delta \ll 1$); putative distributions of *all* c_k's and of largest value $\bar{c}$ in series.

"Least informed/informative": All values c_k equally likely, given upper bound $\bar{c}$ of series.



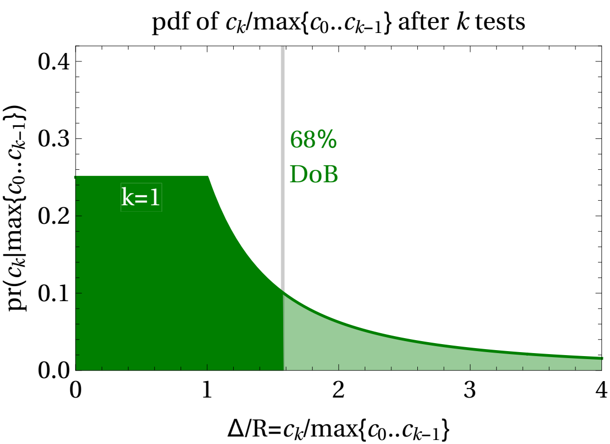
"Any upper bound": $\ln$-uniform prior sets no bias on scale of $\bar{c}$.



Information: Convergence LO$\longrightarrow$NLO$\longrightarrow$N^{2}LO gives probable "largest number" $R = \delta^k \max\{|c_0|\ldots|c_{k-1}|\}$.

Result: **Posterior** $\equiv$ ***Degree of Belief (DoB)*** that next term $c_k\delta^k$ differs from order-k central value by Δ.

$$\mathrm{pr}(\Delta|\max.\,R,\text{order }k) \propto \int_0^\infty \mathrm{d}\bar{c}\;\mathrm{pr}(\bar{c})\;\mathrm{pr}(c_k = \frac{\Delta}{\delta^k}|\bar{c}) \prod_n^{k-1} \mathrm{pr}(c_n|\bar{c}) \to \frac{k}{k+1}\,\frac{1}{2R}\begin{cases} 1 & |\Delta| \le R \\ \left(\frac{R}{|\Delta|}\right)^{k+1} & |\Delta| > R \end{cases}$$



order	DOB in $\pm R$	σ	$\Delta(95\%)$
LO	50%	$1.6\,R$	$11R = 7\sigma$
Gauß	68.27%	$1.0\,R$	2.0σ

Information: Convergence LO→NLO→N^{2}LO gives probable "largest number" $R = \delta^k \max\{|c_0| \dots |c_{k-1}|\}$.

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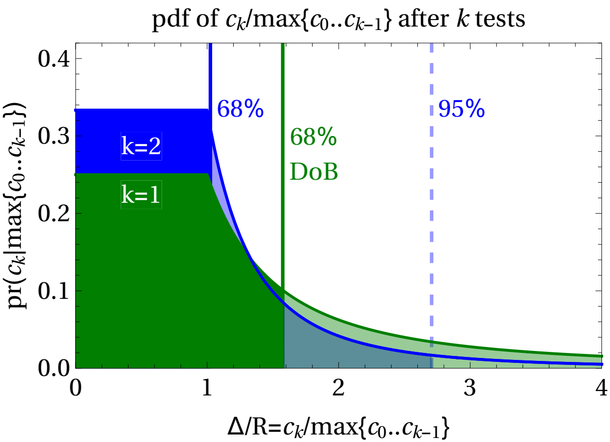

order	DOB in $\pm R$	σ	$\Delta(95\%)$
LO	50%	$1.6\,R$	$11R = 7\sigma$
NLO	66.7%	$1.0\,R$	$2.7R = 2.6\sigma$
Gauß	68.27%	$1.0\,R$	2.0σ

Information: Convergence LO⟶NLO⟶N^{2}LO gives probable "largest number" $R = \delta^k \max\{|c_0|\dots|c_{k-1}|\}$.

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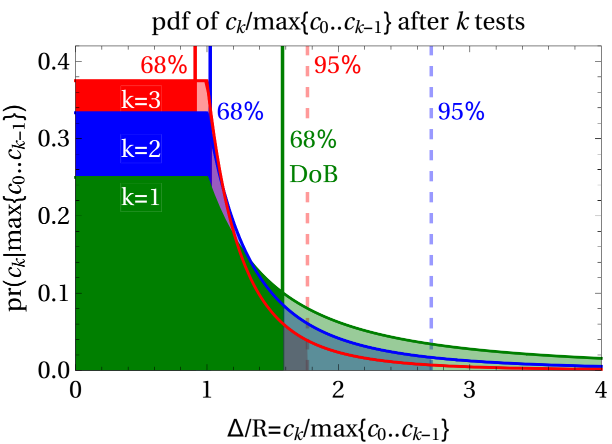


order	DOB in $\pm R$	σ	$\Delta(95\%)$
LO	50%	$1.6\,R$	$11R = 7\sigma$
NLO	66.7%	$1.0\,R$	$2.7R = 2.6\sigma$
N^{2}LO	75%	$0.9\,R$	$1.8R = 1.9\sigma$
k	$\frac{k}{k+1}\%$		
Gauß	68.27%	$1.0\,R$	2.0σ

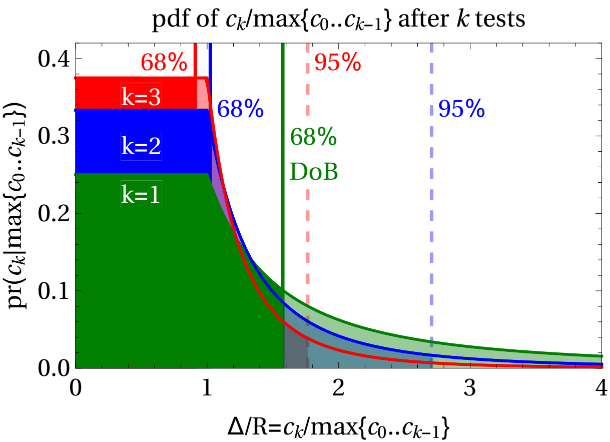
For "high enough" order, largest number R limits $\gtrsim 68\%$ **degree-of-belief interval.**

Quantifying One's Beliefs

Information: Convergence LO⟶NLO⟶N^{2}LO gives probable "largest number" $R = \delta^k \max\{|c_0|\ldots|c_{k-1}|\}$.

Result: Posterior $\equiv$ ***Degree of Belief (DoB)*** that next term $c_k\delta^k$ differs from order-k central value by Δ.

$$\mathrm{pr}(\Delta|\max.\,R,\text{order }k) \propto \int_0^\infty \mathrm{d}\bar{c}\;\mathrm{pr}(\bar{c})\;\mathrm{pr}\Big(c_k = \frac{\Delta}{\delta^k}\Big|\bar{c}\Big)\prod_n^{k-1}\mathrm{pr}(c_n|\bar{c}) \to \frac{k}{k+1}\,\frac{1}{2R}\begin{cases} 1 & |\Delta| \le R \\ \left(\frac{R}{|\Delta|}\right)^{k+1} & |\Delta| > R \end{cases}$$



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k	$\frac{k}{k+1}\%$		
Gauß	68.27%	$1.0\,R$	2.0σ

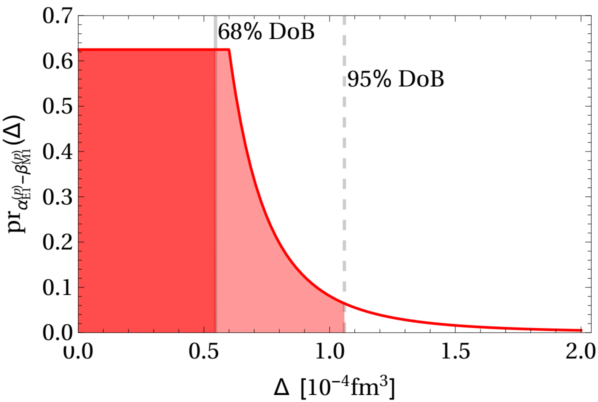
For "high enough" order, largest number R limits $\gtrsim 68\%$ **degree-of-belief interval.**

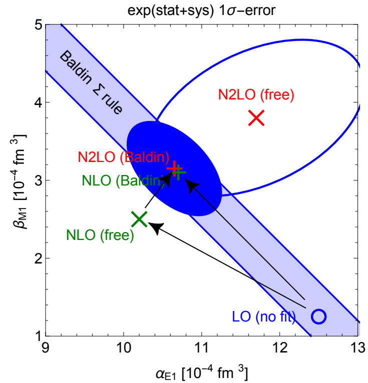
Varying priors: When $k \ge 2$ orders known, DoBs with different assumptions about $\bar{c}$, c_nc vary by $\lesssim \pm 20\%$.

Posterior pdf *not Gauß'ian:* Plateau & power-law tail.

$\Longrightarrow$ ***Interpretation of all theory uncertainties, with these priors; "$A\pm\sigma$": 68% DoB interval $[A-\sigma; A+\sigma]$.***

Uncertainty Profiles of Polarisabilities at the Physical Point



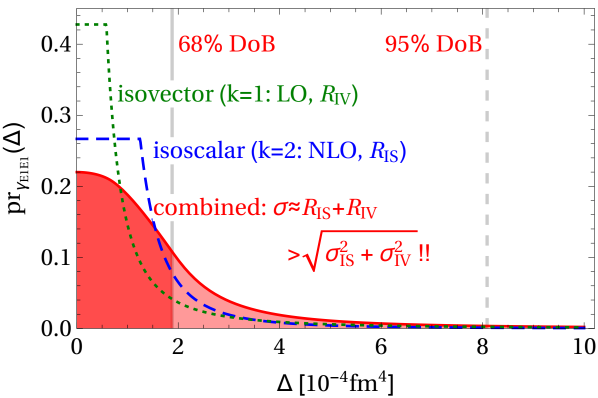


Scalar Pols.:

N^{2}LO $\Longrightarrow k = 3$

Baldin: $\alpha^p_{E1} + \beta^p_{M1}$ fixed.

$\Longrightarrow$ Profile in $\alpha^p_{E1} - \beta^p_{M1}$ translates to $2\Delta_\alpha = 2\Delta_\beta$.



Spin Polarisabilities: LO: $\gamma_i \propto \dfrac{1}{m_\pi^2}$.

$\Longrightarrow$ No $\gamma_i \propto \dfrac{1}{m_\pi(M_\Delta - M_N)}$: $M_\Delta - M_N$ is IR cutoff.

$\Longrightarrow k = 2$ nonzero orders.

Isovector starts at NLO, isospin basis natural in χEFT.

$\Longrightarrow$ **Convolute isoscalar & isovector uncertainties!**

Not Gaußian “add-in-quadrature”, more like linear.

Bayes provides well-defined *procedure*!

(i) Isovector Contributions At The Physical Point

Isovector polarisabilities $\xi^v := \frac{1}{2}(\xi^p - \xi^n)$ at N^2LO; parameter-free. $\Longrightarrow \sim 20\%$ of LO?

Fits:

$$\alpha_{E1}^{p-n} = -0.9 \pm 1.3_{\text{tot}}$$
$$\beta_{M1}^{p-n} = -0.5 \pm 1.3_{\text{tot}}$$

$\Longrightarrow$ Consider m_q-dependence!

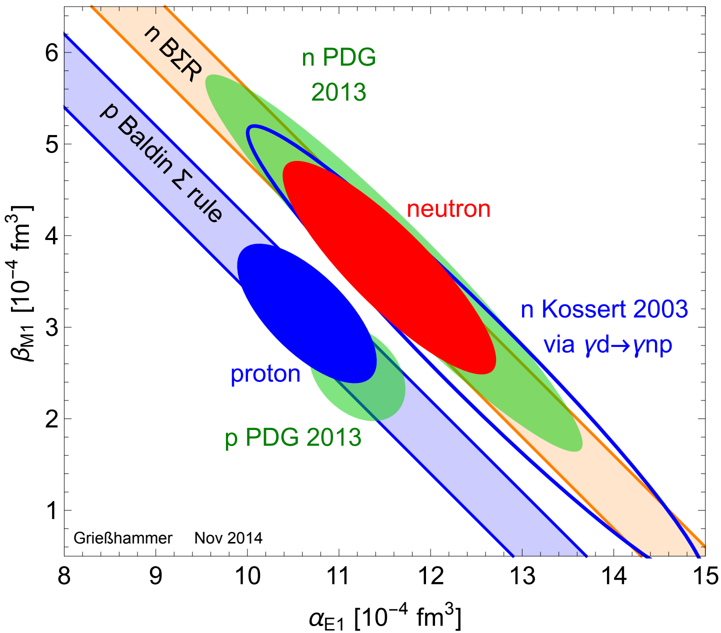
$$\left.\frac{\mathrm{d}\beta_{M1}^v}{\mathrm{d}\ln m_{\mathrm{q}}}\right|_{m_\pi^{\text{phys}}} = 0.65 \pm 0.4_{\text{th}}$$

$$\left.\frac{\mathrm{d}\alpha_{E1}^v}{\mathrm{d}\ln m_{\mathrm{q}}}\right|_{m_\pi^{\text{phys}}} = 0.7 \pm 0.4_{\text{th}}$$

HW: Get σ! Know isovector only at LO: $k = 1$

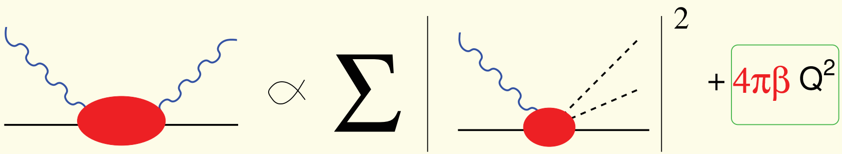
solution: $\sigma = 1.6R = 1.6 \times \text{LO} \times (\delta = 0.4)$

exp(stat+sys)+theory/model 1σ–error in quadrature



Possible fine-tuning at m_π^{phys} (statistically weak signal).

$\Longrightarrow$ **SPECULATION – NO ERROR BARS** hg/JMcG/DRP 1511.01952



Subtracted DR

$$\overline{T}_1(\nu,Q^2) = -\nu^2 \int_{\nu_{th}^2}^{\infty} \frac{d\nu'^2}{\nu'^2} \frac{W_1(\nu',Q^2)}{\nu'^2-\nu^2} + 4\pi\beta Q^2 + O(Q^4)$$

Cottingham Σ Rule: $\beta_{M1}^{\nu} \Longleftrightarrow$ proton-neutron self-energy difference: $M_{\mathrm{p-n}} = M_{\mathrm{p-n}}^{\mathrm{strong}} + M_{\mathrm{p-n}}^{\mathrm{em,elastic}} - A\ \beta_{M1}^{\nu}$

If $-A\beta_{M1}^{\nu} \approx 0.5\ \mathrm{MeV}$ ***and If*** dispersive $A \propto \int\limits_0^{\Lambda} dQ^2\, Q^2 \left(\frac{m_\rho^2}{m_\rho^2+Q^2}\right)^2$ weakly m_π-dependent Walker-Loud/ Carlson/Miller 2012

Then $\left.\frac{dM_{\mathrm{p-n}}^{\beta}(m_\pi)}{d\ln m_{\mathrm{q}}}\right|_{m_\pi^{\mathrm{phys}}} = -0.65\ \mathrm{MeV}$: ***Might not be negligible*** vs. $\left.\frac{dM_{\mathrm{p-n}}^{\mathrm{strong}}}{d\ln m_{\mathrm{q}}}\right|_{m_\pi^{\mathrm{phys}}} \approx -2.1\ \mathrm{MeV}$ Bedaque/Luu/ Platter 2011

$\Longrightarrow$ Impact on neutron lifetime relates to Anthropic Principle. . . (shortened for larger m_q??)

3. Polarisabilities Beyond Physical Pion Masses

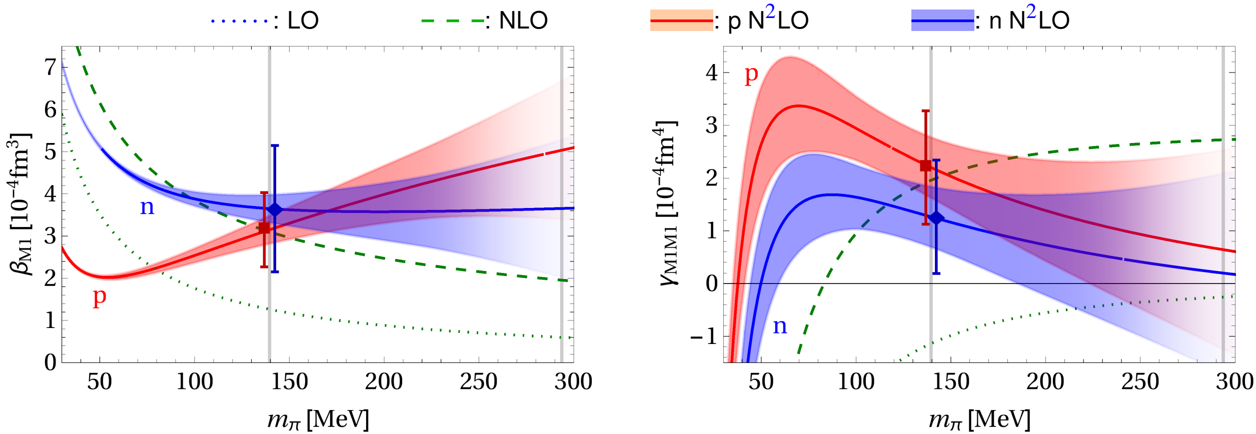
(a) Extending Chiral Corridors of Uncertainties

χEFT: explicit m_π-dependence, parameters fixed at m_π^{phys}.

Propagating Uncertainties:

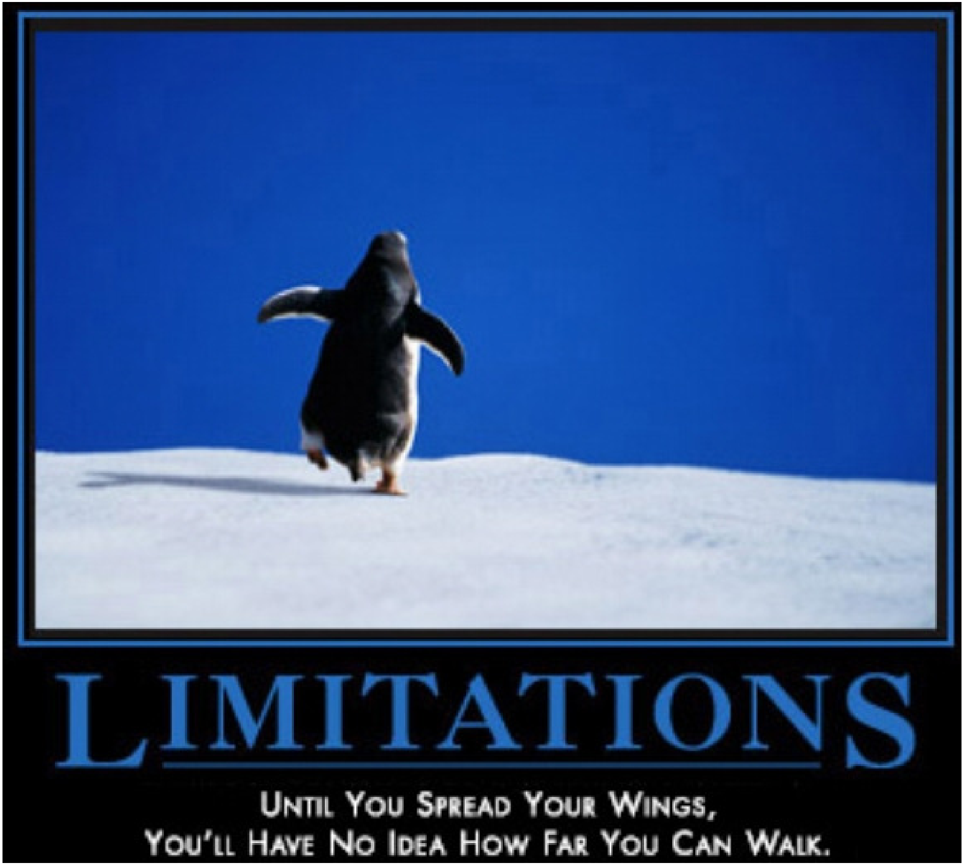
– Bayesian order-by-order as before, now at each m_π.

– Some contributions linear in m_π. $\Longrightarrow$ Conservative expansion parameter $\delta(m_\pi) = 0.4 \times \frac{m_\pi}{m_\pi^{\text{phys}}}$.



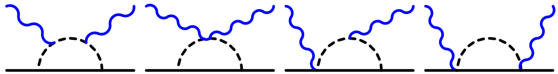
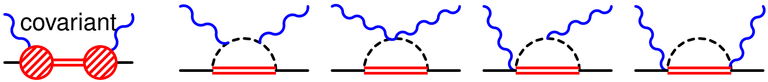
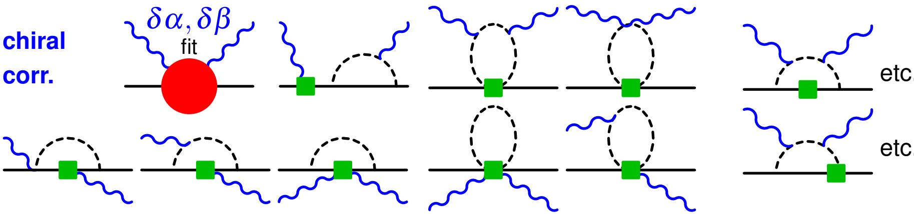
At physical $m_\pi = 140$ MeV: paramagnetic $\Delta(1232)$ fine-tuned against diamagnetic NLO πN loops.

Only physical point without substantial isospin splitting.


LIMITATIONS
Until You Spread Your Wings,
You'll Have No Idea How Far You Can Walk.

(b) It's A Bit More Complicated…

Bernard/Kaiser/Meißner 1992-4, Butler/Savage/Springer 1992-3, Hemmert/…1998
Kumar/McGovern/Birse 2000, McGovern 2001, JMcG/DRP/hg 2013 + 1511.01952

Both *magnitude* and *relative* importance of contributions change with m_π:		$\sim m_\pi^{\text{phys}}$	$\sim M_\Delta - M_N \approx 300\text{ MeV}$
charged pion cloud infinite in chiral limit		$e^2\delta^2$ LO	$e^2\epsilon^1$ **LO** isoscalar only
$\Delta(1232)$ + **its π cloud**		$e^2\delta^3$ NLO	
chiral corr.		$e^2\delta^4$ N^{2}LO	$e^2\epsilon^2$ NLO **incomplete:** no χcorrection to Δ & $\Delta\pi$; isovector incomplete

(i) Close to m_π^{phys} $\Longrightarrow$ $\sqrt{\frac{m_\pi}{\Lambda_\chi \approx 800\text{ MeV}}} \approx \frac{M_\Delta - M_N}{\Lambda_\chi} =: \delta$**-counting** Pascalutsa/Phillips 2002

(ii) Close to 300 MeV $\Longrightarrow$ $\frac{m_\pi \sim (M_\Delta - M_N)}{\Lambda_\chi} =: \epsilon$**-counting** Manohar/Jenkins 1994,…

(iii) Beyond $\Lambda_\chi \approx 800\text{ MeV}$ $\Longrightarrow$ no small parameter, no convergence $\Longrightarrow$ ***at best qualitatively useful!***

Use unified amplitude: $\Longrightarrow$ **Accuracy N^{2}LO ($\sim 6\%$) for $m_\pi \sim 140\text{ MeV}$, LO ($\sim 40\%$) for $m_\pi \sim 300\text{ MeV}$.**

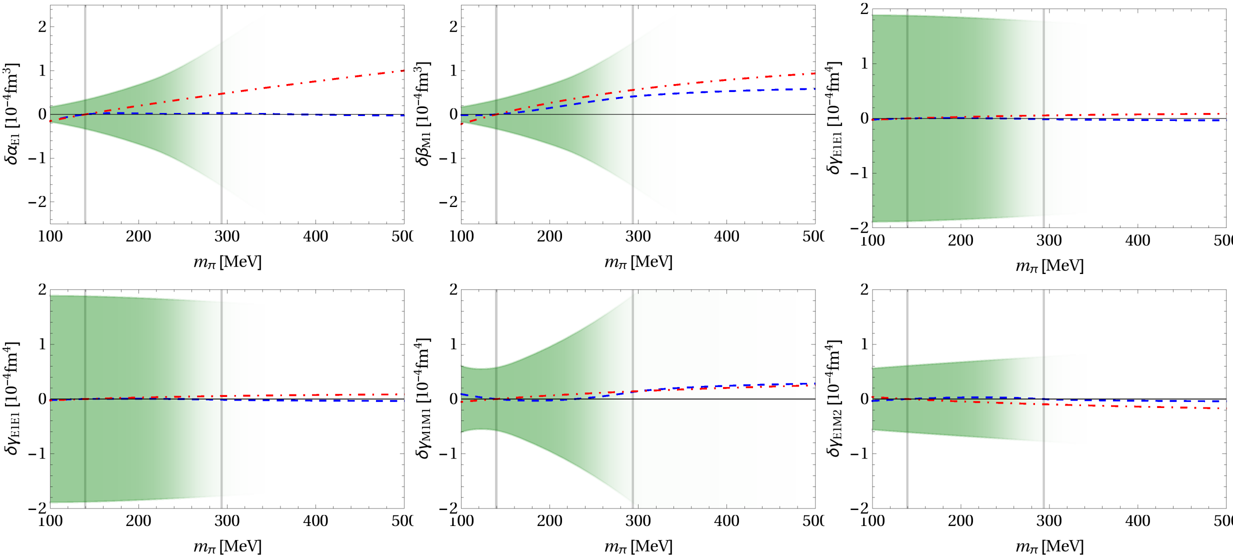
Gradual loss of accuracy, isovector incomplete, only LO more sensitive to Bayesian prior.

$\Longrightarrow$ ***Fade corridors out beyond*** $\sim 250\text{ MeV}$.

At this order, $g_A, f_\pi, M_N, (M_\Delta - M_N), \ldots$ independent of m_π.

(c) Test Uncertainties: Selected Higher-Order Corrections

Theory uncertainty at $m_\pi = 140$ MeV from convergence pattern.
Less strict as $m_\pi \nearrow \Delta$, breakdown as $m_\pi \nearrow \Lambda_\chi$. Confirm by selected higher-order terms.



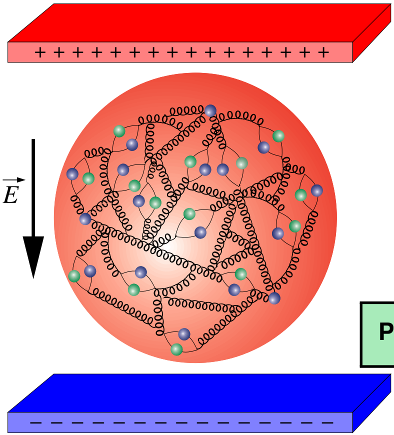
Uncertainties over-estimated?? There could be worse. . .

Constructed *intrinsic* χEFT uncertainties and credibility region. $\Longrightarrow$ Predictive power, falsifiable.

(d) En Route to Static Polarisabilities from Lattice QCD: Chiral Extrapolations

Towards comparable uncertainties in experiment, χEFT and lattice QCD:

χEFT at $\mathcal{O}(e^2\delta^4)$ provides reliable error estimate for $\frac{m_\pi}{\Lambda_\chi}$ extrapolation.



$$\mathcal{L}_{\mathrm{pol}} = 2\pi\, N^\dagger \left[\alpha_{E1}\, \vec{E}^2 + \beta_{M1}\, \vec{B}^2 + \gamma_{E1E1}\, \vec{\sigma}\cdot(\vec{E}\times\dot{\vec{E}}) + \dots \right] N$$

Pick fully dynamical, $m_\pi \ll \Lambda_\chi \approx 800\,\mathrm{MeV}$, ∞ volume: mostly neutron.

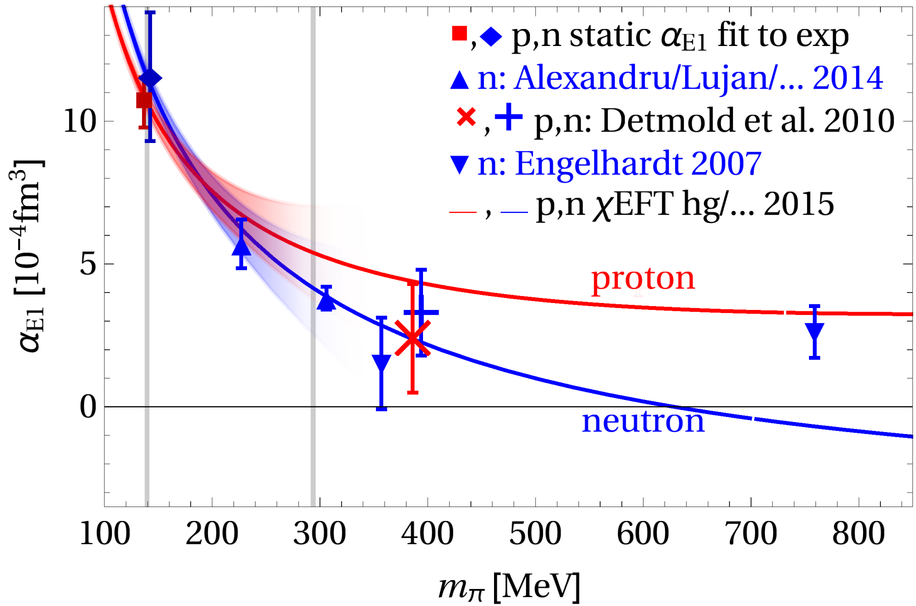
Active lattice groups:

Alexandru/Lee/…2005-;
Engelhardt/LHPC 2006-;
NPLQCD 2006-, 2015;
Leinweber/Primer/Hall/…2013-

(e) Electric Polarisabilities: This Is Not A Fit

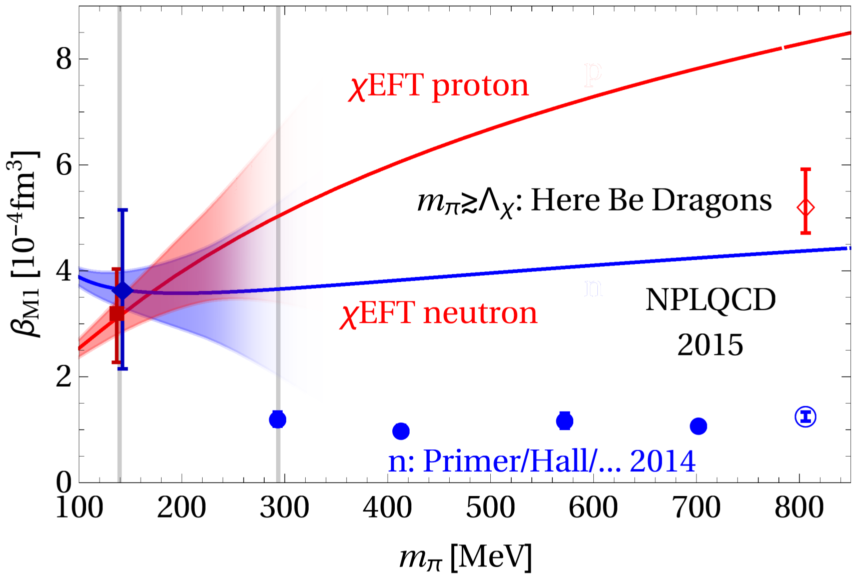
Criteria: $m_\pi \ll \Lambda_\chi \approx 800$ MeV, **extrapolated to infinite volume, fully dynamical (except for charging sea).**

Lattice computations use χEFT for infinite-volume and partial-quenching: Detmold/Tiburzi/Walker-Loud 2006.



χ**EFT insinuates substantial isospin splitting for** $m_\pi \gtrsim 300$ MeV **– beyond credibility region.**

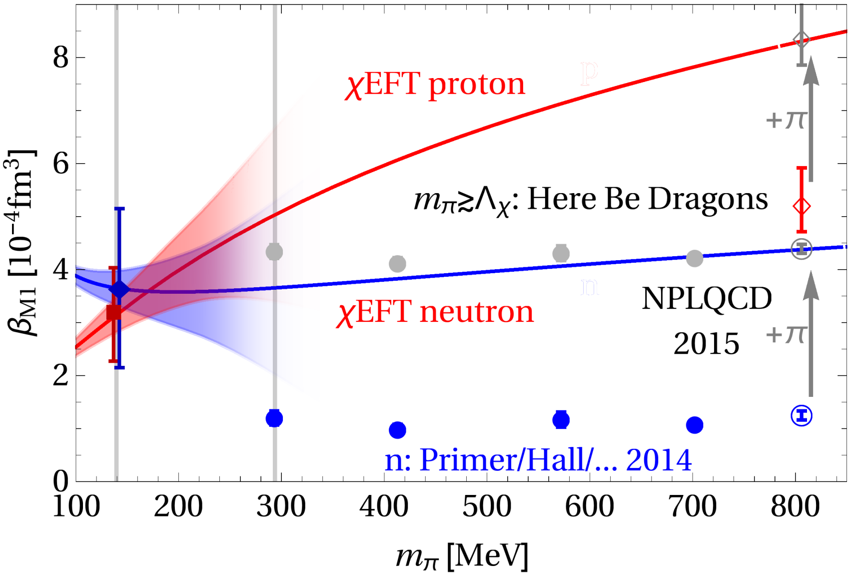
χEFT *predicts* substantial isospin splitting for $m_\pi \gtrsim 200$ MeV:
At $m_\pi = 140$ MeV, paramagnetic $\Delta(1232)$ accidentally fine-tuned against diamagnetic NLO πN loops.



Why m_π-independent offset?

χEFT ***predicts*** **substantial isospin splitting for** $m_\pi \gtrsim 200$ MeV:

At $m_\pi = 140$ MeV, **paramagnetic** $\Delta(1232)$ **accidentally fine-tuned against diamagnetic NLO** π**N loops.**

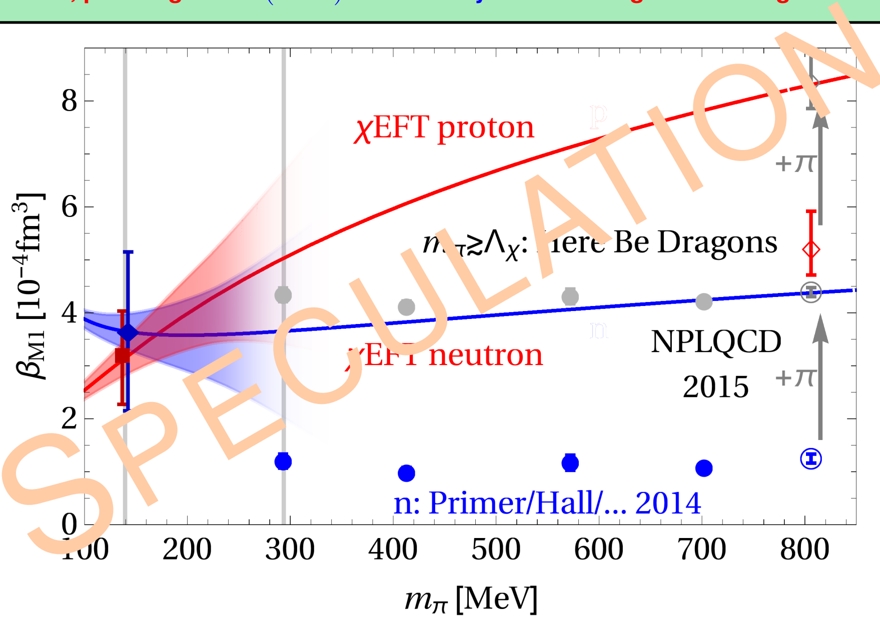


Why m_π**-independent offset?** **Why isoscalar off by "exactly"** $\pi \times 10^{-4}$ fm^3**?**

Why isovector "exactly" matched? ***Principle of Chiral Persistence?***

χEFT ***predicts*** **substantial isospin splitting for** $m_\pi \gtrsim 200\ \text{MeV}$:

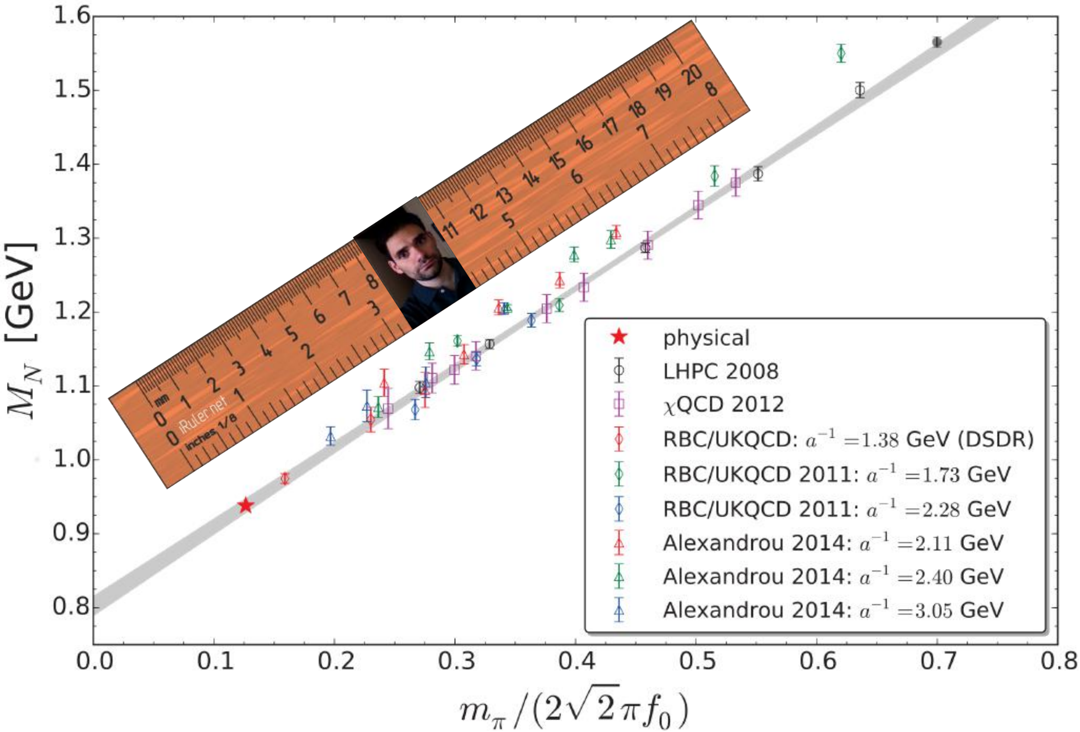
At $m_\pi = 140\ \text{MeV}$, **paramagnetic** $\Delta(1232)$ **accidentally fine-tuned against diamagnetic NLO** π**N loops.**



Why m_π**-independent offset?** **Why isoscalar off by "exactly"** $\pi \times 10^{-4}\ \text{fm}^3$**?**

Why isovector "exactly" matched? ***Principle of Chiral Persistence?***

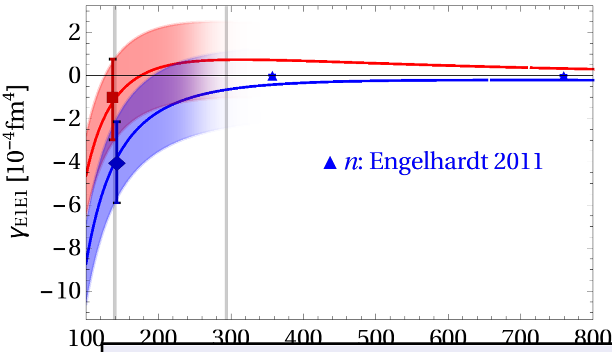
(g) When χEFT Does Not Work: "Ruler Plots"

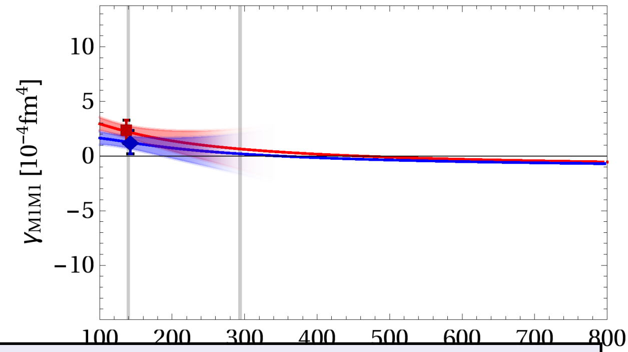


χ**EFT:** $M_N(m_\pi) - M_N(m_\pi = 0) \propto m_q \propto m_\pi^2$.

Lattice: $M_N = 800.0\ \text{MeV} + 1.0 m_\pi$! **WHY??** **Like heavy-mass pion???**

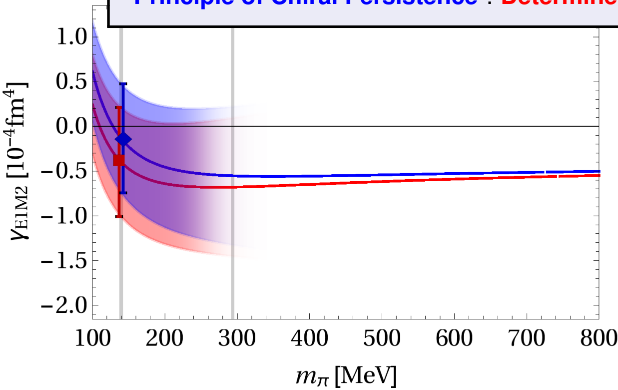
(h) Chiral Extrapolations of Spin Polarisabilities

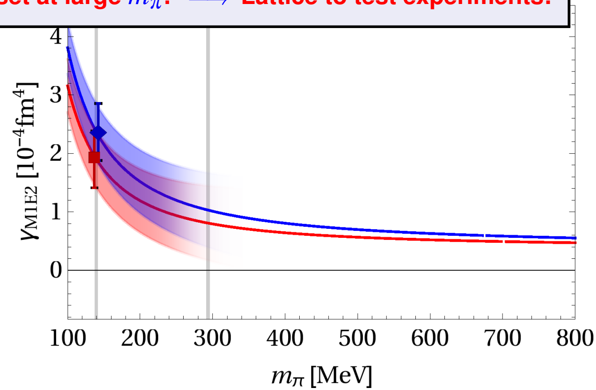




No statistically significant isospin splitting.

"**Principle of Chiral Persistence**": **Determine offset at large m_π.** $\Longrightarrow$ **Lattice to test experiments!**





4. Concluding Questions

Polarisabilities: scales, symmetries & mechanisms of interactions with & among constituents:

χiral symmetry of pion-cloud, iso-spin breaking, $\Delta(1232)$ properties, nucleon spin-constituents.

χEFT relates Lattice QCD (unphysical m_π) to Data: systematic, model-independent, reliable errors.

Compton amplitude to $350\ \text{MeV}$ – Scalar Dipole Polarisabilities from *all Compton data below* $200\ \text{MeV}$:

proton N^{2}LO	$\alpha^p = 10.65 \pm 0.35_{\text{stat}} \pm 0.2_{\Sigma} \pm 0.3_{\text{theory}}$	$\beta^p = 3.15 \mp 0.35_{\text{stat}} \pm 0.2_{\Sigma} \mp 0.3_{\text{theory}}$
neutron NLO	$\alpha^n = 11.55 \pm 1.25_{\text{stat}} \pm 0.2_{\Sigma} \pm 0.8_{\text{theory}}$	$\beta^n = 3.65 \mp 1.25_{\text{stat}} \pm 0.2_{\Sigma} \mp 0.8_{\text{theory}}$

Lattice-QCD needs m_π-dependence. $\Longrightarrow$ Employ *same framework*: χEFT with explicit $\Delta(1232)$.

Theory Uncertainty Corridor of Extrapolation changes with m_π by interrelated effects:

– Expansion parameter $\frac{m_\pi}{\Lambda_\chi}$ increases with m_π. $\Longrightarrow$ Relative error increases.

– $\Delta(1232)$ scale is m_π-independent. $\Longrightarrow$ Reorder contributions at $m_\pi \sim (M_\Delta - M_N)$.

– Pionic d.o.f.s freeze out. $\Longrightarrow$ Magnitude of m_π-contributions decreases.

χEFT predictions for all proton & neutron scalar & spin polarisabilities:

α_{E1}, β_{M1}: substantial isospin splitting for $m_\pi \gtrsim 300\ \text{MeV}$

γ_i: magnitude and m_π-dependence parameter-free.

$\Longrightarrow$ *Experiment, χEFT, lattice with competitive uncertainties?*

Polarisabilities: clean probes to relate lattice-QCD to low-energy phenomena.

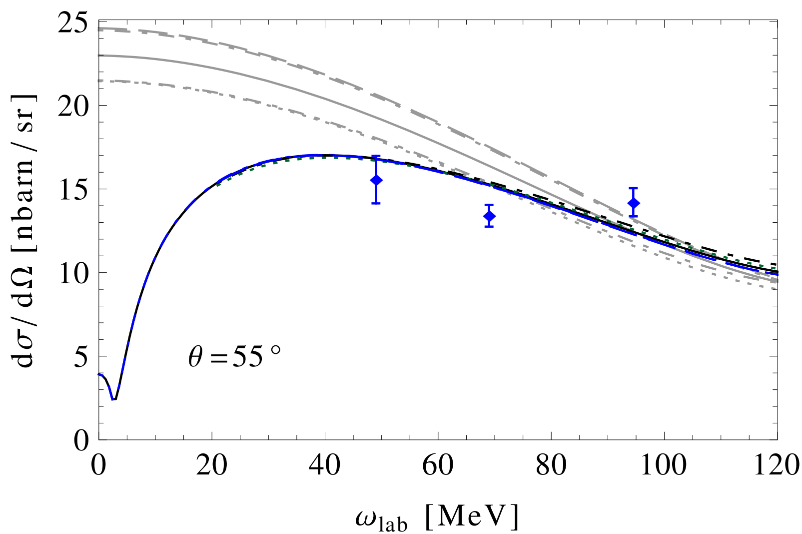

END OF
IMPROVEMENT
THANK YOU
FOR YOUR PATIENCE

(a) NN-Rescattering Leads To An Exact Low-Energy Theorem hg/…2010, 2012

Low-Energy Theorem: Thomson limit $\mathcal{A}(\omega=0) = -\frac{e^2}{M_d}\vec{\epsilon}\cdot\vec{\epsilon}'$.

Thirring 1950, Friar 1975, Arenhövel 1980: Thomson limit $\Longleftrightarrow$ current conservation $\Longleftrightarrow$ gauge invariance.

Exact Theorem $\Longrightarrow$ At each χEFT order $\Longrightarrow$ Checks numerics.



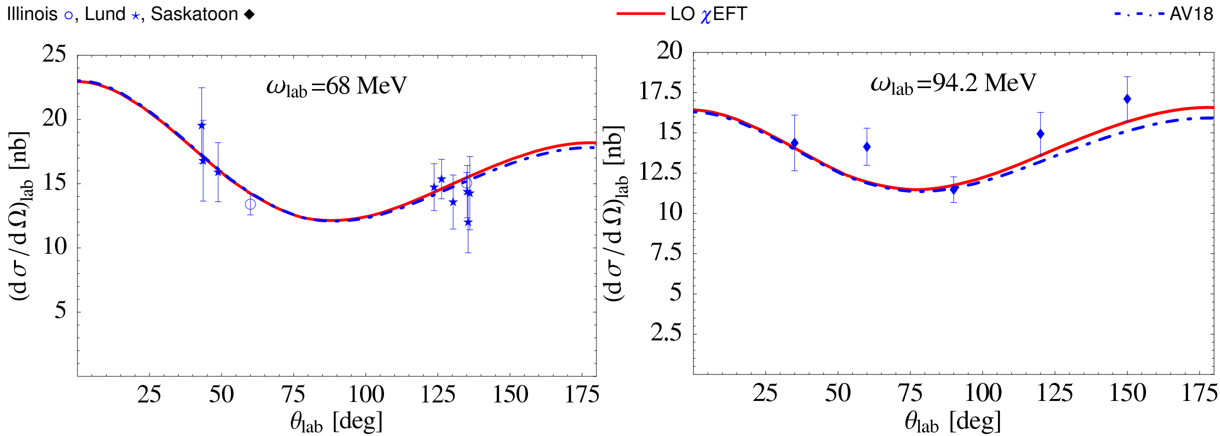
Significantly reduces cross section for $\omega \lesssim 70$ MeV. Urbana, Lund data

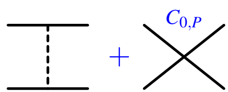
Numerically confirmed to $\lesssim 0.2\%$, irrespective of deuteron wave function & potential. model-independence

Wave function & potential dependence significantly reduced even as $\omega \to 150$ MeV $\Longrightarrow$ **gauge invariance**.

(a) NN-Rescattering Leads To An Exact Low-Energy Theorem

Dependence of T_{NN} on NN-potential $\cong$ short-distance, for $\omega \to 0$ clear from Thomson.



LO χEFT-potential: 

$\sim Q^{-1}$

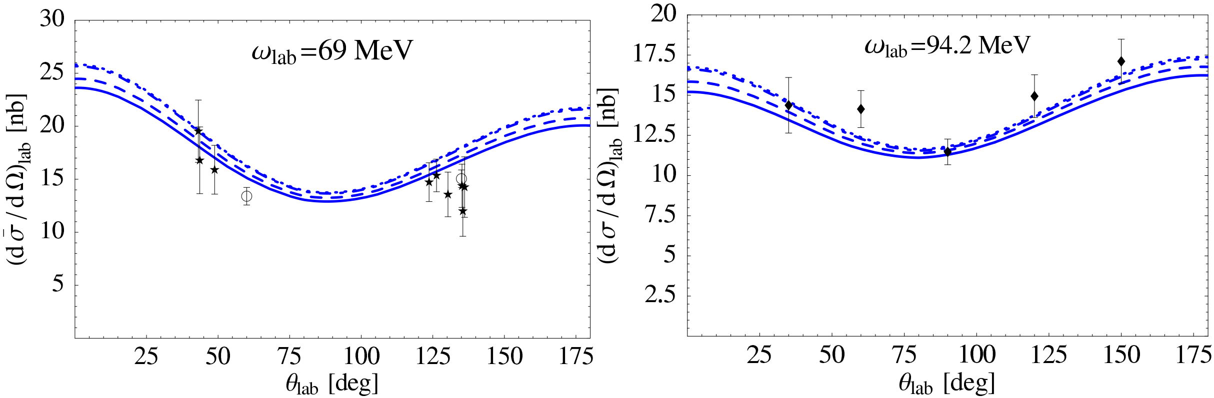
Consistent for Compton at NLO: $\mathcal{O}(Q^0)$-correction of NN-potential presumed zero.

AV18 provides $< 3\%$ corrections $\Longrightarrow$ suggests higher-order indeed $Q^1 \approx \left(\frac{1}{7}\right)^2$.

(a) NN-Rescattering Leads To An Exact Low-Energy Theorem

Wave-function sampling: no major dependence

Illinois ○, Lund ⋆, Saskatoon ◆ — "NNLO χPT" - · - AV18 – – CD-Bonn · · · Nijmegen 93



extreme cases:	"NNLO χPT"	Nijmegen 93
$\chi^2/$d.o.f. unconstrained	1.8	2.5
$\chi^2/$d.o.f. constrained	1.7	2.4

but with $\sim 10\%$ worrisome enough to trigger further investigations. . .

5. The EFT Promise: Serious Theorists Have Error Bars

Scientific Method: Quantitative results with corridor of theoretical uncertainties for *falsifiable predictions*.

"Double-Blind" Theory Errors: Assess with pretense of no/very limited data.

PHYSICAL REVIEW A **83**, 040001 (2011)

Editorial: Uncertainty Estimates

The purpose of this Editorial is to discuss the importance of including uncertainty estimates in papers involving theoretical calculations of physical quantities.

It is not unusual for manuscripts on theoretical work to be submitted without uncertainty estimates for numerical results. In contrast, papers presenting the results of laboratory measurements would usually not be considered acceptable for publication in *Physical Review A* without a detailed discussion of the uncertainties involved in the measurements. For example, a graphical presentation of data is always accompanied by error bars for the data points. The determination of these error bars is often the most difficult part of the measurement. Without them, it is impossible to tell whether or not bumps and irregularities in the data are real physical effects, or artifacts of the measurement. Even papers reporting the observation of entirely new phenomena need to contain enough information to convince the reader that the effect being reported is real. The standards become much more rigorous for papers claiming high accuracy.

The question is to what extent can the same high standards be applied to papers reporting the results of theoretical calculations. It is all too often the case that the numerical results are presented without uncertainty estimates. Authors sometimes say that it is difficult to arrive at error estimates. Should this be considered an adequate reason for omitting them? In order to answer this question, we need to consider the goals and objectives of the theoretical (or computational) work being done. Theoretical papers can be broadly classified as follows:

Workshop *"Predictive Capabilities of Nuclear Theories"*, Krakow (Poland), 25 Aug 2012

Special Issue *J. Phys. G* (Feb 2015):

"Enhancing the Interaction between Nuclear Experiment and Theory through Information and Statistics"

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physical effects not included in the calculation from the beginning, such as electron correlation and relativistic corrections. It is of course never possible to state precisely what the error is without in fact doing a larger calculation and obtaining the higher accuracy. However, the same is true for the uncertainties in experimental data. The aim is to estimate the uncertainty, not to state the exact amount of the error or provide a rigorous bound.

There are many cases where it is indeed not practical to give a meaningful error estimate for a theoretical calculation; for example, in scattering processes involving complex systems. The comparison with experiment itself provides a test of our theoretical understanding. However, there is a broad class of papers where estimates of theoretical uncertainties can and should be made. Papers presenting the results of theoretical calculations are expected to include uncertainty estimates for the calculations whenever practicable, and especially under the following circumstances:

1. If the authors claim high accuracy, or improvements on the accuracy of previous work.
2. If the primary motivation for the paper is to make comparisons with present or future high precision experimental measurements.
3. If the primary motivation is to provide interpolations or extrapolations of known experimental measurements.

These guidelines have been used on a case-by-case basis for the past two years. Authors have adapted well to this, resulting in papers of greater interest and significance for our readers.

The Editors

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Workshop *"Predictive Capabilities of Nuclear Theories"*, Krakow (Poland), 25 Aug 2012

Special Issue *J. Phys. G* (Feb 2015):

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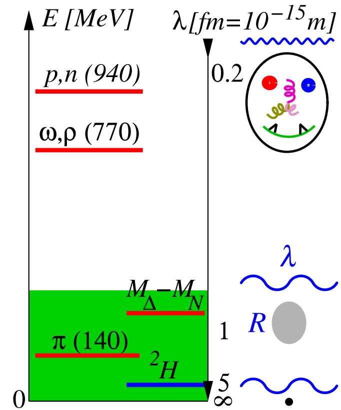
(a) Power-Counting Non-Perturbative EFTs

Correct long-range + symmetries: Chiral SSB, gauge, iso-spin,...
Short-range: ignorance into minimal parameter-set at given order.

Systematic ordering in $Q = \dfrac{\text{typ. momentum } p_{\text{typ}}}{\text{breakdown scale } \overline{\Lambda}_{\text{EFT}}} \ll 1$

Controlled approximation: model-independent, error-estimate.

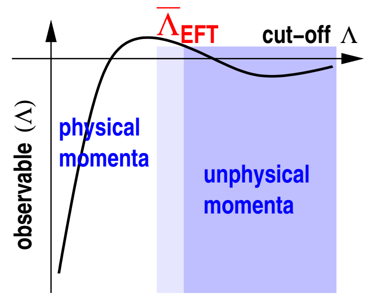
$\Longrightarrow$ **Chiral Effective Field Theory χEFT $\equiv$ low-energy QCD**
$\Longrightarrow$ **Pion-less Effective Field Theory EFT($\not\pi$) $\equiv$ low-energy χEFT**



Shallow real/virtual QCD bound states $\Longrightarrow$ Few-N non-perturbative!

$$T_{\text{LO}} = V_{\text{LO}} + V_{\text{LO}}\, G\, T_{\text{LO}}$$
$$T_{\text{NLO}} = (\mathbb{1} + T_{\text{LO}}^{\dagger})\, V_{\text{NLO}}\, (\mathbb{1} + T_{\text{LO}})$$ strict perturbation about LO

$\Longrightarrow$ Analytic results rare; regularisation by **cut-off** $\Lambda \overset{!!}{\neq} \overline{\Lambda}_{\text{EFT}}$.



$\Longrightarrow$ saturated at $\overline{\Lambda}_{\text{EFT}} \lesssim \Lambda$.

(b) (Some) Ways to Estimate Theoretical Uncertainties at fixed k

Choose most conservative/worst-case error for final estimate! Clearly state your choice!

Expansion parameter $Q = \frac{\text{typ. low scale } p_{\text{typ}}}{\text{typ. high scale } \overline{\Lambda}_{\text{EFT}}} \Longrightarrow \mathcal{O} = \sum_{i=0}^{k-1} c_i(\Lambda)\, Q^i$ complete up to order Q^{k-1} (N^{k}LO).

– A priori: Q^k of LO.

– Convergence pattern of series: smaller corrections LO $\to$ NLO $\to$ N^{2}LO $\to \dots$

$\Longrightarrow$ **Bayesian estimate:** error $Q^k \times \max_i |c_i|$ captures corridor with $\frac{k}{k+1} \times 100\%$ degree of belief.

Furnstahl/Klco/Phillips/Wesolowski (BUQEYE) 2015

– Less dependence on particular low-E data taken for LECs. (e.g. Z-param. vs. ERE; fit H_0 to a_3 vs. $B_3, \dots$)

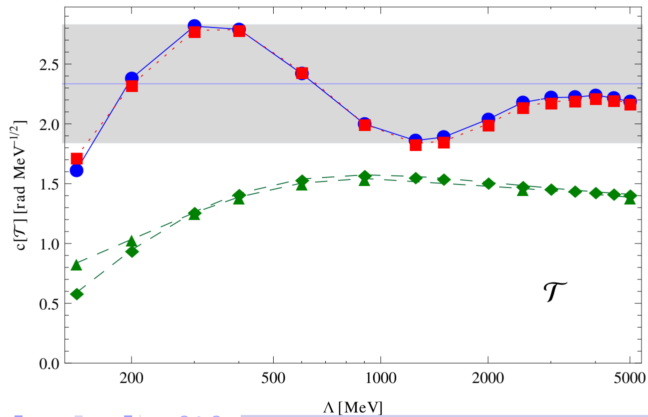
– Include selected higher-order RG- & gauge-invariant effects: *does not increase accuracy.*

– Corridor mapped by cutoff Λ *in wide range*.

Should decrease order-by-order.

Example: PV coefficient in nd at $k = 0$.

hg/Schindler/Springer 2012



(a) Using Cut-Offs to Your Advantage

Observable $\mathcal{O}(k)$ at momentum k, order Q^n in EFT, cut-off Λ:

$$\mathcal{O}_n(k;\mu) = \underbrace{\sum_i^n \left(\frac{k,p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^i \mathcal{O}_i}_{\text{renormalised, } \Lambda\text{-indep.}} + \underbrace{\mathcal{C}(\Lambda;k,p_{\text{typ}},\overline{\Lambda}_{\text{EFT}}) \left(\frac{k,p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1}}_{\text{residual } \Lambda\text{-dependence}}$$

renormalised, Λ-indep.

residual Λ-dependence
parametrically small
$\mathcal{C}$ "of natural size"

$\Longrightarrow$ **Difference between *any two* cut-offs:** $\dfrac{\mathcal{O}_n(k;\Lambda_1)-\mathcal{O}_n(k;\Lambda_2)}{\mathcal{O}_n(k;\Lambda_1)} = \left(\dfrac{k,p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1} \times \dfrac{\mathcal{C}(\Lambda_1)-\mathcal{C}(\Lambda_2)}{\mathcal{C}(\Lambda_1)}$

Isolate breakdown scale $\overline{\Lambda}_{\text{EFT}}$, order n by double-ln plot of "derivative of observable w. r. t. cut-off".

Test consistency: Does numerics match predicted convergence pattern?

Renormalisation Group Evolution: $\Lambda_1 \to \Lambda_2 \Longrightarrow \dfrac{\Lambda}{\mathcal{O}}\dfrac{\mathrm{d}\mathcal{O}}{\mathrm{d}\Lambda} = \left(\dfrac{k,p_{\text{typ.}}}{\overline{\Lambda}_{\text{EFT}}}\right)^{n+1} \dfrac{\mathrm{d}\ln\mathcal{C}(\Lambda)}{\mathrm{d}\ln\Lambda} \to 0$ if exact RGE.

Residual Λ-dependence decreases parametrically order-by-order.

Complication: Several intrinsic low-energy scales in few-N EFT:
scattering momentum k, m_π, inverse NN scatt. lengths $\gamma(^3\mathrm{S}_1) \approx 45\ \mathrm{MeV}$, $\gamma(^1\mathrm{S}_0) \approx 8\ \mathrm{MeV}$,...