Parton Distribution Functions from Large Momentum Effective Theory

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Outline

- Review of parton distribution functions (PDF)
	- **Parton physics and the infinite momentum frame**
	- Why it is difficult to calculate PDFs in lattice QCD?
- A new way of calculating parton physics
	- **Example 3 Lesson from the gluon polarization**
	- Large momentum effective theory approach
- **PDFs from large momentum effective theory**
	- From a quasi-PDF to the normal PDF
	- Requirements for the computation resources

Parton distribution functions

- **PDFs characterize the momentum distributions of quarks and** gluons inside the nucleon.
- They are important inputs for making predictions in high-energy scattering experiments. hadrons

 p, \bar{p} –

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Parton distribution functions

 $(J. Campbell, HCP2012)$

- **PDFs induce large uncertainty in Higgs production at the LHC.**
- **A** better understanding and precise determination of PDFs required to help us understand the standard model and disentangle new physics effects.

Parton distribution functions

- **PDFs are the universal properties of nucleons**
	- **•** Nucleons are bound states of quarks and gluons described by the fundamental theory of QCD.
	- The quarks and gluons are strongly interacting and relativistic, PDFs are intrinsically non-perturbative.
- Our knowledge of PDFs is mainly driven by the data from state-of-the-art high-energy programs, BCDMS, SLAC, NMC, JLab, HERA, E866, CDF, DØ, COMPASS, RHIC, JPARC, LHC, ..., (see talks by Nocera and Schmidt), but a first principle study of PDFs is still behind.

Physics of PDFs

- \blacksquare The nucleon wavefunction $|P>$ is usually defined at equal time and includes lots of complexities: valence quarks, sea quarks, gluons;
- An equal-time wavefunction is not frame independent:
	- Under a Lorentz boost Λ, |P> transforms as

 $\langle P \rangle = U(\Lambda(p)) |P=0\rangle$

 \blacksquare U(\land) depends on the interaction, so the transformation U(\land (P)) is not kinematic, but dynamical.

Physics of PDFs

- **In high energy scattering, the nucleon is travelling at a** large momentum relative to the probe;
	- **The time scale of the interaction between quarks and gluons is** dilated due to relativistic effects;
	- When the nucleon exchanges a hard momentum with the probe within an "impulse", the struck quark or gluon almost does not feel the other constituents;
	- **•** In the infinite momentum frame (IMF), this picture becomes simple as the quarks and gluons appear like free particles, i.e., partons.

x n^{*y*} *dx d n i D*¹ *d e*¹

- **•** However, the IMF can never be achieved in reality, so this picture is an approximation for high-energy scattering: with the second all diagrams with the second and all diagrams of the second and all di we see that the parton momentum distribution refers to the gauge-invariant kinetic momentum distribution refer
The gauge-invariant kinetic momentum distribution refers to the gauge-invariant kinetic momentum distribution
	- Leading twist effect
	- x y x − y y z x − y − z higher-twist • Sensitive to the soft scales at
		- One has to fix to the light-cone gauge $A^+=0$ to eliminate the longitudinal gluons, and thus have a clear partonic picture.
- AP (Altarelli, Parisi, 1977) equation and longitudinal the longitudinal term in the longitudinal term was derived in the IMF. **From Ji, Xu**, and

DDEs in light-song quantization tion is the form of the form of the time-independent perturbation the IMF μ

Actually the light-front formulation of a dynamical theory goes back more than half a cen-

- **The IMF** picture of parton physics is replaced by lightcone quantization, in which the probe is moving at the speed of light and the nucleon is at rest: field at the canonical plane-wave expressed, the canonical plane-wave expansions of the Fock particles. The Fock particles of the Fock particles. The Fock particles of the Fock particles. The Fock particles of the Fock par cone quantization, in which the prope is moving at the
cannot of light and the purlean is at rest. tion theory as in the mechanic mechanics. One can then go on the set of the set of the set of the set of determine the set of the set
- In light-cone quantization, the wavefunction is defined at equal light-cone time $\xi^+=(x^0+x^3)/\sqrt{2}=0$;
- The Hamiltonian H_{LC} =P[−] can be used to formulate "time"independent perturbation theory as in non-relativistic quantum mechanics; is most convenient to work with α are particles are
- **The nucleon wavefunction can have a Fock space expansion**

$$
|P\rangle = \sum_{n\alpha,\lambda_i} \int \Pi_i \frac{dx_i d^2 k_{\perp i}}{\sqrt{2x_i}(2\pi)^3} \psi_{n\alpha}(x_i, k_{\perp i}, \lambda_i) |n\alpha : x_i P^+, k_{\perp, i}, \lambda_i\rangle ,
$$

PDFs in light-cone quantization ⇠*[±]* ⁼ ¹ (⇠⁰ *[±]* ⇠³

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■ The PDFs can be defined as light-cone correlations along the "new spatial direction" $\xi = (x^0{-}x^3)/\sqrt{2}$: and the DDEscape of other *defined* as light sepace-treations **For the four secrive active as fight-cone cone actions**

$$
q(x,\mu^2) = \int \frac{d\xi^-}{4\pi} e^{-ix\xi^-P^+} \langle P|\overline{\psi}(\xi^-)\gamma^+ \times \exp\left(-ig \int_0^{\xi^-} d\eta^- A^+(\eta^-)\right) \psi(0)|P\rangle ,
$$

- The light-cone gauge link ensures its gauge invariance;
- **Invariant under the boost along the nucleon momentum** direction; renormalization scale. Here the infinite boost factors all disappeared, and IMF physics is in the international reflective introduction throughough the boost and gluon operator. The above matrix element of matrix elements
The above matrix elements and gluon operator. The above matrix elements are matrix elements and above matrix e is now independent of the hadron external momentum P $\bf p}$ $\bf p}$ $\bf p}$ $\bf p}$ $\bf p}$ $\bf p}$
- \blacksquare In the light-cone gauge $A^+ \! = \! 0$, has clear interpretation as parton $density:$

$$
q(x) \sim \int d^2k_{\perp} dk^+ n(k^+,k_{\perp}) \delta(x - k^+ / P^+)
$$

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) *,* (1)

First principle calculation of PDFs

- Despite its nice features, calculating PDFs in light-cone quantization is difficult,
	- No systematic approximation has been found in solving QCD in light-cone quantization in 3+1 dimensions;
	- AdS/QCD: no exact correspondence can be made.
- **The only practical approach to solve non-perturbative** QCD is the lattice theory.
	- **Defined in the Euclidean space, with** $\tau = it$ **being real;**
	- \blacksquare A^4 = *i* A^0 is real;
	- Has been successful in calculating hadron spectroscopy, nucleon form factors, finite temperature QCD, etc.

the physical property of the parton density operator. For example, the quark longitudinalmomentum distribution distribution distribution di unit *^q*(*x*) = ¹ 2*P* ⁺ ! *d*λ 2π *^eⁱ*λ*^x*⟨*P S|*Ψ(λ*n*)γ⁺Ψ(0)*|P S*⟩ *,* (18) where *n* is a light-like four-vector *n*² = 0 and the nucleon momentum is *P ^µ* = (*P*⁰*,* 0*,* 0*, P*³),

- Lattice QCD cannot directly access light-cone quantities due to their dependence on real time: attice QCD cannot dif *a*⊂D cannot unectry access ngr
neir denendence on real time:
- Lattice can calculate the moments of $\overbrace{ }^{0.5}$ PDFs which are local operators; • Lattice can calculate the moments of _[

$$
\left|x^{n-1}q(x)dx\right|\sim n_{\mu_1}...n_{\mu_n}\langle P|\overline{\psi}(0)\gamma^{\mu_1}iD^{\mu_2}...iD^{\mu_n}\psi(0)|P\rangle\right|\quad\text{as}\quad\text{and}\quad\text
$$

- Parametrize the PDFs with a smooth $\frac{1}{2}$ or $\frac{1}{2}$ functional form a parton with kinetic momentum \int_{a+}^a \sim the kinetic momentum structure is clearly seen through Feynman diagrams in Fig.
	- Determine the parameters from the lattice computed moments $\begin{array}{ccc} \text{Fig. 5. R} \\ \text{A Gov2 T} \end{array}$
- Problem: number of calculable moments is limited (H.W. Lin's talk) $\boxed{xf(x) = A}$

 α and α at α and α at α

Detmold et al., Mod.Phys.Lett.A 03'

$$
\text{moments is limited (H.W. Lin's talk)} \quad \boxed{xf(x) = Ax^b(1-x)^c(1+\epsilon\sqrt{x}+\gamma x)}
$$

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Traditional way of calculating PDFs in Lattice QCD

 \blacksquare The number of calculable moments strongly limits the power of lattice QCD to calculate PDFs;

Given that lattice QCD can only calculate the matrix elements of time-independent operators at zero or finite nucleon momentum, can we find a way to obtain PDFs directly from lattice QCD?

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	- Why it is difficult to calculate PDFs in lattice QCD?
- A new way of calculating parton physics
	- **Example 3 Lesson from the gluon polarization**
	- Large momentum effective theory approach
- Lattice QCD calculation of PDFs
	- From a quasi-PDF to the normal PDF
	- Requirements for the computation resources

The gluon polarization ΔG **contributes to the longitudinal** nucleon spin, and is measured in polarized deep inelastic scattering (DIS) and proton-proton collisions.

Longitudinal proton spin structure

```
ΔΣ(Q²=10 GeV<sup>2</sup>) = 0.242,
de Florian et al., 2009
\Delta G(Q^2=10 \text{ GeV}^2) \sim 0.2,
de Florian et al., 2014
SLAC	
HERMES (DESY)
COMPASS (CERN)
JLab
RHIC
```
■ *ΔG* is defined as the first moment of the polarized gluon distribution function (A.V. Manohar, PRL 1990):

$$
\Delta g(x) = \frac{i}{2xP^+} \int d\xi^- e^{-ixP^+\xi^-} \langle PS | F_a^{+,i}(\xi^-) L^{ab}(\xi^-, 0) \tilde{F}_{b,i}^+(0) | PS \rangle
$$

In light-cone quantization,

$$
\Delta g(x) = \begin{array}{|c|c|} \hline \quad \bullet & \quad \bullet \\ \hline \quad \bullet & \quad \bullet \end{array}
$$

- **E** However, the first moment of $\Delta g(x)$ does not correspond to any local gauge-invariant operators;
- As a result, ΔG cannot be directly calculated on the lattice, and has been a problem for a long time.

 \blacksquare A recent proposal tries to define the gauge-invariant gluon spin operator at equal time in the rest frame of the nucleon (Chen et al., PRL 2008).

$$
S_g = \int d^3x \, \vec{E} \times \vec{A}_{\perp}
$$

In Abelian gauge theory,

$$
\vec{A} = \vec{A}_{\perp} + \vec{A}_{\parallel}, \quad \vec{\nabla} \cdot \vec{A}_{\perp} = 0, \quad \vec{\nabla} \times \vec{A}_{\parallel} = 0
$$

 A_{perp} is gauge invariant under a gauge transformation.

- **•** However, A_{perp} is not frame independent. In QED, its transformation under a Lorentz boost is dynamical.
- This makes the physical meaning of $E\times A_{\text{perp}}$ as spin not clear.

- **E** Actually, in the rest frame of the nucleon, due to the existence of unphysical degrees of freedom (d.o.f.), there is no physical meaning of gluon spin at all;
- Nevertheless, if we boost the nucleon to the IMF, longitudinal gluon field strength is suppressed, A_{perp} fully captures the physical d.o.f., and therefore $E \times A_{\text{perp}}$ can be regarded as the spin of free gluon fields in this limit.
- **This is the famous Weizsäcker-Williams approximation (or** equivalent photon method) in classical electrodynamics (Jackson, Classical Electrodynamics).

Moreover, we can prove that $(J_i, Zhang, Zhao, PRL)$ 2013)

$$
\vec{E} \times \vec{A}_{\perp} \xrightarrow{P \to \infty} \int dx \frac{i}{x} \int d\xi^{-} e^{-ixP^{+}\xi^{-}} F_{a}^{+,i}(\xi^{-}) L^{ab}(\xi^{-},0) \tilde{F}_{b,i}^{+}(0)
$$

- The nucleon matrix element of $E \times A_{\text{perp}}$ is just the gaugeinvariant gluon spin defined from the first momentum of $\Delta g(x)$.
- ExA_{perp} is time independent, and therefore can be readily calculated in lattice QCD.
- One can obtain ΔG by studying the IMF limit of the matrix element of ExA_{perp} .

- **However, since** $E \times A_{\text{perp}}$ **is frame dependent, its matrix** element will depend on the nucleon momentum P , which becomes a large scale of theory;
- \blacksquare In perturbation theory, if one does the ultraviolet (UV) regularization first, the matrix element of E×A_{perp} can contain $logarithms$ of P , which is singular in the IMF limit.
- **The Instead, if one takes the IMF limit of** $E \times A_{\text{perp}}$ **first, and then** does the UV regularization, one obtains the correct ΔG .
- \blacksquare The tricky point is that lattice QCD can only calculate finite momentum matrix elements.

Matching to the physical results through a LaMET One-loop example (Ji, Zhang, and Zhao, 2013) the physical results that

One loop example:

p Ji, Zhang, Zhao, PRL 2013 *k* + *p*

One-loop example (Ji, Zhang, and Zhao, 2013)

• At large but finite \vec{p} , taking the loop momentum $k \to \infty$ first

At large but finite p, taking the loop momentum
$$
\kappa \to \infty
$$
 first
\n
$$
\langle p, s | (\vec{E} \times \vec{A}_{\perp})^3 | p, s \rangle = \frac{\alpha_S C_F}{4\pi} \left[\frac{5}{3\epsilon} + \frac{5}{3} \ln \frac{\mu^2}{m^2} + \frac{4}{3} \ln \frac{4\vec{p}^2}{m^2} - \frac{1}{9} \right] u^{\dagger} \Sigma^3 u + O(\frac{m^2}{\vec{p}^2}),
$$

• Taking $\vec{\rho}^2 \rightarrow \infty$ first,

$$
\bra{\boldsymbol{\rho},\boldsymbol{s}}\left(\vec{E}\times\vec{A}_\perp\right)^3\ket{\boldsymbol{\rho},\boldsymbol{s}}=\frac{\alpha_{\boldsymbol{S}}C_{\boldsymbol{\digamma}}}{4\pi}(\frac{3}{\epsilon}+3\ln\frac{\mu^2}{m^2}+7)\;u^\dagger\Sigma^3u
$$

• The light-cone gauge result,

$$
\bra{\boldsymbol{\rho},\boldsymbol{s}}\left(\vec{E}\times\vec{A}\right)^3\ket{\boldsymbol{\rho},\boldsymbol{s}}\stackrel{\scriptscriptstyle A^+=0}{=}\frac{\alpha_{\boldsymbol{S}}C_{\boldsymbol{\digamma}}}{4\pi}(\frac{3}{\epsilon}+3\ln\frac{\mu^2}{m^2}+7)\;u^{\dagger}\Sigma^3u
$$

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- By observation we find that the IMF limit does not change the collinear or infrared (IR) divergence in the matrix element, and the difference is just in the UV divergent part.
- \blacksquare The UV divergent part is calculable in perturbative QCD, and therefor we can take this advantage to match the matrix element at finite momentum and in the IMF.
- **A systematic matching procedure is provided by the** large momentum effective theory (LaMET).

Large momentum effective theory

The large momentum effective theory (LaMET) is a theory that expands in powers of $1/P$, where P is the nucleon momentum (Ji, Sci. China Phys. Mech. Astro., 2014):

- 1. Construct a Euclidean quasi-observable \tilde{O} which can be calculated in lattice QCD;
- 2. The IMF limit of $\ddot{\text{O}}$ is constructed to be a parton observable O ;
- 3. The matrix elements of $\ddot{\text{O}}$ is dependent on the lattice cut-off Λ and generally P, i.e., $\tilde{O}(P/\Lambda)$, while the parton observable depends on the renormalization scale μ (if in the MSbar scheme), i.e., $O(\mu)$;

Large momentum effective theory

4. Taking the P->∞ limit of $\tilde{O}(P/\Lambda)$ is generally ill-defined due to the singularities in quantum field theory, but it can be related to $O(\mu)$ through a factorization formula:

$$
\tilde{O}(P/\Lambda) = Z(P/\Lambda, \mu/\Lambda)O(\mu) + \frac{c_2}{P^2} + \frac{c_4}{P^4} + \dots
$$

- P is much larger than Λ_{OCD} as well as the nucleon mass M;
- **•** O(μ) captures all the infrared (IR) and collinear divergences in $\ddot{O}(P/\Lambda)$, and thus the leading term can be factorized into $O(\mu)$ and a perturbative coefficient Z.

How matching works?

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partons at finite nucleon momentum [7]. Such correlations are time-dependent and intrinsi-Quaci narton dictribution been mainly focused on the evaluation of local moments of the distributions. However, the

Example 1 Consider the spatial correlation of two quarks in the nucleon with momentum P along the z direction: light-cone distribution, a related quantity, which may be called quasi-distribution [10]. In - Inucleon with momentum P along the \boldsymbol{z} direction

$$
\tilde{q}(x,\Lambda,P^z)=\int_{-\infty}^{\infty}\frac{dz}{4\pi}e^{izk^z}\langle P|\overline{\psi}(0,0_+,z)\gamma^z\exp\left(-ig\int_0^zdz'A^z(0,0_+,z')\right)\psi(0)|P\rangle
$$

 \overline{E} = kz/Pz is the longitudinal momentum fraction, and V \overline{V} is the z-component of \overline • Equal-time correlation; entity of the U is the Nucleon of Ti, PRL 2013

different for the technical reasons $\mathcal{B}(\mathcal{S})$.

- The matrix element of the ζ^* is the ζ^* is the ζ^* quasi distribution depends on $\begin{array}{ccc} & | & \sqrt{2} \gamma \ \chi & | & | \end{array}$ can be significant for a lattice for any $\sum_{i=1}^{\infty}$ is the nucleon momentum P^z: the nucleon momentum P^z ;
- In the IMF limit, $z \rightarrow \xi$ ⁻, and A^z and $\left| \begin{array}{cc} | & | \end{array} \right|$ me approaches normal PDF. dy Z "^x $-$ > A^+ , the quasi PDF

Ji, PRL 2013

Matching quasi PDF and normal PDF Now we are ready to construct a factorization formula at one-loop order. In the infinite

■ According to LaMET, the quasi PDF is related to the normal PDF through a
perturbative matching formula for the non-singlet case (Xiong, Ji, Zhang, and $Zhao, PRD 2014$: According to LaMET, the quasi-DDF is related to the normal PDF through a

$$
\tilde{q}(x,\Lambda,P^z) = \int_{-1}^1 \frac{dy}{|y|} Z\left(\frac{x}{y},\frac{\Lambda}{P^z},\frac{\mu}{P^z}\right) q(y,\mu) + \mathcal{O}\left(\Lambda_{\text{QCD}}^2/(P^z)^2, M^2/(P^z)^2\right)
$$

momentum frame or on the light-cone, the momentum fraction in parton distributions and

- \blacksquare Integration range is determined by the support of light-cone parton distribution; the light cone-loop one-loop o
- Both light-cone and quasi PDFs contain soft divergences, but they cancel in ${\sf themselves}$;
- **They also have the same collinear divergences;** 。
。 $\overline{\mathbf{S}}$ Λ ne (bllinear divergences;
- Z factor captures the difference in their UV behavior, and is thus perturbatively calculable; $B = 2$ factor captores the unreferred in their OV behavior, and is thos pertondatively columnic, the double pole 1/(1 \geq)2 in the double pole 1/(2) in the seen from Eqs. (8)2 in the seen from Eqs. (8)3 in the seen from Eqs. (
- **Higher twist effects suppressed by the large momentum.** Oct. 22, 2015 **INT-15-3, Seattle** 28 and (9) above. In the previous section we worked in the axial gauge, and the axial gauge, and the linear divergence α \blacksquare comes from the last term in the numerator of the numerator of the axial gauge $\lceil \text{N}\rceil$

 $\frac{1}{2}$ this double pole in axial gauge computations see e.g. Ref. [16]. If one e.g. Ref. $\frac{1}{2}$

Matching quasi PDF and normal PDF

In this section, we consider the one-loop correction in the case of unpolarized quasi-quark

- **Take unpolarized quark distribution as an example [Ji, Xiong, Zhang and** Zhao PRD, 13] $\overline{}$ Take unnolarized quark distribution as an example $\overline{\text{Li}}$ $\overline{\text{Yion}}$ $\overline{\text{Zhano}}$ and sime emporanted quark discribution as an example property, theng, thang and σ $N(t, t, \theta)$ is incomplete and the detailed result the detailed
	- \blacksquare @LO

Both quasi and light-cone quark distribution yield a simple delta function ω LO

$$
\tilde{q}^{(0)}(x) = q^{(0)}(x) = \delta(1-x)
$$

■ @NLO The one-loop calculation can in principle be carried out in any gauge since the result is

The computation can be carried out in any gauge:

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Matching quasi PDE and normal PDE function renormalization constant. 4) All soft divergences are cancelled. However, there are

■ One-loop matrix element of the normal PDF: α → α → α . This is done for α . This is done for α \blacksquare One-loop matrix element of the normal PDF

$$
q(x,\Lambda) = (1 + Z_F^{(1)}(\Lambda) + \dots)\delta(x-1) + q^{(1)}(x,\Lambda) + \dots
$$

 $\overline{}$ on the other hand, with the same regularization, one can calculate the light-cone parton between partons $\overline{}$

$$
q^{(1)}(x,\Lambda) = \frac{\alpha_S C_F}{2\pi} \begin{cases} 0, & x > 1 \text{ or } x < 0 ,\\ \frac{1+x^2}{1-x} \ln \frac{\Lambda^2}{m^2} - \frac{1+x^2}{1-x} \ln (1-x)^2 - \frac{2x}{1-x} , & 0 < x < 1 , \end{cases}
$$

$$
Z_F^{(1)}(\Lambda) = \frac{\alpha_S C_F}{2\pi} \int dy \begin{cases} 0, & y > 1 \text{ or } y < 0 ,\\ -\frac{1+y^2}{1-y} \ln \frac{\Lambda^2}{m^2} + \frac{1+y^2}{1-y} \ln (1-y)^2 + \frac{2y}{1-y} , & 0 < y < 1 , \end{cases}
$$

Remarks:

with

and

In the IMF or on the light-cone, partons can only have momentum fraction 0<x<1; $t_{\rm H}$ and $t_{\rm H}$ and $t_{\rm H}$ and $t_{\rm H}$ \mathcal{F} (with \mathcal{F} the renormalization scale) are slightly different, and are slightly different, and and and and and and and are slightly different, and are slightly different, and are slightly different, and and an can be obtained from the above ones by making the above ones by making the replacement ln \mathcal{N}_c

Matching quasi DDE and normal DDE malization renormalization renormalization divergences are cancelled. However, there are cancelled and the cancel

One-loop matrix element of the normal PDF: distribution by taking the limit σ . This is done for σ σ and σ σ σ σ σ σ σ

$$
q(x,\Lambda) = (1 + Z_F^{(1)}(\Lambda) + \dots)\delta(x-1) + q^{(1)}(x,\Lambda) + \dots
$$

 $\overline{}$ on the same regularization, with the same regularization, one can calculate the light-cone parton parton parton on

$$
q^{(1)}(x,\Lambda) = \frac{\alpha_S C_F}{2\pi} \left\{ \underbrace{\begin{pmatrix} 0\\ \frac{1+x^2}{1-x} \end{pmatrix}}_{m} \underbrace{\frac{\Lambda^2}{m^2} - \frac{1+x^2}{1-x} \ln(1-x)^2 - \frac{2x}{1-x}, \ 0 < x < 1, \end{pmatrix}}_{x > 1 \text{ or } x < 0, \ y > 1 \text{ or } y < 0, \ y > 1 \text{ or } y < 0, \right\}
$$

Remarks:

 Z

with

and

- **Example 20 Light-cone singularity cancels between real and virtual contributions;**
- Collinear divergence $\ln(m^2)$ exists in the PDF. **Eight concent surgeranty cancers between rear and virtual continuotions,** confirmed the above ones by the above ones by making the replacement ln 4π μ

hing quasi DDE and n integration is logarithmically divergent, in the above result we leave it unintegrated, in order Matching quasi PDF and normal PDF integration, a momentum cutoff is also needed in the z direction. It is interesting to see that \mathcal{L} integration is logarithmically divergent, in the above result we leave it unintegrated, in order itching quasi PDF and normal PDF integration, a momentum cutoff is also needed in the z direction. It is interesting to see that \mathcal{L}

the collinear divergence exists only for \mathcal{C} , which is the basis for \mathcal{C} and \mathcal{C} and \mathcal{C}

• One-loop matrix element of the quasi PDF: contribution and ignore the power-suppressed ones. This is in principal also be the case of the case o ric roop maanx cicment or the quasi PDT.

$$
\tilde{q}(x,\Lambda,P^z) = (1 + \tilde{Z}_F^{(1)}(\Lambda,P^z))\delta(x-1) + \tilde{q}^{(1)}(x,\Lambda,P^z) + \dots ,
$$

FIG. 1: One-loop corrections to quasi quark distribution.

the collinear divergence exists only for 0 <x< 1, which is the basis for factorization.

$$
\tilde{q}^{(1)}(x,\Lambda,P^z) = \frac{\alpha_S C_F}{2\pi} \left\{ \begin{array}{l} \frac{1+x^2}{1-x} \ln \frac{x}{x-1} + 1 + \frac{\Lambda}{(1-x)^2 P^z}, & x > 1 ,\\ \frac{1+x^2}{1-x} \ln \frac{(P^z)^2}{m^2} + \frac{1+x^2}{1-x} \ln \frac{4x}{1-x} - \frac{4x}{1-x} + 1 + \frac{\Lambda}{(1-x)^2 P^z} , & 0 < x < 1 ,\\ \frac{1+x^2}{1-x} \ln \frac{x-1}{x} - 1 + \frac{\Lambda}{(1-x)^2 P^z} , & x < 0 , \end{array} \right\}
$$

$$
\tilde{Z}_{F}^{(1)}(\Lambda, P^{z}) = \frac{\alpha_{S}C_{F}}{2\pi} \int dy \begin{cases} -\frac{1+y^{2}}{1-y} \ln \frac{y}{y-1} - 1 - \frac{\Lambda}{(1-y)^{2}P^{z}} \\ -\frac{1+y^{2}}{1-y} \ln \frac{(P^{z})^{2}}{m^{2}} - \frac{1+y^{2}}{1-y} \ln \frac{4y}{1-y} + \frac{4y^{2}}{1-y} + 1 - \frac{\Lambda}{(1-y)^{2}P^{z}} \end{cases},
$$

Ramarks.

Remarks:

■ In the finite momentum frame, the quark momentum fraction $-\infty < x < +\infty$. e r $10m$ omentum frame, the quark momentum fraction –∞ <x< +∞. Eq. (8) is valid for x well below Λ/P^z. In a nucleon with large but finite momentum P^z th the finite momentum frame, the quark momentum fraction –∞ <x< +∞.

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(1−y)2Pz , y 2Pz , y 2P

hing quasi DDE and n integration is logarithmically divergent, in the above result we leave it unintegrated, in order Matching quasi PDF and normal PDF integration, a momentum cutoff is also needed in the z direction. It is interesting to see that \mathcal{L} to see the match between the structures of Z˜(1) ^F and ˜q(1). If one chooses to perform the y latching quasi PDF and normal PDF \blacksquare the collinear divergence exists included to the basis of \sim

the collinear divergence exists only for \mathcal{C} , which is the basis for \mathcal{C} and \mathcal{C} and \mathcal{C}

One-loop matrix element of the quasi PDF: contribution and ignore the power-suppressed ones. This is in principal also be the case of the case o one loop matrix element of the quasi for .

⎧

1
1
1

$$
\tilde{q}(x,\Lambda,P^z) = (1 + \tilde{Z}_F^{(1)}(\Lambda,P^z))\delta(x-1) + \tilde{q}^{(1)}(x,\Lambda,P^z) + \dots ,
$$

FIG. 1: One-loop corrections to quasi quark distribution.

In field theory calculations, the ultraviolet cut-off shall be larger than any other scale in

$$
\tilde{q}^{(1)}(x,\Lambda,P^z) = \frac{\alpha_S C_F}{2\pi} \begin{cases} \frac{1+x^2}{1-x} \ln \frac{x}{x-1} + 1 + \frac{\Lambda}{(1-x)^2 P^z} , & x > 1 ,\\ \frac{1+x^2}{1-x} \ln \frac{(P^z)^2}{m^2} + \frac{1+x^2}{1-x} \ln \frac{4x}{1-x} - \frac{4x}{1-x} + 1 + \frac{\Lambda}{(1-x)^2 P^z} , & 0 < x < 1 ,\\ \frac{1+x^2}{1-x} \ln \frac{x-1}{x} - 1 + \frac{\Lambda}{(1-x)^2 P^z} , & x < 0 , \end{cases}
$$

$$
\tilde{Z}_{F}^{(1)}(\Lambda, P^{z}) = \frac{\alpha_{S}C_{F}}{2\pi} \int dy \begin{cases} \frac{-\frac{1+y^{2}}{1-y} \ln \frac{y}{y-1}}{\frac{1+y^{2}}{1-y} \ln \frac{(P^{z})^{2}}{m^{2}}} - \frac{1-\frac{\Lambda}{(1-y)^{2}P^{z}}}{\frac{1+y^{2}}{1-y} \ln \frac{4y}{1-y} + \frac{4y^{2}}{1-y} + 1 - \frac{\Lambda}{(1-y)^{2}P^{z}}}, \\ - \frac{1+y^{2}}{1-y} \ln \frac{y-1}{y} + 1 - \frac{\Lambda}{(1-y)^{2}P^{z}} \end{cases}
$$

Remarks:

• No UV divergence (logΛ). Logarithm of P^z only exists in o<x<1. —
erg ence ■ No UV divergence (logA). Logarithm of P^z only exists in o<x<1. $(1, 2, 2)$

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the collinear divergence exists only for \mathcal{C} , which is the basis for \mathcal{C} and \mathcal{C} and \mathcal{C}

One-loop matrix element of the quasi PDF: contribution and ignore the power-suppressed ones. This is in principal also be the case of the case o one loop matrix element of the quasi for .

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$$
\tilde{q}(x,\Lambda,P^z) = (1 + \tilde{Z}_F^{(1)}(\Lambda,P^z))\delta(x-1) + \tilde{q}^{(1)}(x,\Lambda,P^z) + \dots ,
$$

FIG. 1: One-loop corrections to quasi quark distribution.

In field theory calculations, the ultraviolet cut-off shall be larger than any other scale in

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$$

Remarks:

• Axial singularity cancels between real and virtual contributions. ⎪⎪⎪⎪⎪⎪⎪⎪⎩ ity ca ■ Axial singularity cancels between real and virtual contributions. $(1,2,2)$

Oct. 22, 2015 INT-15-3, Seattle 34 Oct. 22, 2015 **INT-15-3, Seattle INT-15-3, Seattle** t_1 and t_2 10. t_3 is expected to be negligible. The negligible t_1 is expected to be negligible. To be

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$$

Remarks:

• Collinear divergence exists only in o<x<1, and is the same as the normal PDF. ⎪⎪⎪⎪⎪⎪⎪⎪⎩ rgen ■ Collinear divergence exists only in o<x<1, and is the same as the normal PDF.

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latching −1 |y| Z i, $\overline{}$ Λ \blacksquare µ P^z # **F** and normal PDF

 \blacksquare By comparing their one-loop matrix elements, the matching factor can be read as: The MS subtraction scheme with $\frac{1}{2}$ The Z factor can be read as.

$$
Z\left(\xi,\frac{\Lambda}{P^z},\frac{\mu}{P^z}\right) = \delta(\xi-1) + \frac{\alpha_s}{2\pi}Z^{(1)}\left(\xi,\frac{\Lambda}{P^z},\frac{\mu}{P^z}\right) + \dots
$$

where the integration range is determined by the support of the support of the support of the quark distribution α

$$
Z^{(1)}(\xi)/C_F = \frac{1+\xi^2}{1-\xi} \ln \frac{\xi}{\xi-1} + 1 + \frac{1}{(1-\xi)^2} \frac{\Lambda}{P^z}, \quad \xi > 1
$$

$$
Z^{(1)}(\xi)/C_F = \frac{1+\xi^2}{1-\xi} \ln \frac{(P^z)^2}{\mu^2} + \frac{1+\xi^2}{1-\xi} \ln[4\xi(1-\xi)] - \frac{2\xi}{1-\xi} + 1 + \frac{1}{(1-\xi)^2} \frac{\Lambda}{P^z}, \quad 0 < \xi < 1
$$

$$
Z^{(1)}(\xi)/C_F = \frac{1+\xi^2}{1-\xi} \ln \frac{\xi-1}{\xi} - 1 + \frac{1}{(1-\xi)^2} \frac{\Lambda}{P^z}, \quad \xi < 0
$$

to the lack of a cut-off scale, and the term leading to the term leading to the linear divergence becomes, after

$M = L - L$!1 + ξ² " ln (P^z)² !1 + ξ² " [−] ²^ξ Λ ^µ² ⁺ 1 − και του $\overline{}$

 $1 - 5$

Near $\xi = I$, there is also virtual contribution: int_{10} contrib<mark>u</mark> <u>ibution:</u>

 $1 - 5$

$$
Z^{(1)}(\xi) = \delta Z^{(1)}(2\pi/\alpha_s)\delta(\xi - 1)
$$
\nwith

\n
$$
\delta Z^{(1)} = \frac{\alpha_S C_F}{2\pi} \int dy \begin{cases}\n-\frac{1+y^2}{1-y} \ln \frac{y}{y-1} - 1 - \frac{\Lambda}{(1-y)^2 P^z}, & y > 1, \\
-\frac{1+y^2}{1-y} \ln \frac{(P^z)^2}{\mu^2} - \frac{1+y^2}{1-y} \ln \left[4y(1-y)\right] + \frac{2y(2y-1)}{1-y} + 1 - \frac{\Lambda}{(1-y)^2 P^z}, & 0 < y < 1, \\
-\frac{1+y^2}{1-y} \ln \frac{y-1}{y} + 1 - \frac{\Lambda}{(1-y)^2 P^z}, & y < 0,\n\end{cases}
$$
\n(18)

Program of lattice calculation

Renormalizability of quasi PDF

- Quasi PDF is the matrix element of a non-local operator
	- **•** In the renormalization of local operators, all UV divergences can be removed by local counterterms.
	- Renormalization of non-local operator matrix element is rather distinct. Needs non-local counterterms, and to deal with axial singularities.

Renormalizability of quasi PDF

- **•** In the continuum theory, quasi-PDF can be multiplicatively renormalized at two-loop order in dimensional regularization. [Ji and Zhang, PRD, 15']
- Vertex contribution contains sub-divergences only, which can be removed by counterterms from interaction (UV counterterms in the axial gauge).
- **EXE** Renormalization of quasi PDF reduces to the renormalization of two separate (axial gauge) quark fields, which is equivalent to the renormalization of the heavy-light quark vector current in HQET.
- Such features are expected to hold for all orders, to be checked.

Renormalizability of quasi PDF

- **The renormalization of nonlocal operators in lattice QCD is** desirable, but difficult. It is not tested HYP smearing can make the renormalization factor $~1$;
- With a proper renormalization scheme in the lattice theory, we will be able to calculate the matching between lattice and continuum theory matrix elements with perturbative QCD.

Higher twist corrections

- Similar to target mass corrections from the trace of the nucleon matrix element, which is of order $\sim O(M^2/(P^2)^2)$, has been derived (J.W. Chen, in preparation);
- **•** Higher twist correction to the correlation operator itself, which is of order ~O(Λ_{QCD}²/(P^z)²), has been derived, and its matrix element is to be calculated in lattice QCD (J.W. Chen, in preparation).

Exploratory study of PDFs in lattice QCD

- Isovector (sea) quark distribution calculated with the LaMET approach
- HYP smearing applied to smoothen Wilson line gauge links
- One-loop matching + mass corrections included

Alexandrou et. al. PRD 15'

Requirements for lattice resources

- **Parton with momentum xPz has correlation length** \sim **1/** xP^z , so with a boosted nucleon the distribution the valence quark is shrunk by the boost factor ν .
- \blacksquare Needs higher resolution along the z direction, lattice spacing should be 1/γ to transverse directions, or, with the same box size lattice sites must be γ times that of the transverse directions.

Ji, Sci. China Phys. Mech. Astro., 2014

Small x PDFs

- **The smallest** x for PDF is $\sim \Lambda_{\text{QCD}}/P^z$. The correlation length of valence quark does not change under the Lorentz boost, but to achieve extremely small x one needs extremely large P^z.
- For example, for $x \sim 10^{-4}$, $P^z \sim 3$ TeV, which means that the lattice sites must be 3000 times of those in the transverse directions.
- This requires a lot of computation resources! Current computation power is good at studying large x PDFs, while small x can be probed in experiments.

Lattice size along the time axis

- Since the energy difference between the lowest excited state and the ground state will be suppressed by a factor of ν , one also requires a long evolution to extract the matrix elements.
- \blacksquare For a typical 24⁴ lattice, the ideal choice of lattice size for such calculation will be $24^2 \times (24\gamma)^2$.

Application of the LaMET

- LaMET can be applied to the lattice QCD

calculation of all the other parton physics.
	- **Light-cone wavefuntions**
	- Pion distribution amplitude [Ji, Schäfer, Xiong and Zhang, 15']
	- **Transverse momentum distributions**
	- Generalized parton distributions [Ji, Schäfer, Xiong and Zhang, 15']
	- **Wigner distributions**
	- **Fragmentation functions**

§ …

Summary

- LaMET enables us to extract parton observables from the lattice matrix elements of certain timeindependent, frame-dependent operators.
- The normal PDF is related to quasi PDF through a perturbative matching condition that has been calculated using LaMET.
- A program for the lattice calculation of PDFs is outlined.

Back up slides

■ For QED,

$$
A^i_{\perp} = A^i - \partial^i \frac{1}{\vec{\partial}^2} \vec{\partial} \cdot \vec{A} \rightarrow A^i - \partial^i \frac{1}{\partial^+} A^+
$$

$$
\frac{1}{\partial^+} f(\xi^-) = \frac{1}{2} \int d\xi^- K(\xi^- - \xi^-) f(\xi^-),
$$

$$
K(\xi^- - \xi^-) = \text{sgn}(\xi^- - \xi^-), \text{ or } \pm 2\theta(\xi^- - \xi^-).
$$

■ For QCD, the non-Abelian Coulomb condition leads to the IMF limit of A_{phys}

$$
A_{\text{phys}}^* = 0, \quad A_{\text{pure}}^* = A^*.
$$

Polarized gluon distribution

• Polarized gluon distribution:

$$
\Delta g(x) = \frac{i}{2xP^+} \int d\xi^{-} e^{-ixP^+\xi^-} \langle PS | F_a^{+,i}(\xi^-) L^{ab}(\xi^-, 0) \tilde{F}_{b,i}^+(0) | PS \rangle
$$

 $\Delta q(x)$ can be measured in polarized DIS and proton-proton collisions.

Total gluon polarization:

$$
\Delta G = \int dx \, \Delta g(x)
$$

= $\frac{1}{2P^+} \langle PS \vert \int \frac{dx}{x} \int d\xi^{-} e^{-ixP^+\xi^{-}} F^{+,i}_{a}(\xi^{-}) L^{ab}(\xi^{-}, 0) \tilde{F}^+_{b,i}(0) \vert PS \rangle$
= $-\frac{1}{2P^+} \langle PS \vert [A^{i,a} - \frac{1}{\partial^+} \partial^i A^{+,b}(\xi^{-}) L^{ba}(\xi^{-}, \xi^{-})] \tilde{F}^+_{i,b}(0) \vert PS \rangle$
Bac_{kt. 22, 2015}

Moment of the quark GTMD

Zhao, Liu, Yang, 2015

$$
\int dx d^2 k_{\perp} k^i_{\perp} f(x, \vec{k}_{\perp}, \vec{\Delta}_{\perp}) = \frac{1}{2\overline{P}^+} \langle P^{\dagger} S | \overline{\psi}(0) \gamma^{\dagger} i \overline{D}^i_{\text{pure}} \psi(0) | PS \rangle
$$

$$
\varepsilon^{ij} \lim_{\Delta \to 0} \frac{\partial}{\partial i \Delta^i} \int dx d^2 k_{\scriptscriptstyle T} k^j_{\scriptscriptstyle T} f(x, k_{\scriptscriptstyle T}, \Delta_{\scriptscriptstyle T}) = \frac{1}{2\overline{P}^+} \frac{\langle PS | \int d^3 \xi \overline{\psi}(\xi) \gamma^{\dagger} \varepsilon^{ij} \xi^i i \overline{D}^j_{\text{pure}} \psi(\xi) | PS \rangle}{(2\pi)^3 \delta^{(3)}(0)},
$$

$$
\lim_{p \to \infty} \langle L_q^c \rangle = \varepsilon^{ij} \lim_{\Delta \to 0} \frac{\partial}{\partial i \Delta^i} \int dx d^2 k_{\scriptscriptstyle T} k^j_{\scriptscriptstyle T} f(x, k_{\scriptscriptstyle T}, \Delta_{\scriptscriptstyle T})
$$

$$
= - \int dx d^2 k_{\scriptscriptstyle T} \frac{k_{\scriptscriptstyle T}^2}{M^2} F^q_{14}(x, 0, k_{\scriptscriptstyle T}^2, 0, 0).
$$

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Moment of the quark GTMD

F₁₄ is a GTMD whose measurement is unknown right now, but its possibility has been studied (Courtoy et al., 2014);

$$
L_q^{\text{can}}(x) = \varepsilon^{ij} \lim_{\Delta \to 0} \frac{\partial}{i \partial \Delta} \left(\int \frac{d\xi}{2\pi} e^{-ix\overline{P} + \xi^-} \left\langle P^{\dagger} S \middle| \overline{\psi}(\xi^-) \gamma^+ i \overline{D}_{\text{pure}}^j \psi(0) \middle| PS \right\rangle \right)
$$

■ The canonical quark OAM density is related to a twist-three GPD measureable in hard exclusive processes (Ji et al., 2012; Hatta, 2012).

■ The gluon GTMD

Zhao, Liu, Yang, 2015

$$
g(x,\vec{k}_{\perp},\vec{\Delta}_{\perp}) = -\frac{i}{2x\overline{P}^{+}} \int \frac{dz^{-}d^{2}z_{\perp}}{(2\pi)^{3}} e^{ix\overline{P}^{+}z^{-}} \langle P^{*}S|F^{+\alpha}(-\frac{z^{-}}{2}, -\frac{z_{\perp}}{2})\gamma^{+} \times W^{-}_{-z^{-}/2,\pm\infty}W^{T}_{-z_{\perp}/2,z_{\perp}/2}W^{-}_{\pm\infty,z^{-}/2}F^{+}_{\alpha}(\frac{z^{-}}{2}, \frac{z_{\perp}}{2})|PS\rangle
$$

The canonical gluon OAM density is also related to a twistthree GPD (Hatta et al., $2012, 2013$).

$$
\lim_{P^z \to \infty} \left\langle L_g^c \right\rangle = \varepsilon^{ij} \lim_{\Delta \to 0} \frac{\partial}{\partial i \Delta^i} \int dx d^2 k_T \ k_T^j g(x, k_T, \Delta_T)
$$