

Multiparticle processes from the lattice



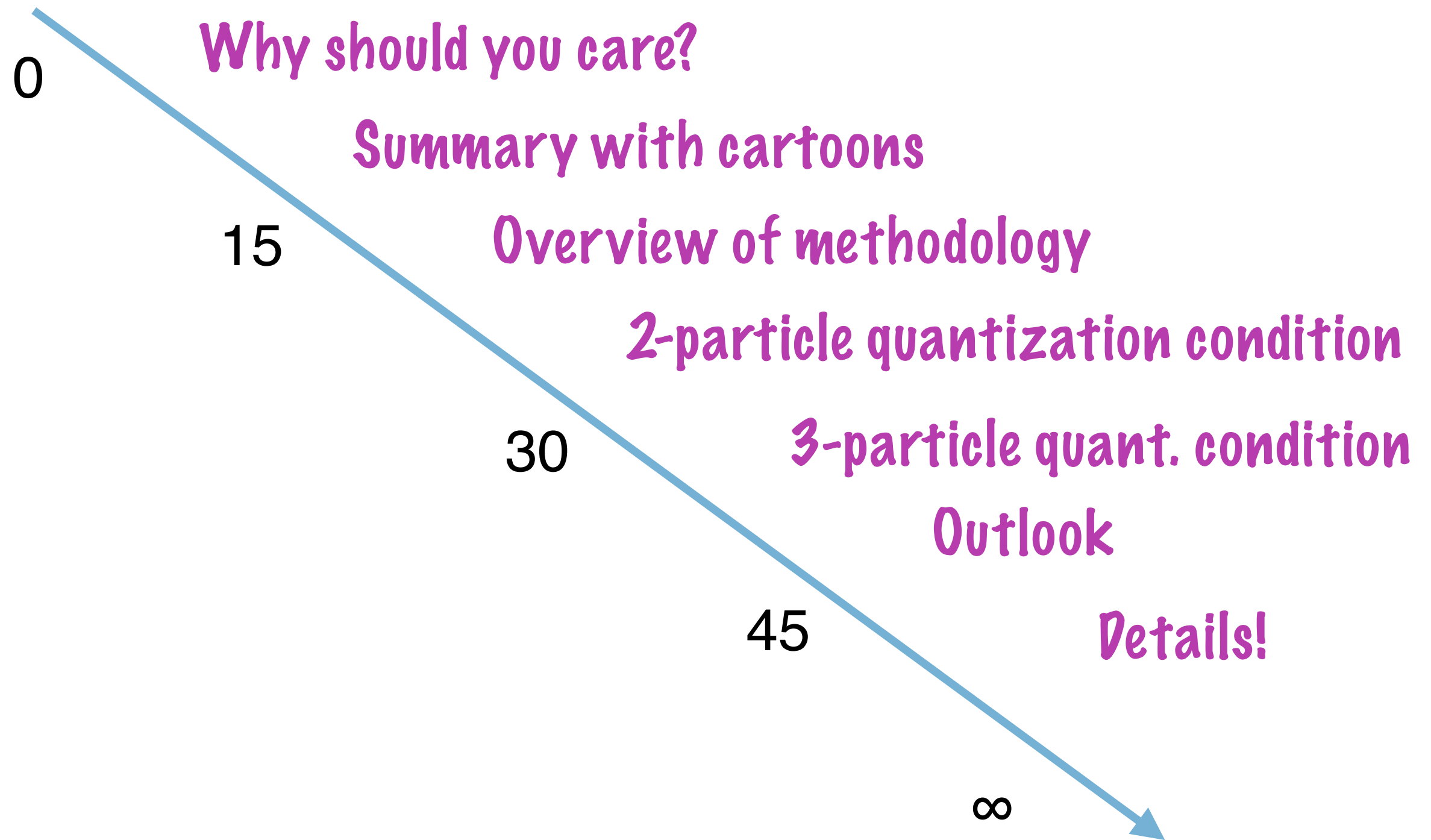
Steve Sharpe
University of Washington



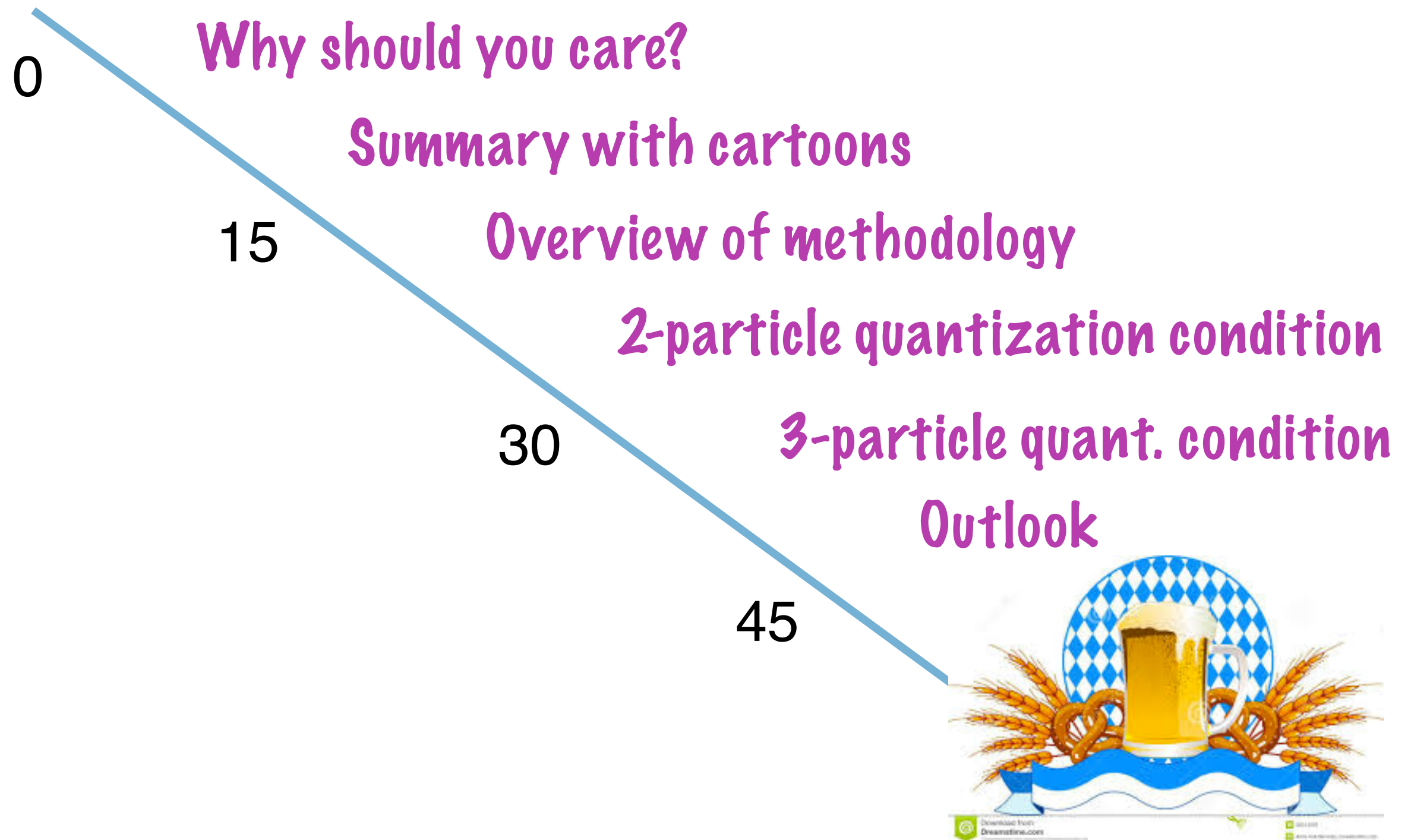
Overview

- This talk is more like a review
 - Aim to show where field stands and the major obstacles to progress
- Focus will be on theoretical issues (not simulations)
 - Though I will not go into many technical details
- The key theoretical issue is the impact of working in a finite spatial volume (always the case in simulations)
 - Discretization of space-time is not the issue, so can use continuum QFT
 - Use of Euclidean time in simulations is not (directly) the issue either

Outline



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What is the relation to
“QCD FOR NEW PHYSICS AT THE PRECISION FRONTIER”?

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- Weak decay and mixing amplitudes allow tests of the SM and thus indirect searches for BSM physics
- Many of these involve multiparticle final or intermediate states, e.g.
 - $K \rightarrow \pi\pi\pi$ ($\Delta I = 1/2$ rule and ϵ'/ϵ)
 - $K_0 - \bar{K}_0$ mixing (ΔM_K)
 - $B \rightarrow K\pi$ (+ $\tilde{\ell}^+ \tilde{\ell}^-$)
 - $K \rightarrow \pi\pi\pi\pi$
 - $D \rightarrow \pi\pi, KK, \eta\eta, 4\pi, \dots$
 - $D_0 - \bar{D}_0$ mixing
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- Tests require lattice QCD (LQCD) calculations of corresponding hadronic matrix elements
- These need both theoretical developments (finite-volume QFT) and advances in simulations

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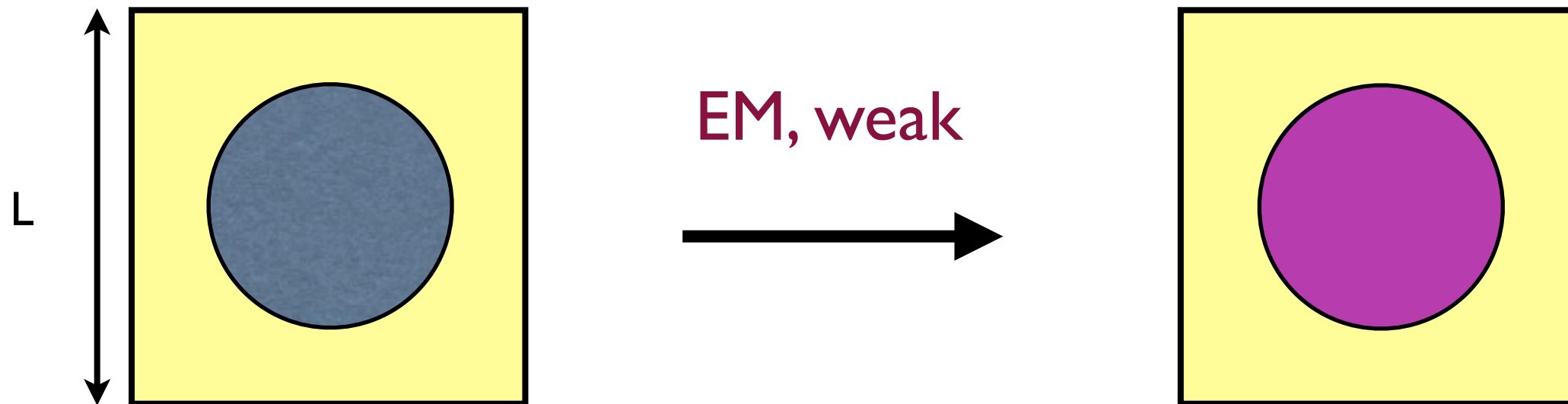
Theory remains to be developed

- Tests require lattice QCD (LQCD) calculations of corresponding hadronic matrix elements
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Summary with cartoons

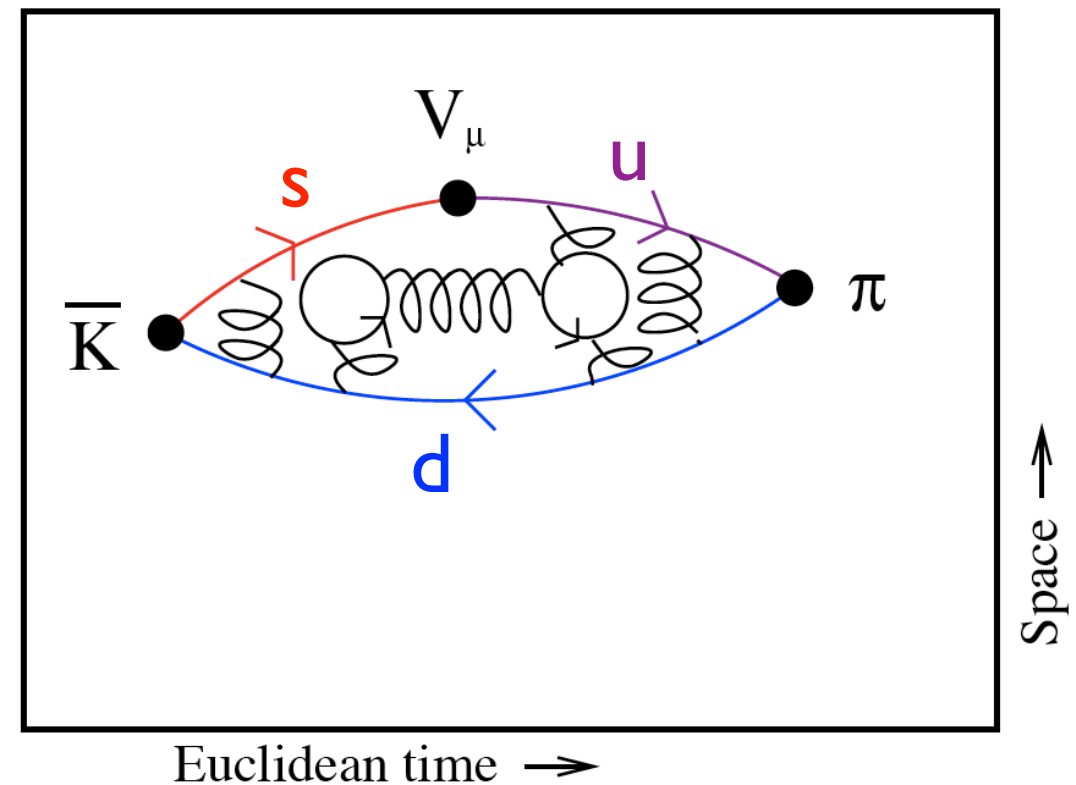
Well-controlled LQCD calculations

Discussed in Aida's talk; averages for many quantities presented by FLAG



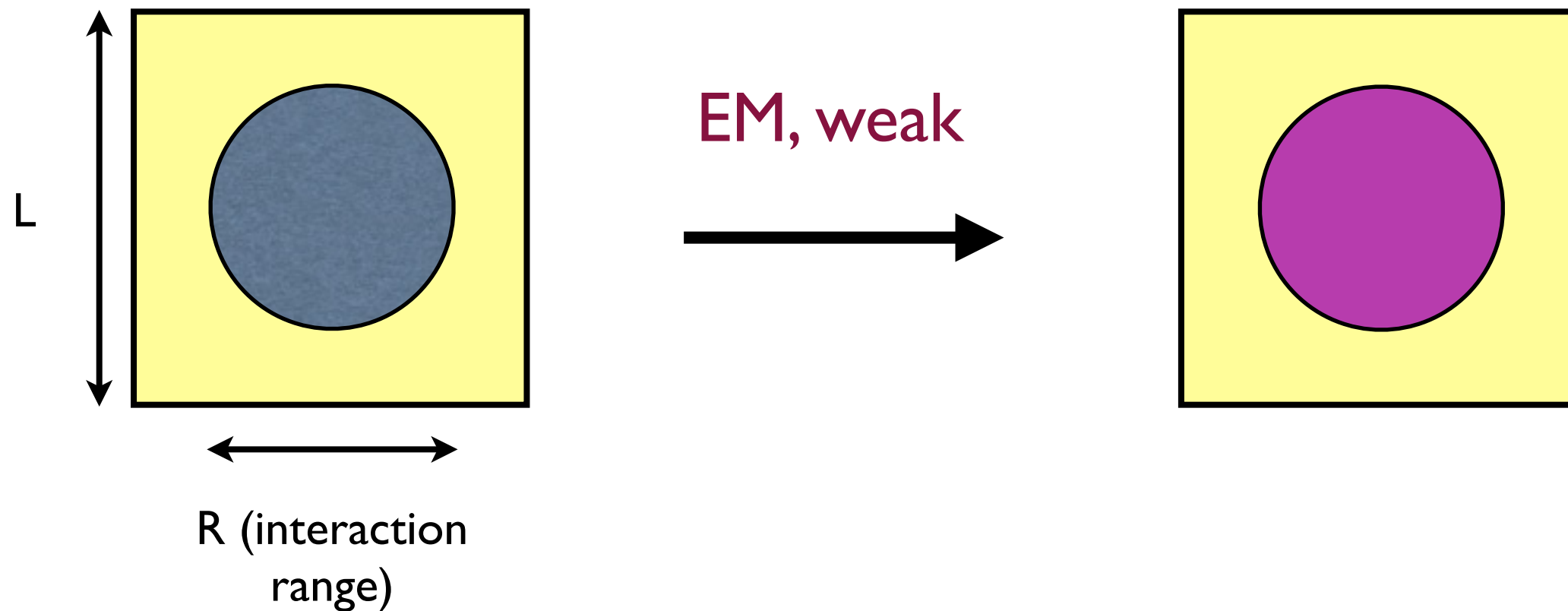
Example:
 $K \rightarrow \pi$ form factor

$$\langle \pi(\vec{p}_2) | V_\mu(0) | \bar{K}(\vec{p}_1) \rangle$$



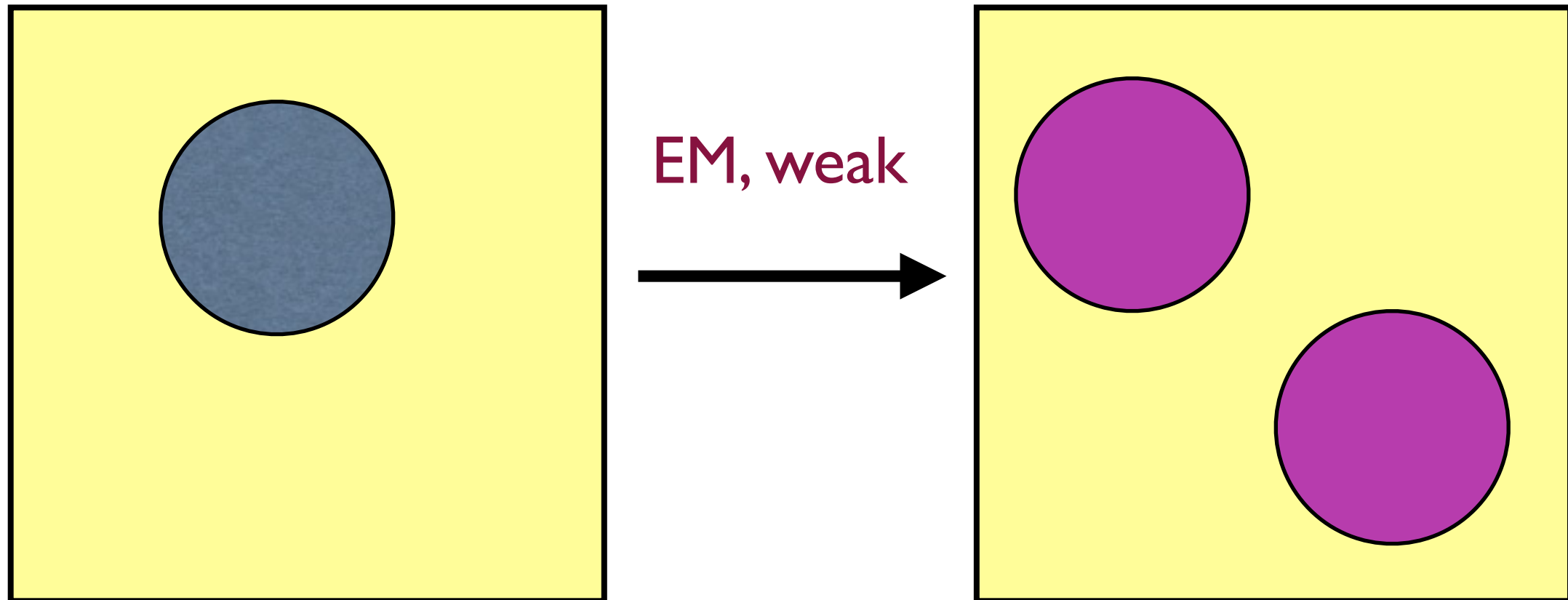
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For large enough boxes ($L > 2R$) dominant finite-volume effects for single-particle states fall as $\exp(-M_\pi L)$ [Lüscher] and can be made small

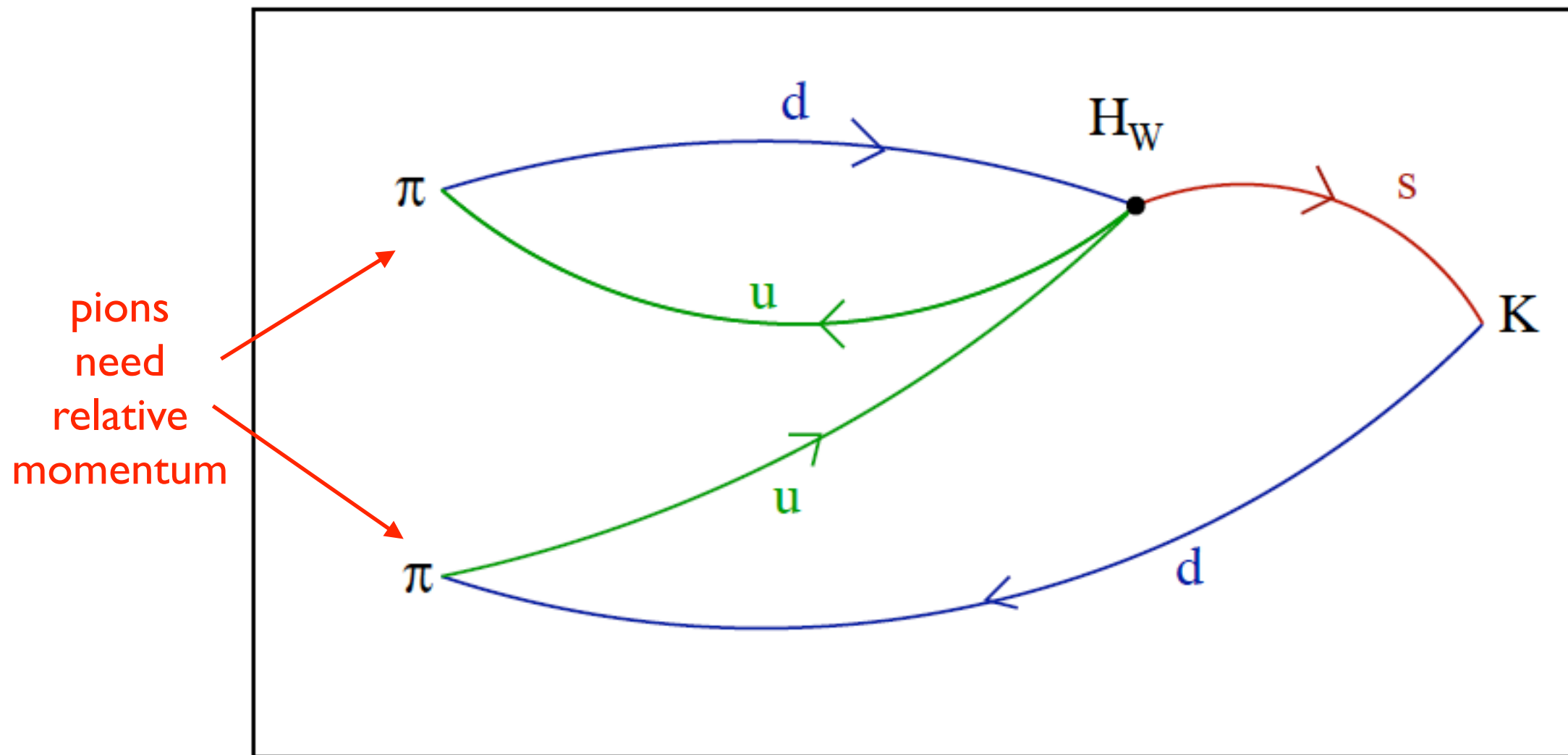
Present frontier (1)



e.g. $K \rightarrow \pi\pi$ ($I=0$ & 2) [RBC/UKQCD]

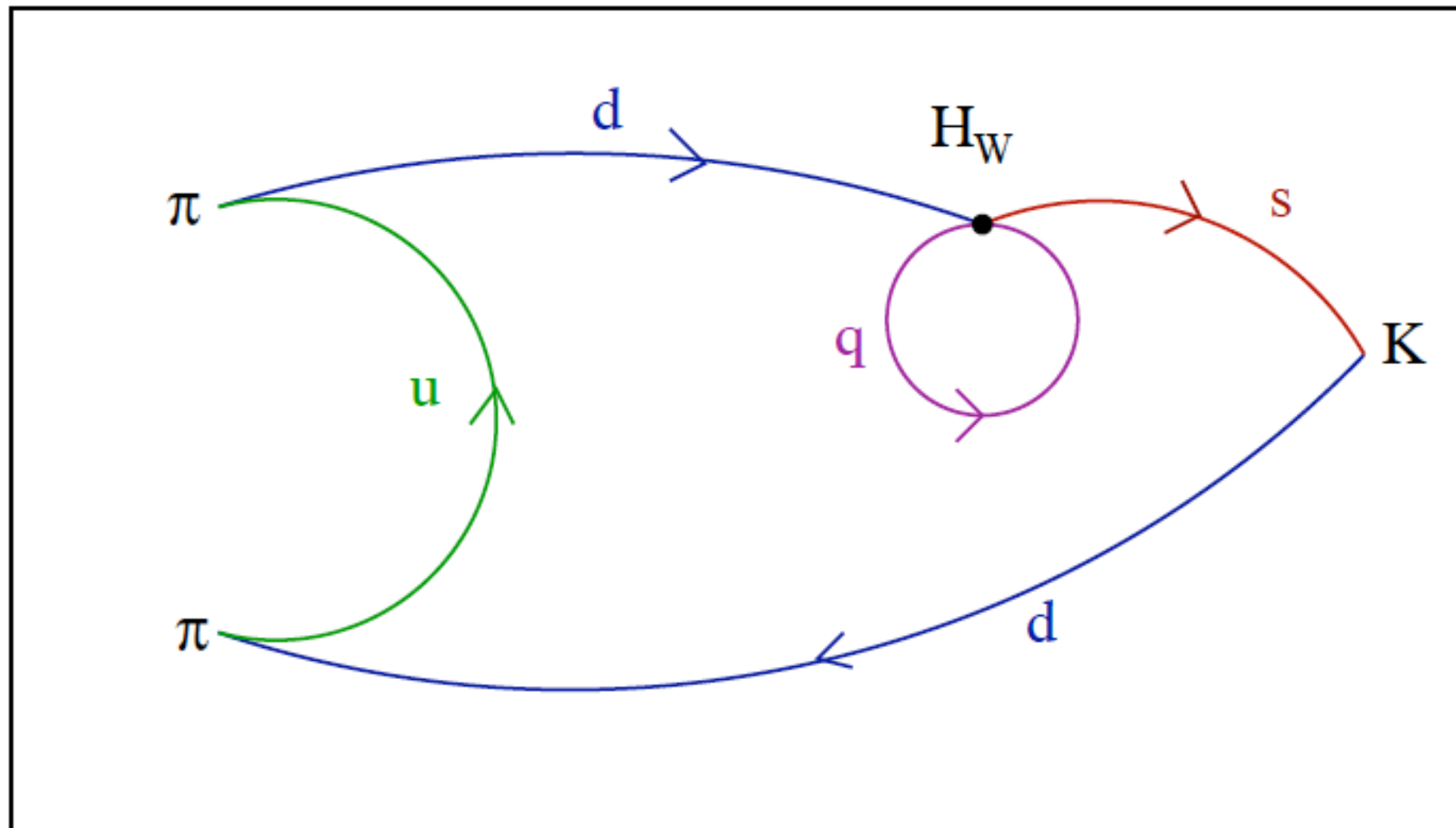
- Issues associated with 2 particles ($1/L^n$ finite-volume effects,...) are theoretically understood [Lüscher, Lellouch & Lüscher, ...]
- Numerical implementations expanding despite computational challenges

Computational challenges



Four-point function

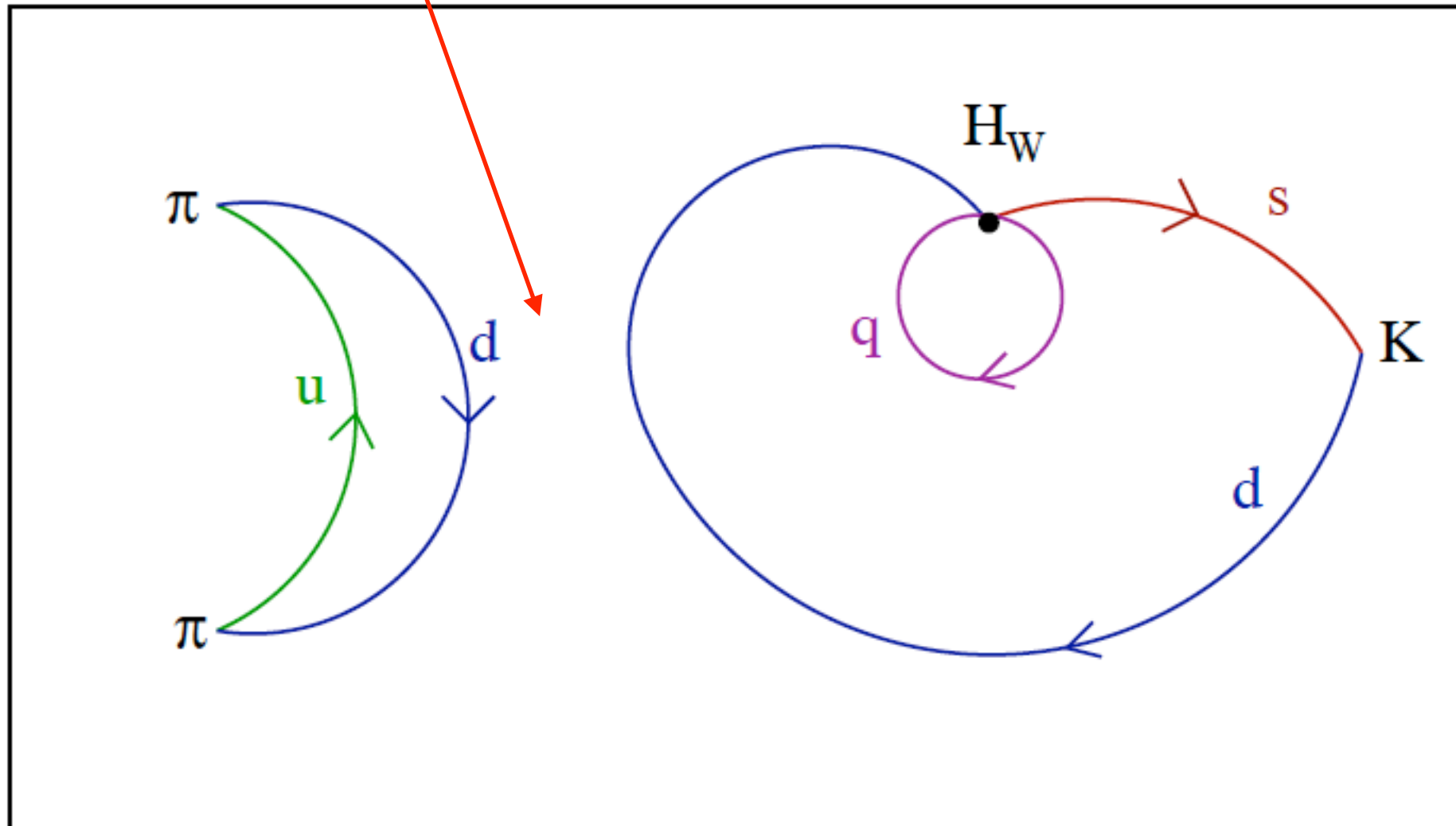
Computational challenges



Partially quark-disconnected contractions

Computational challenges

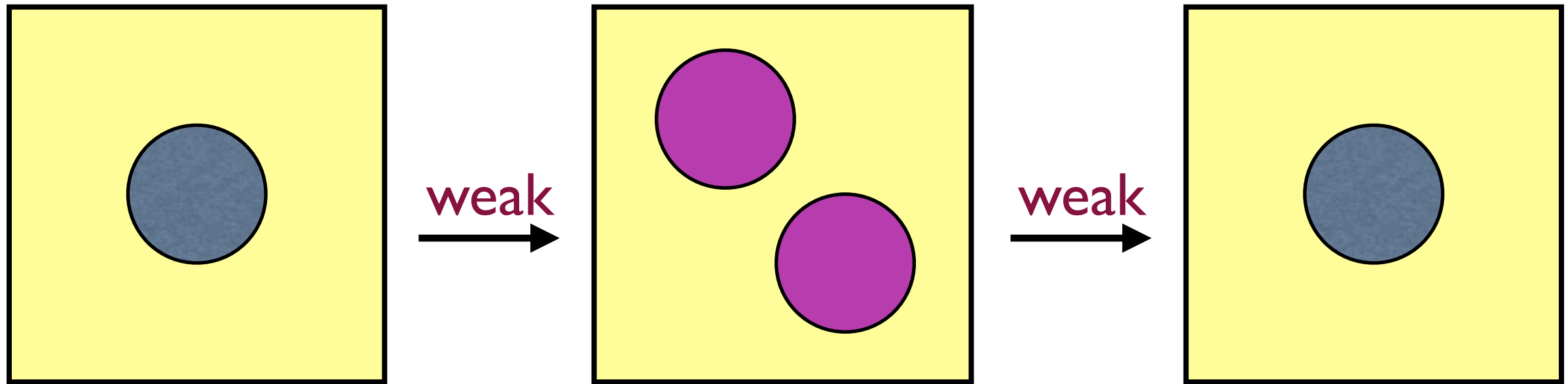
gluons are here!



Fully quark-disconnected contractions

Present frontier (2)

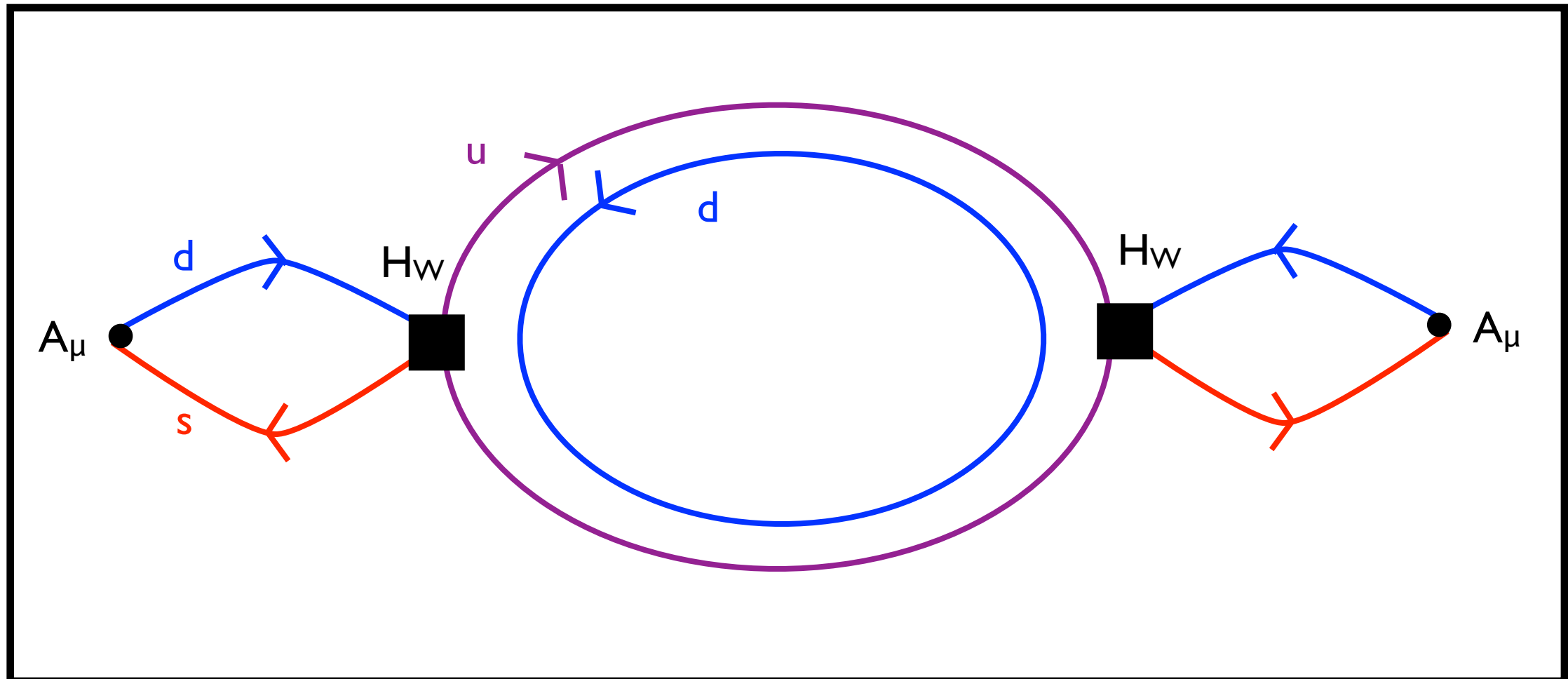
$\Delta M_K: K_0 \rightarrow \pi\pi$ (virtual) $\rightarrow \bar{K}_0$ [RBC/UKQCD]



- Issues associated with intermediate two-particle states are theoretically understood [Christ, Feng, Martinelli, Sachrajda]
- Pilot numerical calculation [RBC/UKQCD]

Present frontier (2)

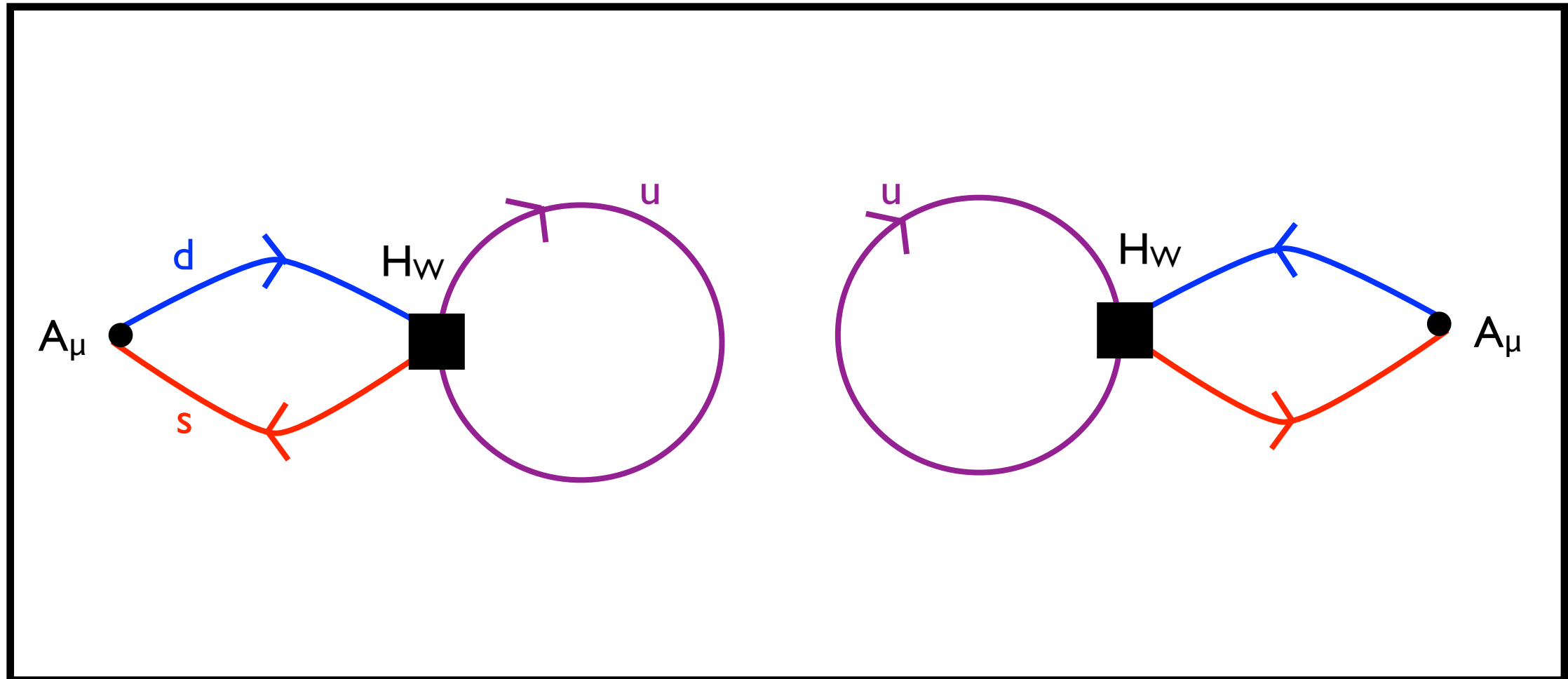
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Four-point function involving two weak currents

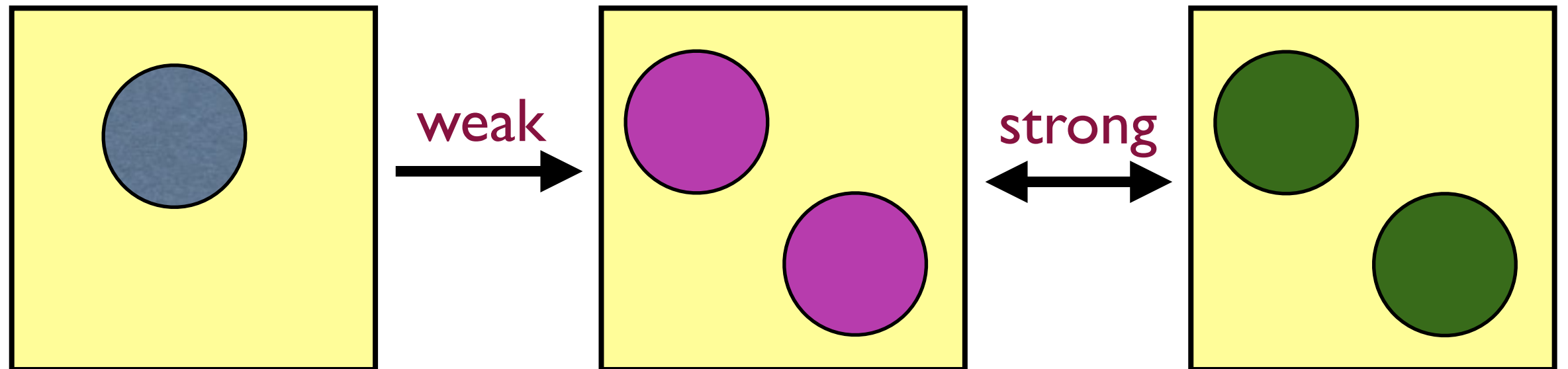
Present frontier (2)

$\Delta M_K: K_0 \rightarrow \pi\pi$ (virtual) $\rightarrow \bar{K}_0$ [RBC/UKQCD]



Quark-disconnected contributions

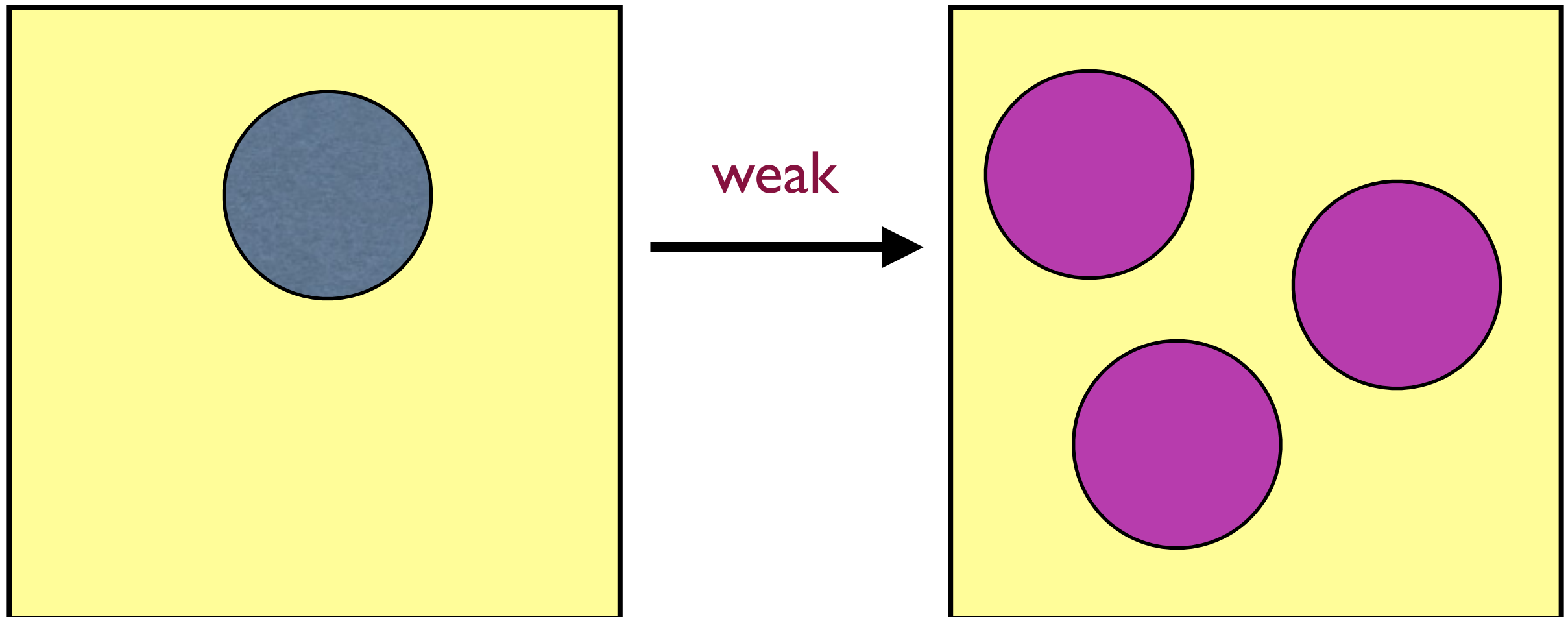
Present frontier (3)



e.g. $K_0 \rightarrow \pi_+ \pi_-$ & $\pi_0 \pi_0$ in presence of isospin breaking

- Theoretical challenge: finite-volume eigenstates are inevitably mixtures of final states
- Formalism for disentangling this mixing and obtaining correct (infinite volume) normalization and phases for amplitudes exists [Hansen & SS; Briceño, Hansen & Walker-Loud]

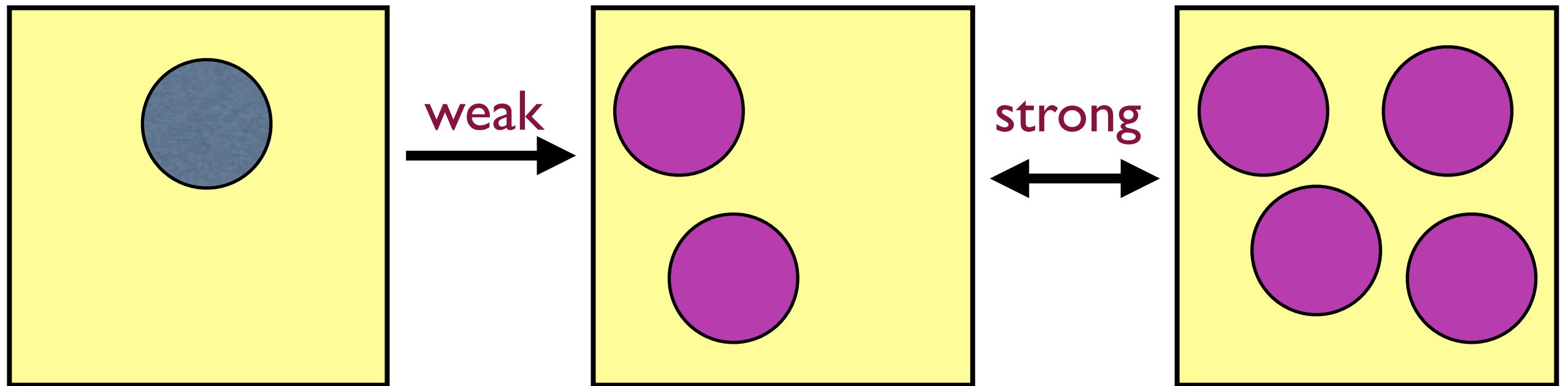
Pushing the frontier



e.g. $K_0 \rightarrow \pi\pi\pi$

- Theoretical challenge: removing $1/L^n$ finite-volume effects
- Partial progress (3 particle quantization condition) [Hansen & SS]

Beyond the frontier



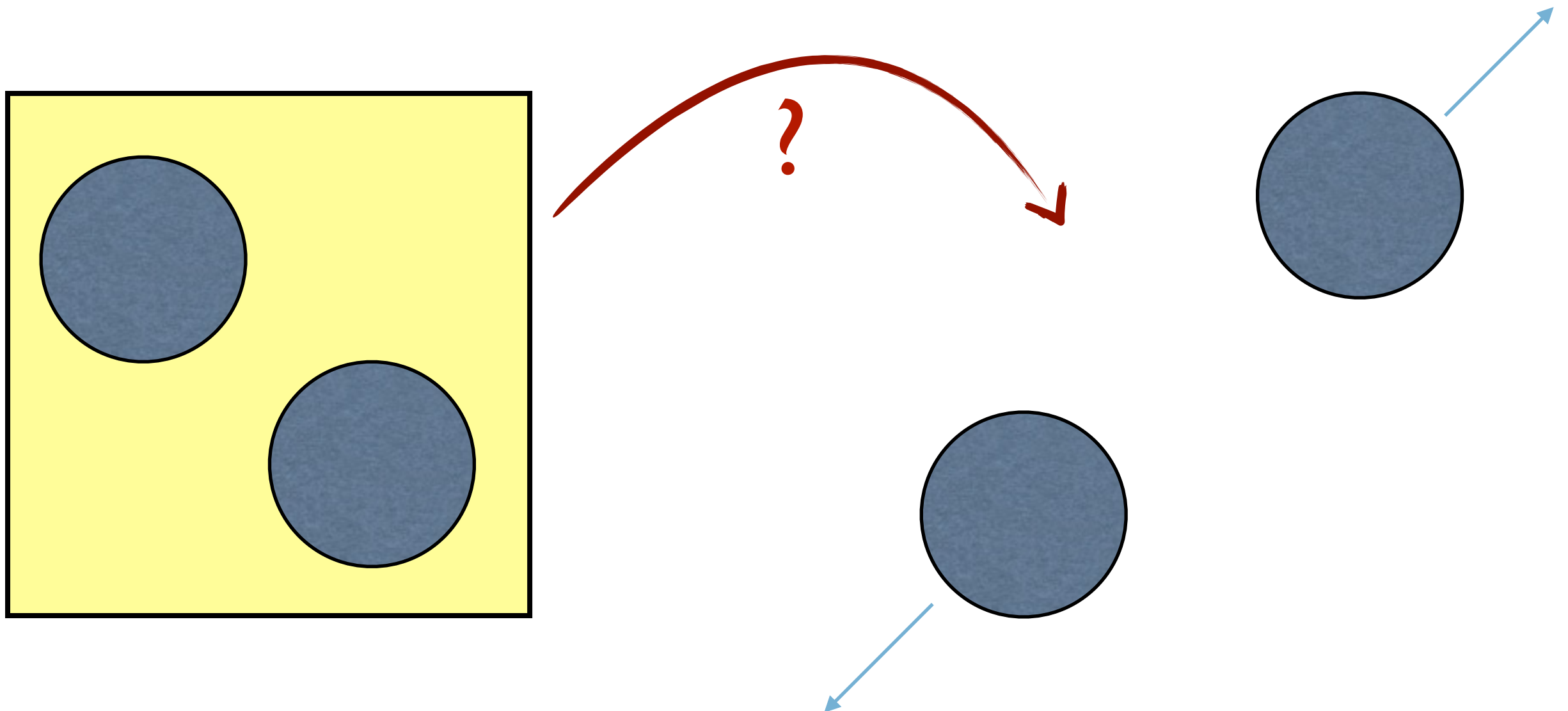
e.g. $D \rightarrow \pi\pi$ & 4π & $K\bar{K}$ & $\eta\eta$ & 6π

- Finite-volume eigenstates at $E=M_D$ are inevitably mixtures of all open channels
- Need to separate these (to get individual “out” states)
- Similar issues arise in $D-\bar{D}$ mixing for virtual intermediate states
- Open theoretical problem
- Issues for simulations are also challenging (but less so?)

Overview of methodology

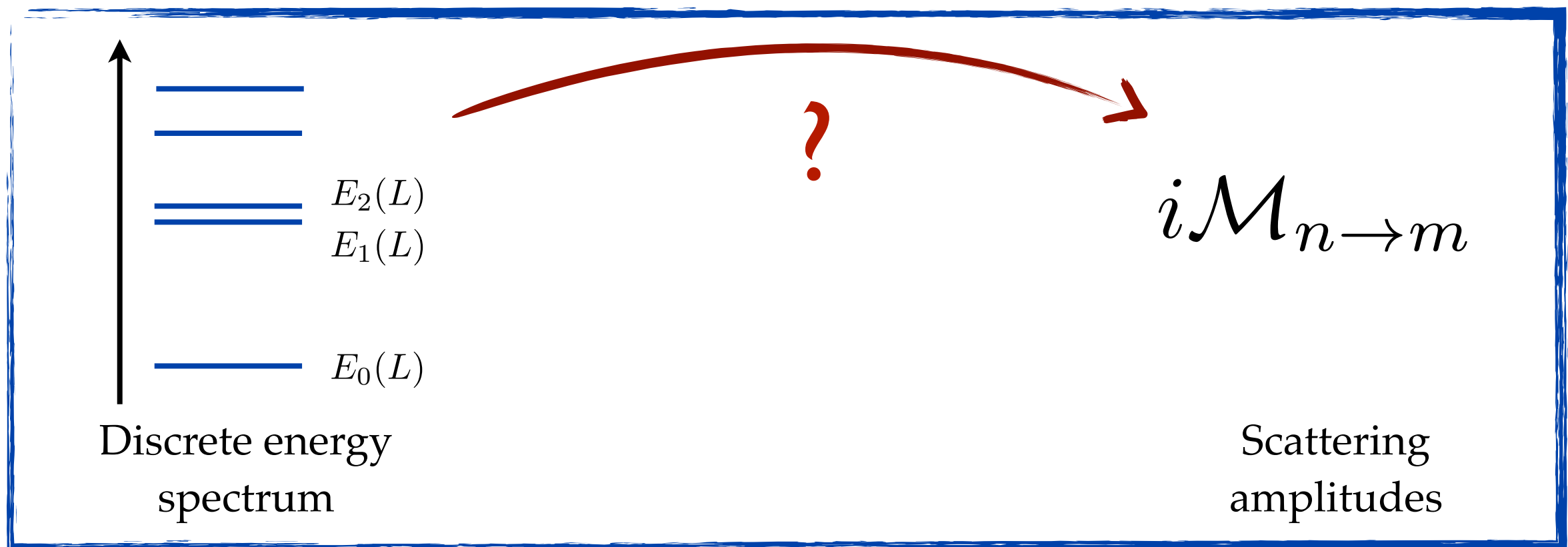
The fundamental issues

- Lattice QCD can calculate energy levels of multiparticle systems in a box and matrix elements involving these states
- How are these related to infinite-volume scattering amplitudes and decay amplitudes?



1st fundamental issue

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2nd fundamental issue

- Lattice QCD can calculate energy levels of multiparticle systems in a box and matrix elements involving these states
- How are these related to infinite-volume scattering amplitudes and decay amplitudes?

$$L \langle n | \mathcal{H}_W | D \rangle_L$$

finite-volume state interpolated by two pion operator with $E_n = M_D$

$$\text{out} \langle \pi\pi | \mathcal{H}_W | D \rangle$$

desired infinite-volume matrix element including phase

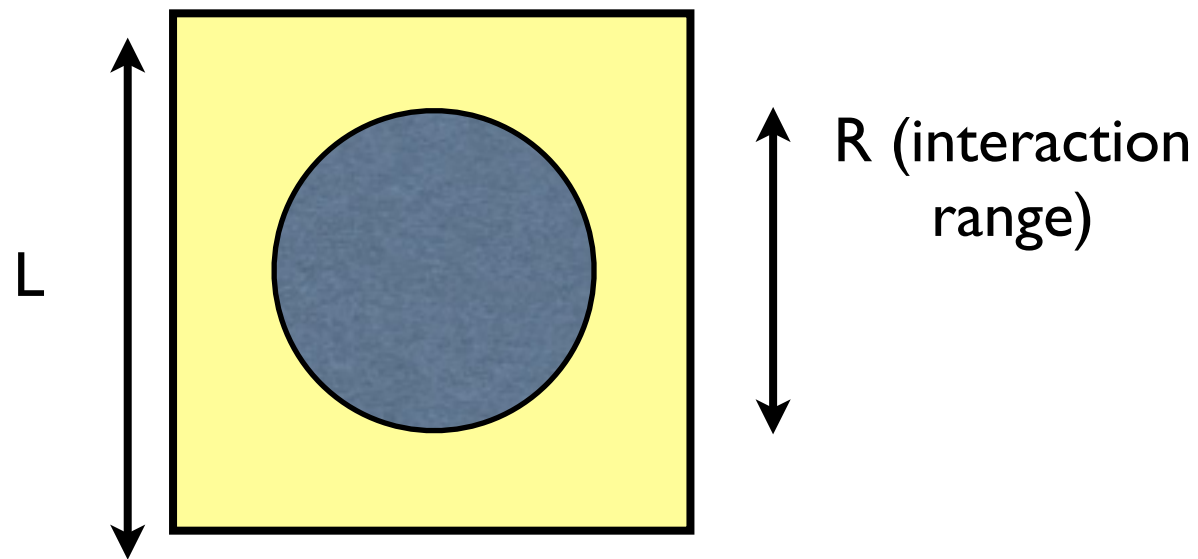
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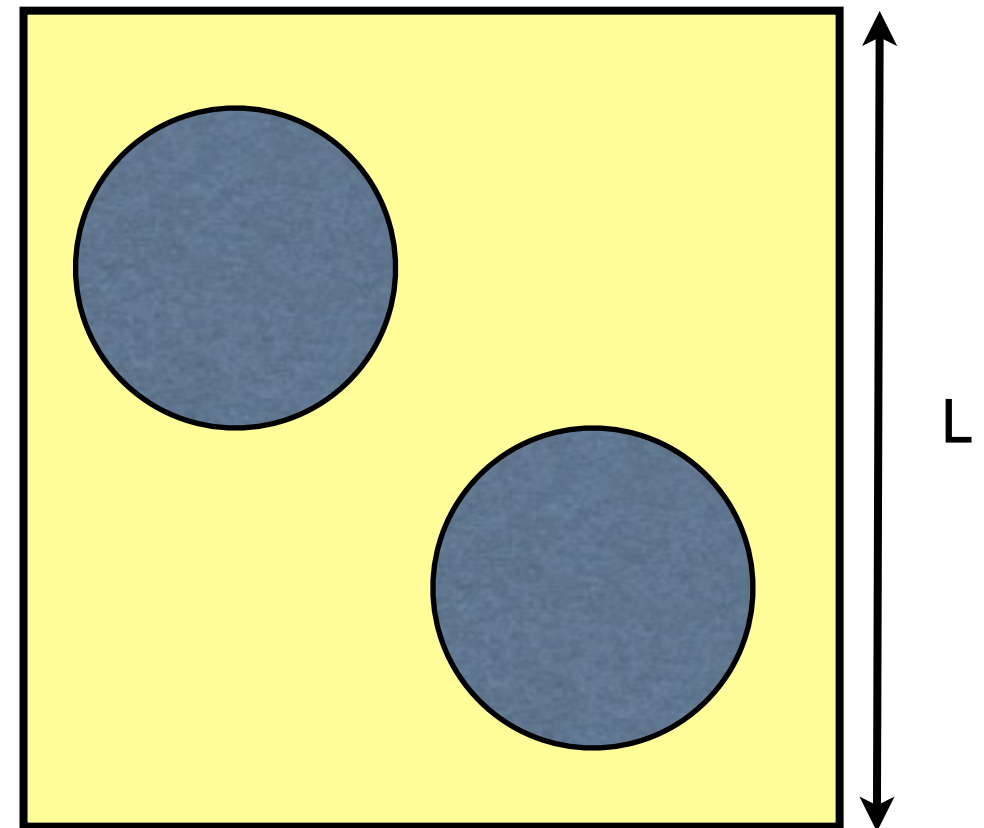
Must understand first issue before one can address the second

Two-particle quantization condition

When is spectrum related to scattering amplitudes?



Single (stable) particle with $L > R$
 Particle not “squeezed”
 Spectrum same as in infinite volume up
 to corrections proportional to $e^{-M_\pi L}$
 [Lüscher]



$L > 2R$
 There is an “outside” region.
 Spectrum is related to scatt. amps.
 up to corrections proportional to $e^{-M_\pi L}$
 [Lüscher,...]

Single-channel 2-particle quantization condition

[Lüscher 86 & 91; Rummukainen & Gottlieb 85; Kim, Sachrajda & SS 05; ...]

- Two particles (say pions) in cubic box of size L with PBC and total momentum \mathbf{P}
- Below inelastic threshold (4 pions), the finite-volume spectrum E_1, E_2, \dots is given by solutions to a secular equation in partial-wave (l, m) space (up to exponentially suppressed corrections)

$$\det \left[(F_{\mathbf{P}\mathbf{V}})^{-1} + \mathcal{K}_2 \right] = 0$$

- $\mathcal{K}_2 \sim \tan \delta/q$ is the K-matrix, which is diagonal in l, m space
- $F_{\mathbf{P}\mathbf{V}}$ is a known kinematical zeta-function, depending on the box shape & E ; It is an off-diagonal matrix in l, m , since the box violates rotation symmetry

Single-channel 2-particle quantization condition

$$\det \left[(F_{\tilde{P}\tilde{V}})^{-1} + \mathcal{K}_2 \right] = 0$$

- Infinite dimensional determinant must be truncated to be practical; truncate by assuming that \mathcal{K}_2 vanishes above l_{max}
- If $l_{max}=0$, obtain one-to-one relation between energy levels and \mathcal{K}_2

$$\mathcal{K}_{2,s}(E_n^*) = - \frac{1}{F_{\tilde{P}\tilde{V};00;00}(E_n, \vec{P}, L)}$$

determines scattering amplitude at
CM energy $E_n^* = \sqrt{E_n^2 - \vec{P}^2}$

“measured” energy-level

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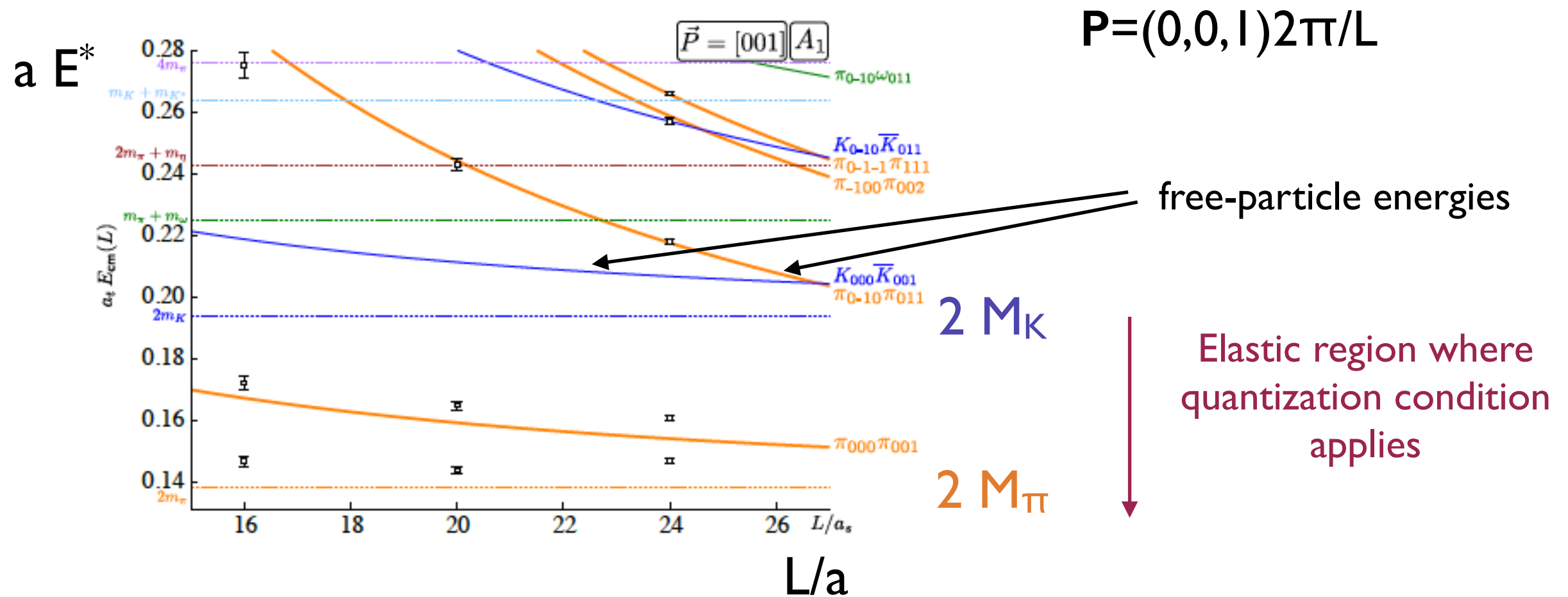
$$\mathcal{K}_{2,s}(E_n^*) = - \frac{1}{F_{\vec{P}\vec{V};00;00}(E_n, \vec{P}, L)}$$

Equivalent to: $\tan[\delta(q^*)] = -\tan[\phi^P(q^*)],$

Application to ρ meson

[Dudek, Edwards & Thomas, 1212.0830]

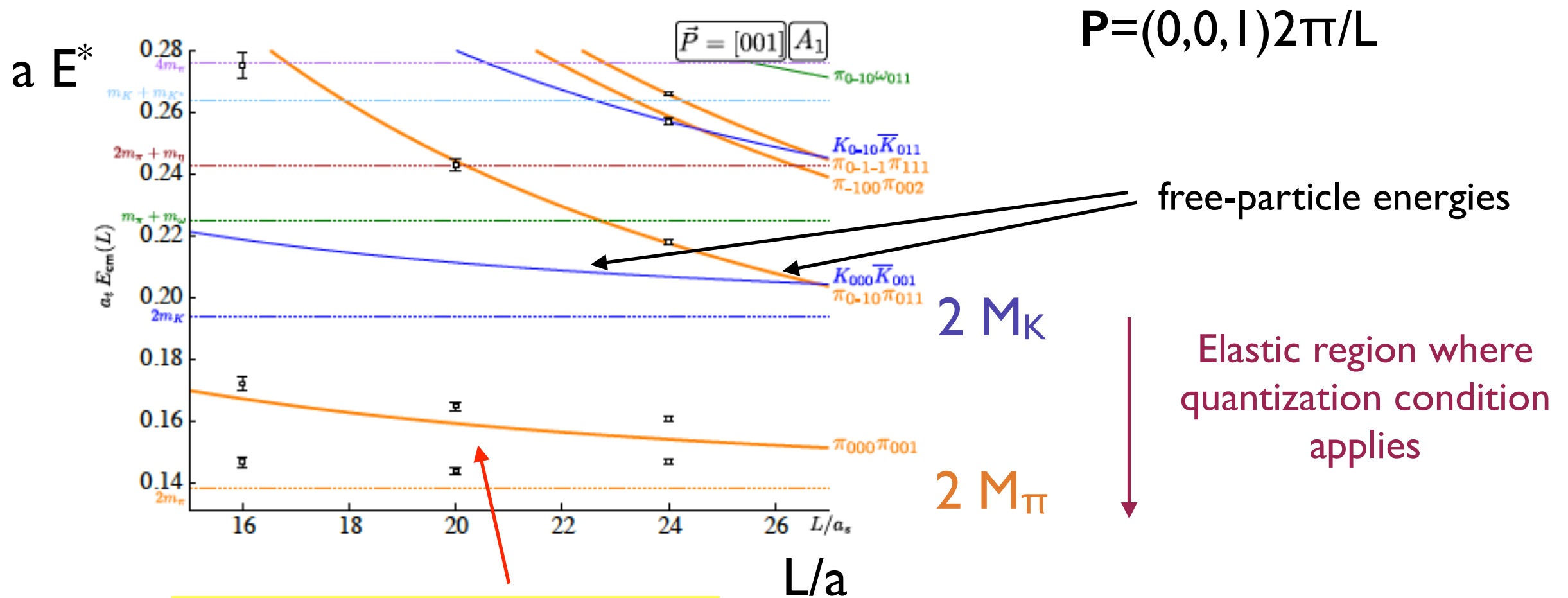
- Proof of principle calculation with $M_\pi \sim 400$ MeV, several \mathbf{P} , many spectral levels



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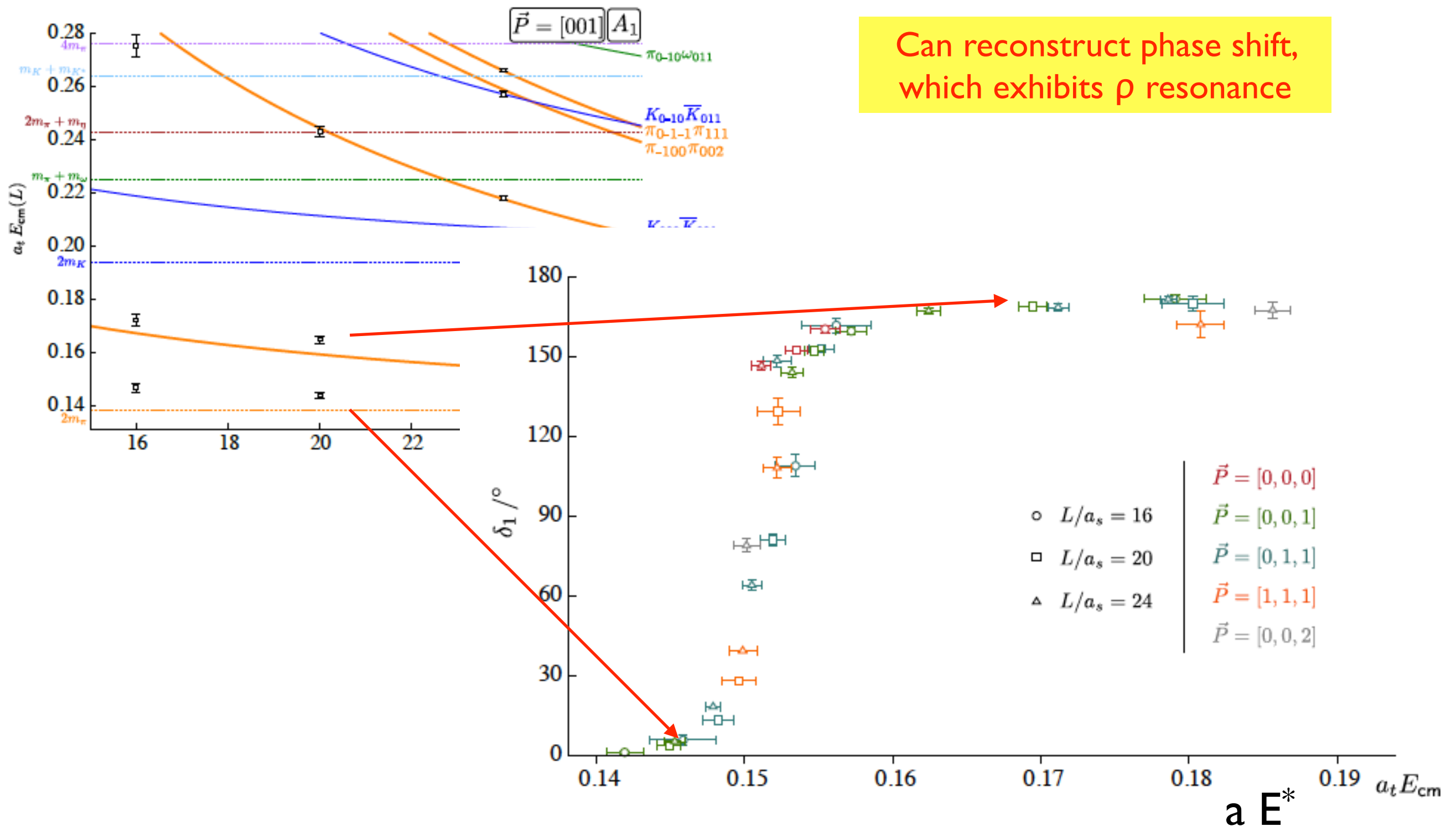
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KEY POINT: there are “extra” levels here, and neither are close to the free levels

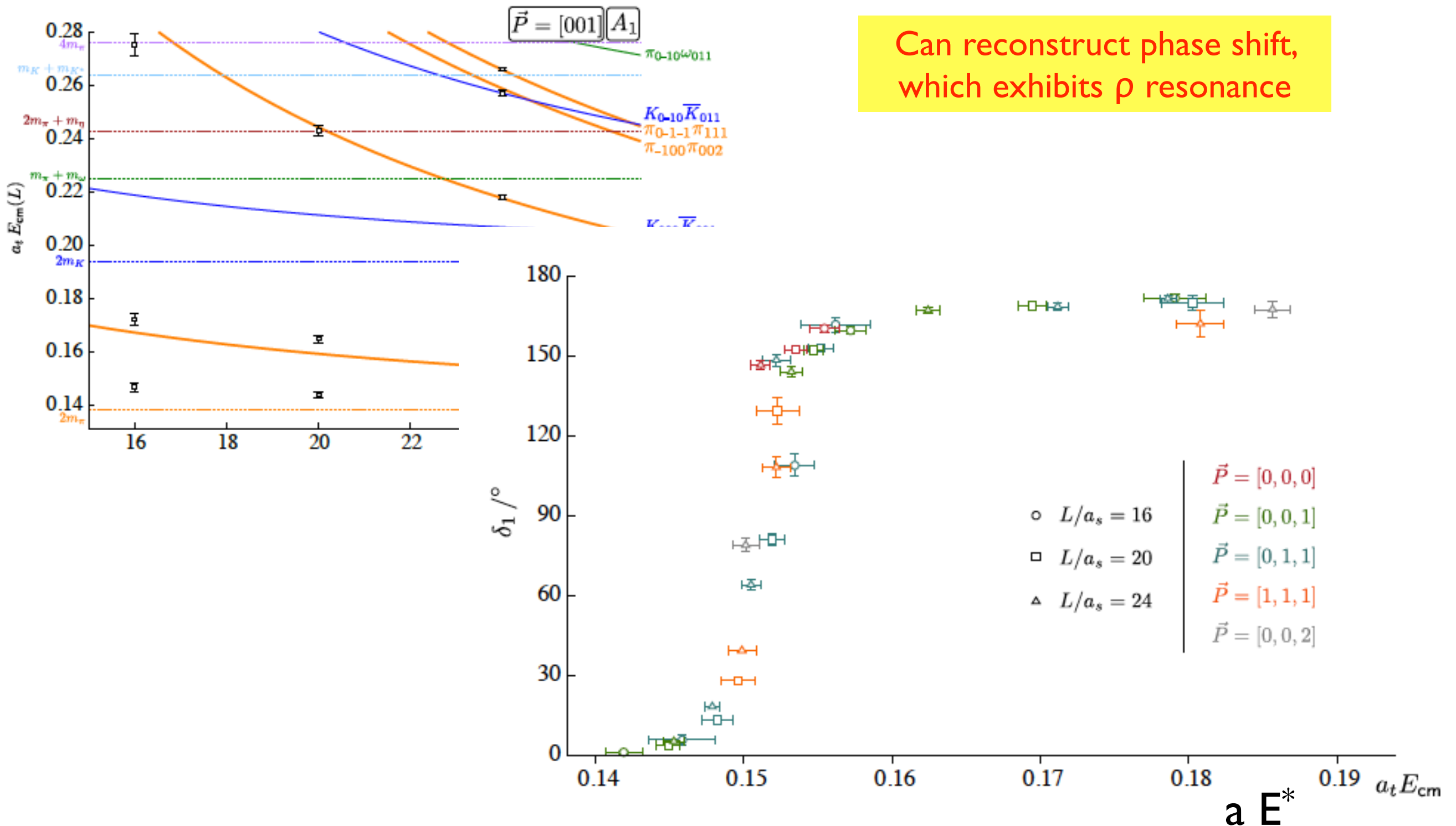
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Application to ρ meson

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Can reconstruct phase shift, which exhibits ρ resonance

Status for 2 particles

1st
issue

- Long understood in NRQM [Huang & Yang 57, ...]
- Quantization formula in QFT for energies below inelastic threshold converted into NRQM problem and solved by [Lüscher 86 & 91]
- Solution generalized to arbitrary total momentum \mathbf{P} , multiple (2 body) channels, general BCs and arbitrary spins [Rummukainen & Gottlieb 85; Kim, Sachrajda & SS 05; Bernard, Lage, Meißner & Rusetsky 08; Hansen & SS 12; Briceño & Davoudi 12; ...]

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2nd
issue

- Relation between finite volume $1 \rightarrow 2$ weak amplitude (e.g. $K \rightarrow \pi\pi$) and infinite volume decay amplitude determined [Lellouch & Lüscher 00]
- LL formula generalized to general \mathbf{P} , to multiple (2 body) channels, to arbitrary currents, general BCs & arbitrary spin (e.g. $\gamma^* \pi \rightarrow \rho \rightarrow \pi\pi$, $\gamma^* N \rightarrow \Delta \rightarrow \pi N$, $\gamma D \rightarrow NN$) [Kim, Sachrajda & SS 05; Christ, Kim & Yamazaki 05; Meyer 12; Hansen & SS 12; Briceño & Davoudi 12; Agadjanov, Bernard, Meißner & Rusetsky 14; Briceño, Hansen & Walker-Loud 14; Briceño & Hansen 15;...]

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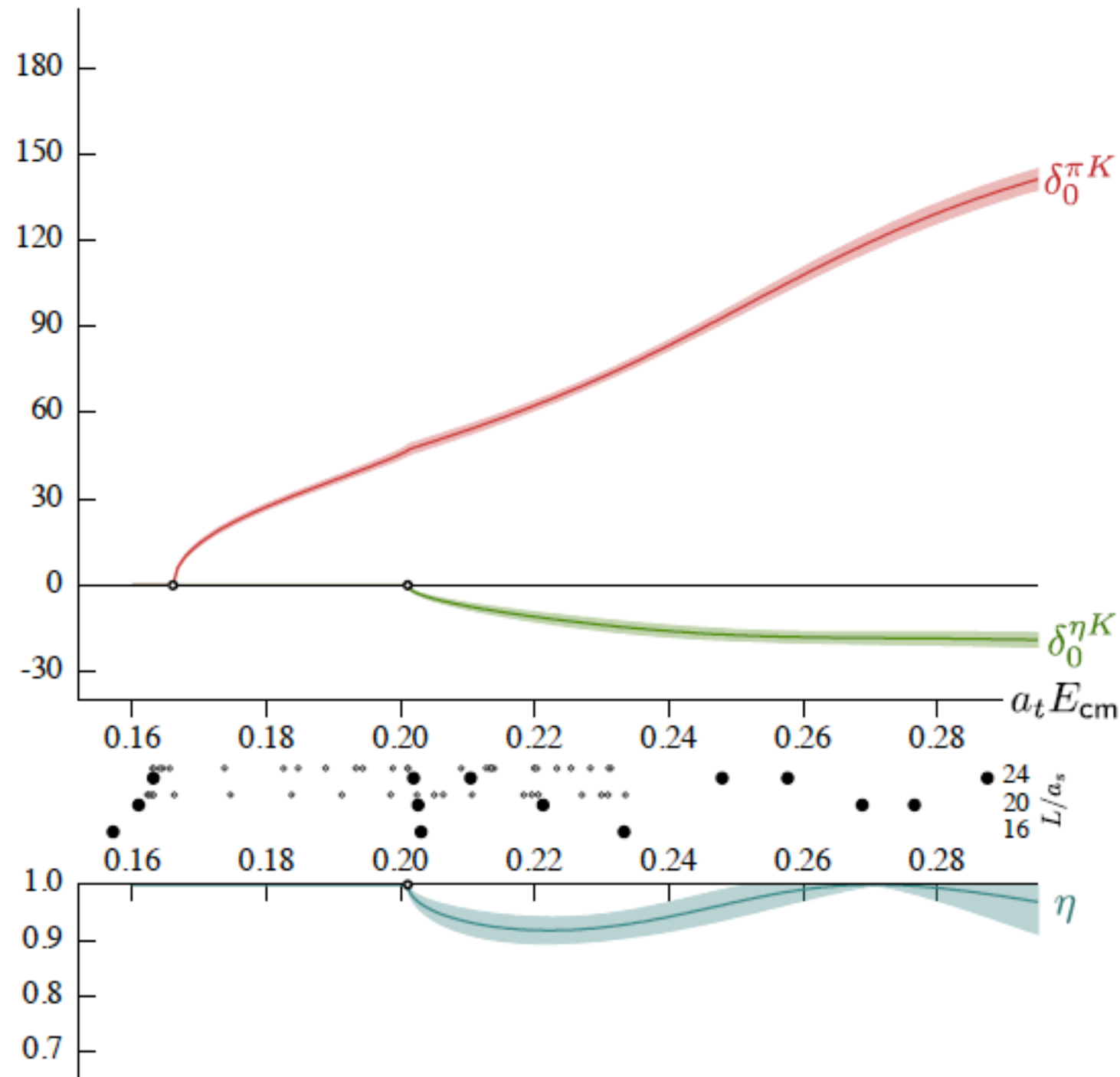
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- Leading order QED effects on quantization condition determined; do NOT fit into general formalism [Beane & Savage 14]

State of the art: multiple 2-particle channels

S-WAVE $\pi K / \eta K$ SCATTERING

[Dudek, Edwards,
Thomas & Wilson 14]

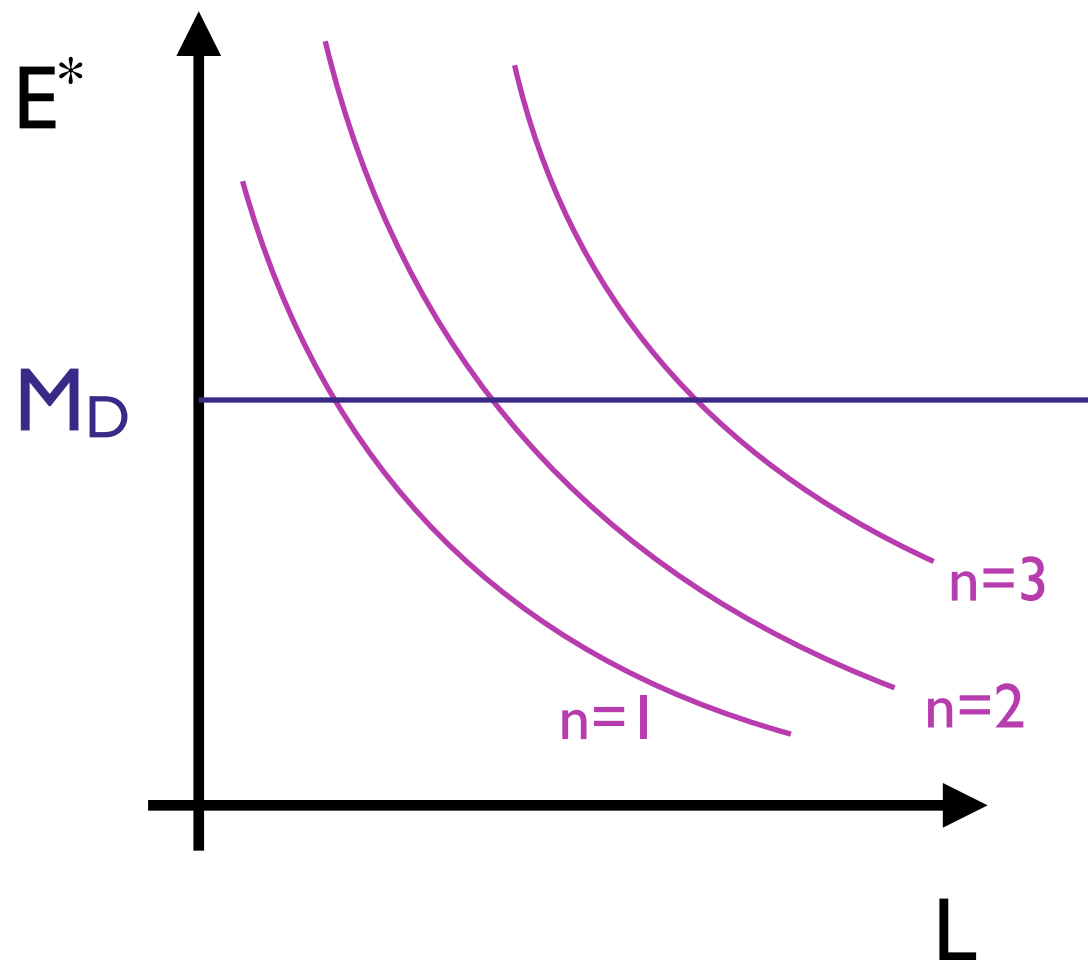


Coupled two-body
channels

$$m_\pi \sim 391 \text{ MeV}$$

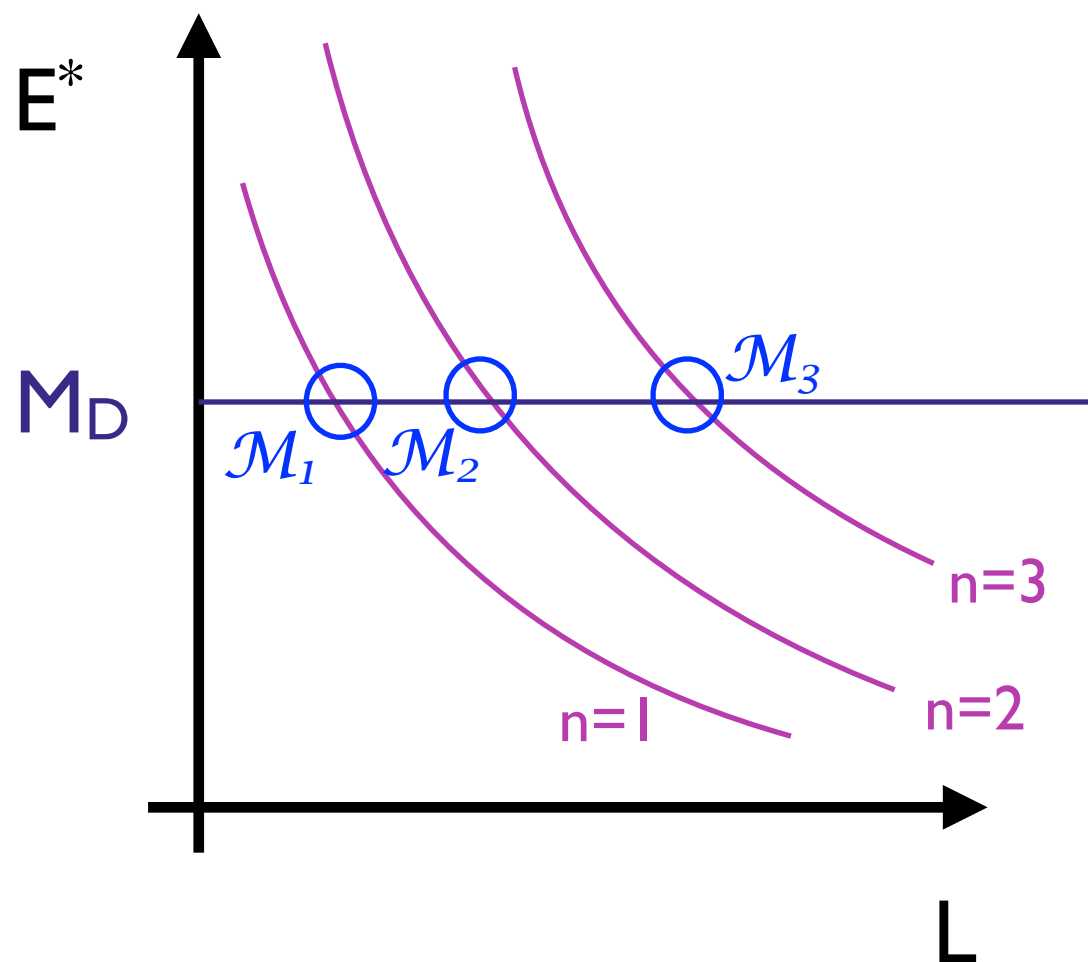
2nd issue: decays to multiple 2-particle channels

- Illustrate with an unphysical scenario:
 - Consider $D \rightarrow$ strange weak decays, and assume $D \rightarrow K\pi$ & $K\eta$ only
- Calculate amplitudes $\mathcal{M}_{n=L} \langle D | H_W | n \rangle_L$ for three finite-volume states with $E_n^* = M_D$
- Calculate values of L_n and slopes S_n of same three spectral lines at $E_n^* = M_D$



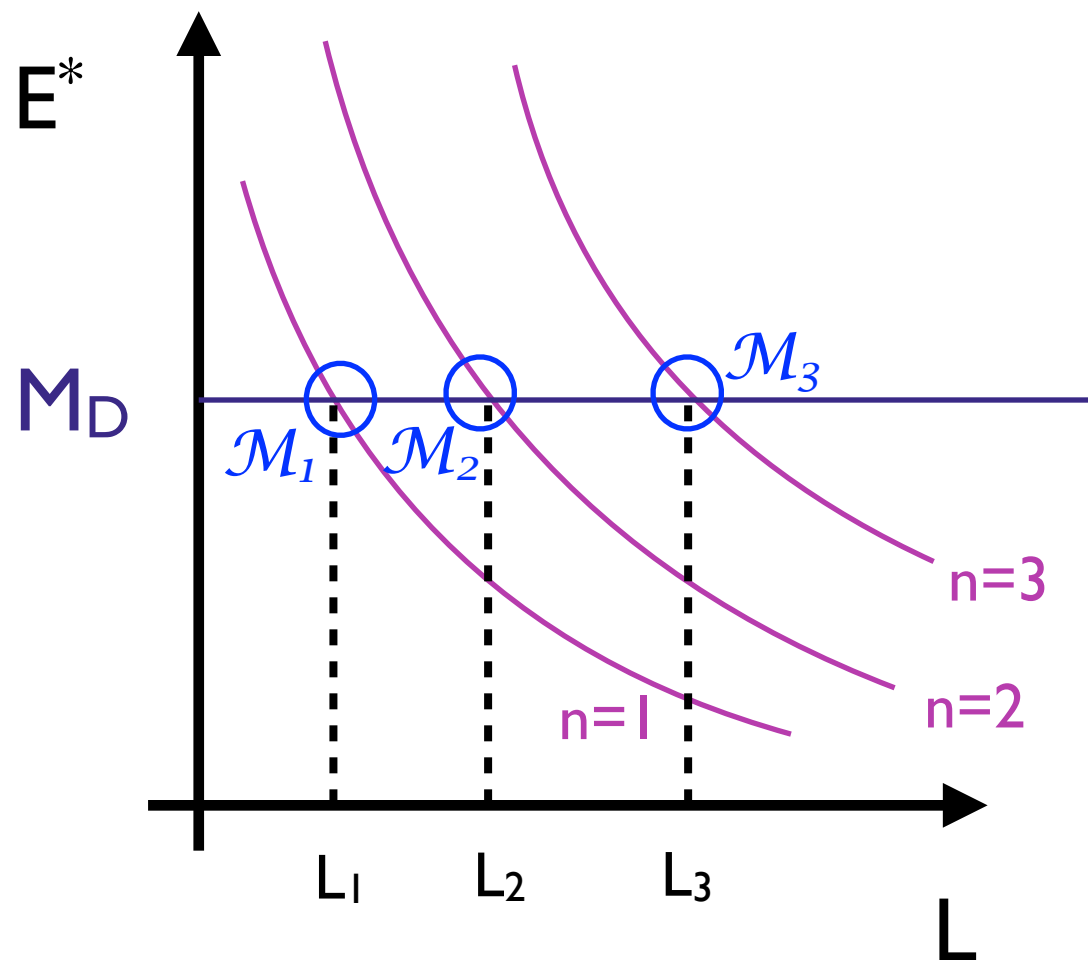
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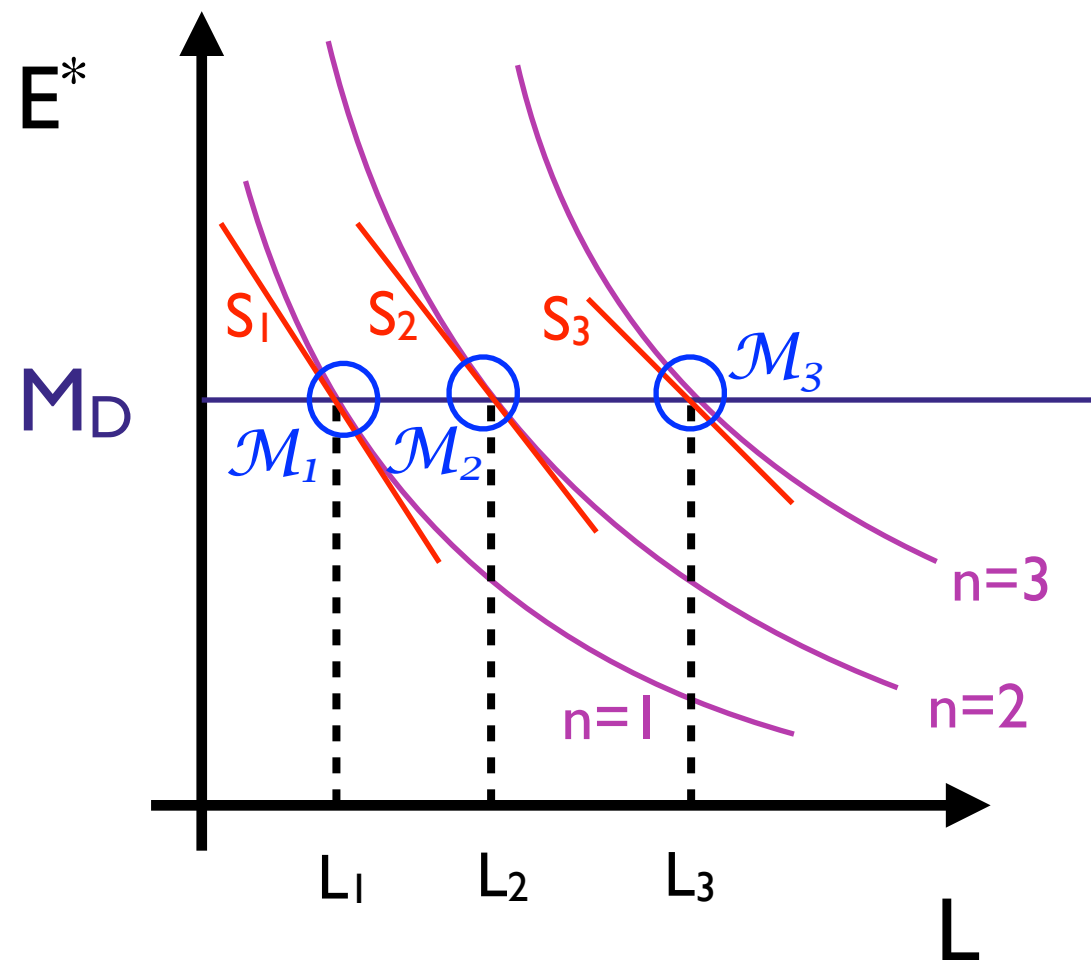
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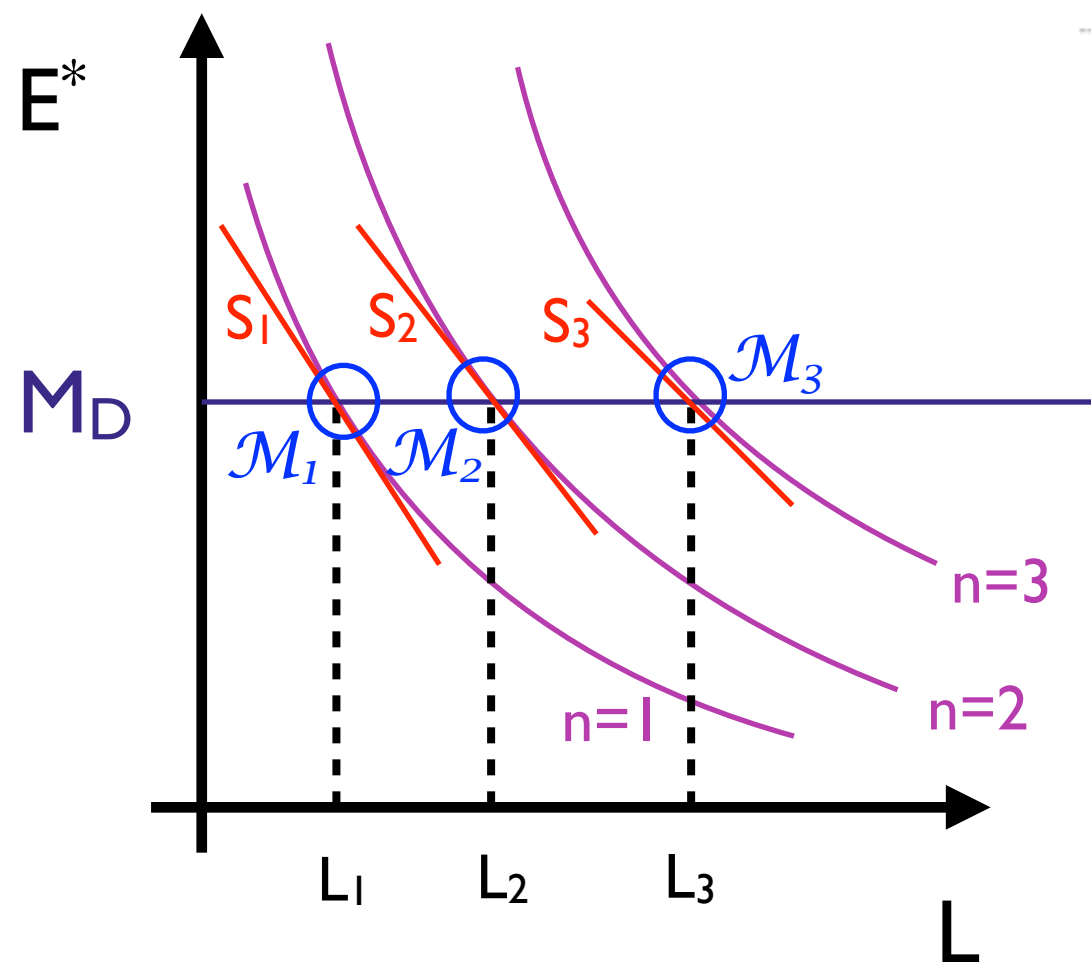
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Combining this information can determine infinite-volume matrix elements (including phases)!
[Hansen & SS]

$$\mathcal{A}_{D \rightarrow K\pi} \quad \text{and} \quad \mathcal{A}_{D \rightarrow K\eta}$$

Three-particle quantization condition

Applications of 3-particle formalism

- Studying resonances with three particle decay channels

$$\omega(782) \rightarrow \pi\pi\pi \quad K^* \longrightarrow K\pi\pi \quad N(1440) \rightarrow N\pi\pi$$

- Calculating weak decay amplitudes/form factors involving 3 particles, e.g. $K \rightarrow \pi\pi\pi$
- Determining NNN interactions
 - Input for effective field theory treatments of larger nuclei & nuclear matter
 - Similarly, $\pi\pi\pi$, $\pi K\bar{K}$, ... interactions needed for study of pion/kaon condensation

Status for 3 particles

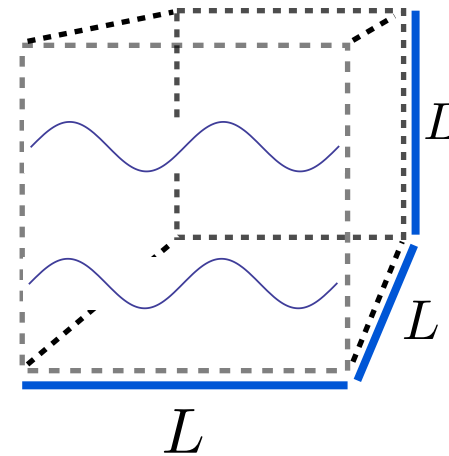
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- [Briceño & Davoudi 12] used a dimer approach in NREFT, with s-wave interactions only, to determine relation between spectrum and a finite volume quantity, itself related to infinite-volume amplitudes by an integral equation
- [Hansen & SS 14, 15] derived quantization condition in (fairly) general, relativistic QFT relating spectrum and \mathcal{M}_2 and 3-body scattering quantity $K_{df,3}$; relation between $K_{df,3}$ & \mathcal{M}_3 via integral equations now known
- [Meißner, Rios & Rusetsky 14] determined volume dependence of 3-body bound state in unitary limit
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Theory considered for 3 particles

- Work in continuum (assume that LQCD can control discretization errors)



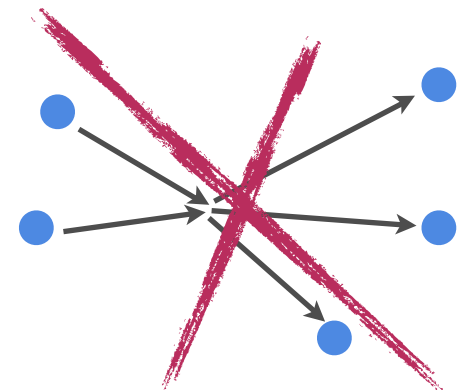
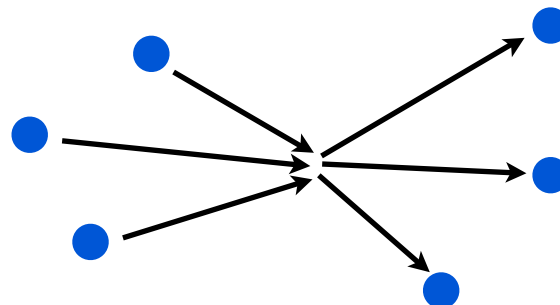
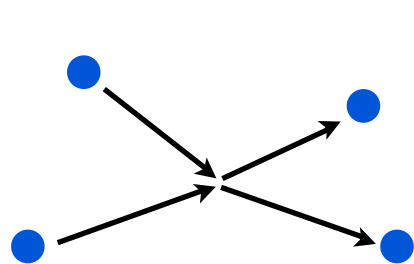
- Cubic box of size L with periodic BC, and infinite (Minkowski) time

- Spatial loops are sums: $\frac{1}{L^3} \sum_{\vec{k}}$ $\vec{k} = \frac{2\pi}{L} \vec{n}$

- Consider identical particles with physical mass m, interacting arbitrarily except for a Z_2 (G-parity-like) symmetry

- Only vertices are $2 \rightarrow 2$, $2 \rightarrow 4$, $3 \rightarrow 3$, $3 \rightarrow 1$, $3 \rightarrow 5$, $5 \rightarrow 7$, etc.

- Even & odd particle-number sectors decouple



3-particle quantization condition [Hansen & SS]

- Spectrum is determined (for given L, \mathbf{P}) by solutions of

$$\det [F_3^{-1} + \mathcal{K}_{\text{df},3}] = 0$$

$$F_3 = \frac{F_{\widetilde{\text{PV}}}}{2\omega L^3} \left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1} \mathcal{K}_2 F_{\widetilde{\text{PV}}}} \right]$$

$$G_{p,\ell',m';k,\ell,m} \equiv \left(\frac{k^*}{q_p^*} \right)^{\ell'} \frac{4\pi Y_{\ell',m'}(\hat{k}^*) H(\vec{p}) H(\vec{k}) Y_{\ell,m}^*(\hat{p}^*)}{2\omega_{kp}(E - \omega_k - \omega_p - \omega_{kp})} \left(\frac{p^*}{q_k^*} \right)^\ell \frac{1}{2\omega_k L^3}$$

- Superficially similar to 2-particle form ...

$$\det [(F_{\widetilde{\text{PV}}})^{-1} + \mathcal{K}_2] = 0$$

- ... but F_3 contains both kinematical, finite-volume quantities (F_{PV} & G) and the dynamical, infinite-volume quantity \mathcal{K}_2

3-particle quantization condition [Hansen & SS]

- Spectrum is determined (for given L, \mathbf{P}) by solutions of

$$\det [F_3^{-1} + \mathcal{K}_{\text{df},3}] = 0$$

Infinite volume
real 3-particle
scattering
quantity

Matrices in the
space describing
3-particle on-shell
kinematics

$$F_3 = \frac{F_{\widetilde{\text{PV}}}}{2\omega L^3} \left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1} \mathcal{K}_2 F_{\widetilde{\text{PV}}}} \right]$$

Known
kinematical
quantity:
essentially
the same
as F_{PV} in
2-particle
analysis

$$G_{p,\ell',m';k,\ell,m} \equiv \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell',m'}(\hat{k}^*) H(\vec{p}) H(\vec{k}) Y_{\ell,m}^*(\hat{p}^*)}{2\omega_{kp}(E - \omega_k - \omega_p - \omega_{kp})} \left(\frac{p^*}{q_k^*}\right)^\ell \frac{1}{2\omega_k L^3}$$

G is known
kinematical
quantity
containing
cut-off
function H

- Superficially similar to 2-particle form ...

$$\det [(F_{\widetilde{\text{PV}}})^{-1} + \mathcal{K}_2] = 0$$

- ... but F_3 contains both kinematical, finite-volume quantities (F_{PV} & G) and the dynamical, infinite-volume quantity \mathcal{K}_2

Final result for 3 particles

$$\det [F_3^{-1} + \mathcal{K}_{\text{df},3}] = 0$$

$$F_3 = \frac{F_{\text{PV}}}{2\omega L^3} \left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1} \mathcal{K}_2 F_{\text{PV}}} \right]$$

- All quantities are (infinite-dimensional) matrices, e.g. $(F_3)_{klm;pl'm'}$, with indices

[finite volume “spectator” momentum: $\mathbf{k}=2\pi\mathbf{n}/L$] \times [2-particle CM angular momentum: l,m]



Three on-shell particles with total energy-momentum (E, \mathbf{P})

- For large \mathbf{k} other two particles are below threshold; must include such configurations by analytic continuation up to a cut-off at $k \sim m$ [provided by $H(\mathbf{k})$]

Final result for 3 particles

$$\det [F_3^{-1} + \mathcal{K}_{\text{df},3}] = 0$$

$$F_3 = \frac{F_{\text{PV}}}{2\omega L^3} \left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1} \mathcal{K}_2 F_{\text{PV}}} \right]$$

- Important limitation: our present derivation requires that all two-particle sub-channels lie below resonance poles at the spectral energy under consideration
- Resonances imply that \mathcal{K}_2 has a pole, and this leads to additional finite volume dependence not accounted for in the derivation
- We only have an ugly solution—searching for something better

Truncation in 3 particle case

$$\det [F_3^{-1} + \mathcal{K}_{\text{df},3}] = 0$$

$$F_3 = \frac{F_{\widetilde{\text{PV}}}}{2\omega L^3} \left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1} \mathcal{K}_2 F_{\widetilde{\text{PV}}}} \right]$$

- For fixed E & \mathbf{P} , as spectator momentum $|\mathbf{k}|$ increases, remaining two-particle system drops below threshold, so F_{PV} becomes exponentially suppressed
 - Smoothly interpolates to $F_{\text{PV}}=0$ due to H factors; same holds for G
- Thus \mathbf{k} sum is naturally truncated (with, say, N terms required)
- l is truncated if both \mathcal{K}_2 and $\mathcal{K}_{\text{df},3}$ vanish for $l > l_{\text{max}}$
- Yields determinant condition truncated to $[N(2l_{\text{max}}+1)]^2$ block

Truncation in 3 particle case

$$\Delta_{L,P}(E) = \det [F_3^{-1} + \mathcal{K}_{df,3}] = 0$$

$$F_3 = \frac{F_{\widetilde{PV}}}{2\omega L^3} \left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1} \mathcal{K}_2 F_{\widetilde{PV}}} \right]$$

- Given prior knowledge of \mathcal{K}_2 (e.g. from 2-particle quantization condition) each energy level E_i of the 3 particle system gives information on $\mathcal{K}_{df,3}$ at the corresponding 3-particle CM energy E_i^*
- Probably need to proceed by parameterizing $\mathcal{K}_{df,3 \rightarrow 3}$, in which case one would need at least as many levels as parameters at given energy
- Given \mathcal{K}_2 and $\mathcal{K}_{df,3}$ one can reconstruct \mathcal{M}_3
- The locality of $\mathcal{K}_{df,3}$ is crucial for this program
- Clearly very challenging in practice, but there is an existence proof....

Isotropic approximation

- Assume $\mathcal{K}_{df,3}$ depends only on E^* (and thus is indep. of \mathbf{k}, l, m)
- Also assume \mathcal{K}_2 only non-zero for s-wave ($\Rightarrow l_{\max}=0$) and known
- Truncated $[N \times N]$ problem simplifies: $\mathcal{K}_{df,3}$ has only 1 non-zero eigenvalue, and problem collapses to a single equation:

$$1 + F_3^{\text{iso}} \mathcal{K}_{df,3}^{\text{iso}}(E^*) = 0$$

Known in terms of
two particle scattering amplitude

$$F_3^{\text{iso}} \equiv \sum_{\vec{k}, \vec{p}} \frac{1}{2\omega_k L^3} \left[F_{\text{PV}}^s \left(-\frac{2}{3} + \frac{1}{1 + [1 + \mathcal{K}_2^s G^s]^{-1} \mathcal{K}_2^s F_{\text{PV}}^s} \right) \right]_{k,p}$$

Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

- Three particle quantization condition depends on $\mathcal{K}_{\text{df},3}$ rather than the three particle scattering amplitude \mathcal{M}_3
- $\mathcal{K}_{\text{df},3}$ is an infinite volume quantity (loops involve integrals) but is not physical
 - Has a very complicated, unwieldy definition
 - Depends on the cut-off function H
 - It was forced on us by the analysis, and is some sort of local vertex
- To complete the quantization condition we must relate $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

Relating $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

Involve only \mathcal{M}_2 and G
so "known"

$$i\mathcal{M}_{L,3\rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[\begin{array}{ccc} \mathcal{L}_L & i\mathcal{K}_{\text{df},3\rightarrow 3} & \frac{1}{1 - iF_3} \\ & & i\mathcal{K}_{\text{df},3\rightarrow 3} \end{array} \mathcal{R}_L \right]$$

$$i\mathcal{M}_{3\rightarrow 3} = \lim_{L \rightarrow \infty} \left. \begin{array}{c} i\mathcal{M}_{L,3\rightarrow 3} \\ i\epsilon \end{array} \right|$$

Sums over k go over
to integrals with $i\epsilon$ pole prescription

- Result is an integral equation giving \mathcal{M}_3 in terms of $\mathcal{K}_{\text{df},3}$
- Requires knowing \mathcal{M}_2 (including continued below threshold)
- Completes formalism—shows that finite volume spectrum is given by infinite-volume scattering amplitudes

Summary & Outlook

Outlook: many challenges remain!

- Tremendous progress over last decade (theory & simulations)
 - Two-particle cases theoretically understood & simulations underway
 - Key examples: $K \rightarrow \pi\pi$ [RBC/UKQCD], $\gamma\pi \rightarrow \rho \rightarrow \pi\pi$ [JLAB]
- Frontier is 3 particles: need to fully develop 3 body formalism
 - Allow two particle sub-channels to be resonant
 - Extend to non-identical particles, particles with spin
 - Generalize LL factors to $1 \rightarrow 3$ decay amplitudes (e.g. for $K \rightarrow \pi\pi\pi$)
 - Include $1 \rightarrow 2, 2 \rightarrow 3, \dots$ vertices
- Develop models of amplitudes so that new results can be implemented in simulations
- Onwards to 4 or more particles?!?

Thank you!
Questions?

Backup slides: the details!

Set-up & main ideas

Methodology

- Calculate (for some $\mathbf{P}=2\pi\mathbf{n}_P/L$)

$$C_L(E, \vec{P}) \equiv \int_L d^4x e^{-i\vec{P}\cdot\vec{x}+iEt} \langle \Omega | T \sigma(x) \sigma^\dagger(0) | \Omega \rangle_L$$

CM energy is
 $E^* = \sqrt{E^2 - P^2}$

- Poles in C_L occur at energies of finite-volume spectrum
- For 2 & 3 particle states, $\sigma \sim \pi^2$ & π^3 , respectively
- E.g. for 2 particles:

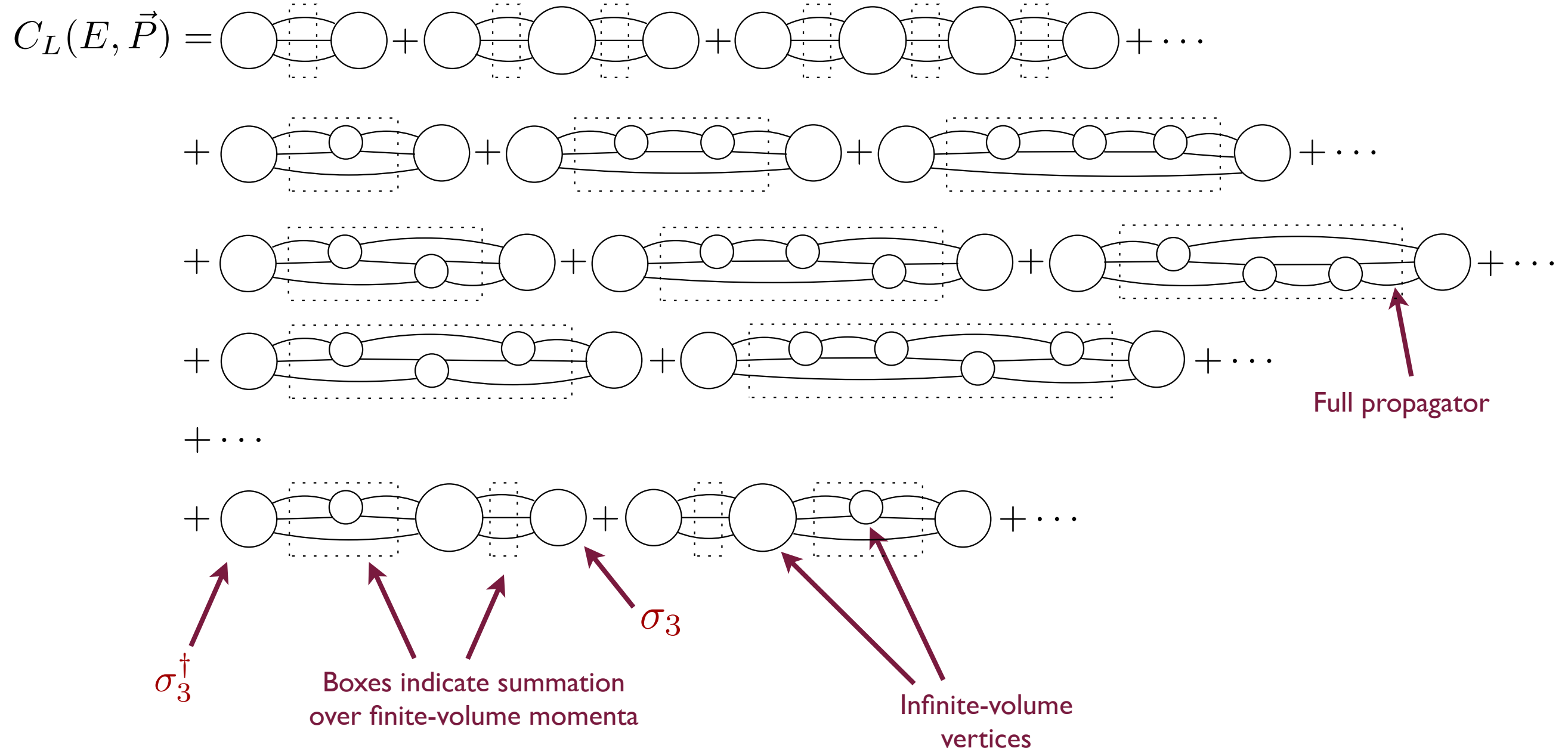
$$C_L(E, \vec{P}) = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} + \dots$$

Boxes indicated summation over finite-volume momenta

Infinite-volume vertices

Full propagators Normalized to unit residue at pole

3-particle correlator



Key step 1

- Replace loop sums with integrals where possible
 - Drop exponentially suppressed terms ($\sim e^{-ML}$, $e^{-(ML)^2}$, etc.) while keeping power-law dependence

$$\frac{1}{L^3} \sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l} \neq \vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l} \cdot \vec{k}} g(\vec{k})$$

Key step 1

- Replace loop sums with integrals where possible
 - Drop exponentially suppressed terms ($\sim e^{-ML}$, $e^{-(ML)^2}$, etc.) while keeping power-law dependence

$$\frac{1}{L^3} \sum_{\vec{k}} g(\vec{k}) = \int \frac{d^3k}{(2\pi)^3} g(\vec{k}) + \sum_{\vec{l} \neq \vec{0}} \int \frac{d^3k}{(2\pi)^3} e^{iL\vec{l} \cdot \vec{k}} g(\vec{k})$$

Exp. suppressed if $g(k)$ is smooth
and scale of derivatives of g is $\sim 1/M$

Key step 2

- Use “sum=integral + [sum-integral]” if integrand has pole, with [Kim,Sachrajda,SS 05]

$$\left(\int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(P - k)^2 - m^2 + i\epsilon} g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*'}} f^*(\hat{q}^*) \mathcal{F}(q^*, q^{*'}) g^*(\hat{q}^{*'}) + \text{exp. suppressed}$$

q^* is relative momentum
of pair on left in CM

Kinematic function

f & g evaluated for ON-SHELL momenta
Depend only on direction in CM

Key step 2

- Use “sum=integral + [sum-integral]” where integrand has pole, with [KSS]

$$\left(\int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(P - k)^2 - m^2 + i\epsilon} g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*'}} f^*(\hat{q}^*) \mathcal{F}(q^*, q^{*'}) g^*(\hat{q}^{*'})$$

- Decomposed into spherical harmonics, \mathcal{F} becomes

$$F_{\ell_1, m_1; \ell_2, m_2} \equiv \eta \left[\frac{\text{Re} q^*}{8\pi E^*} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} + \frac{i}{2\pi EL} \sum_{\ell, m} x^{-\ell} \mathcal{Z}_{\ell m}^P[1; x^2] \int d\Omega Y_{\ell_1, m_1}^* Y_{\ell, m}^* Y_{\ell_2, m_2} \right]$$

$x \equiv q^* L / (2\pi)$ and $\mathcal{Z}_{\ell m}^P$ is a generalization of the zeta-function

Kinematic functions

$Z_{4,0}$ & $Z_{6,0}$ for $P=0$ [Luu & Savage, '11]

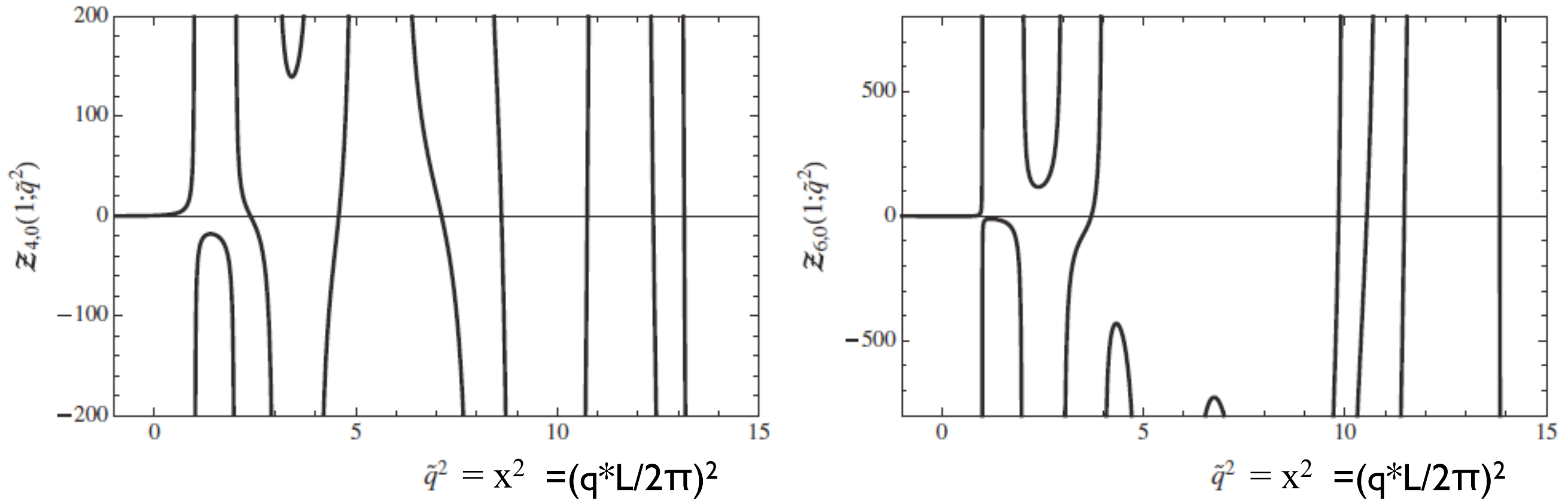


FIG. 29. The functions $Z_{4,0}(1; \tilde{q}^2)$ (left panel) and $Z_{6,0}(1; \tilde{q}^2)$ (right panel).

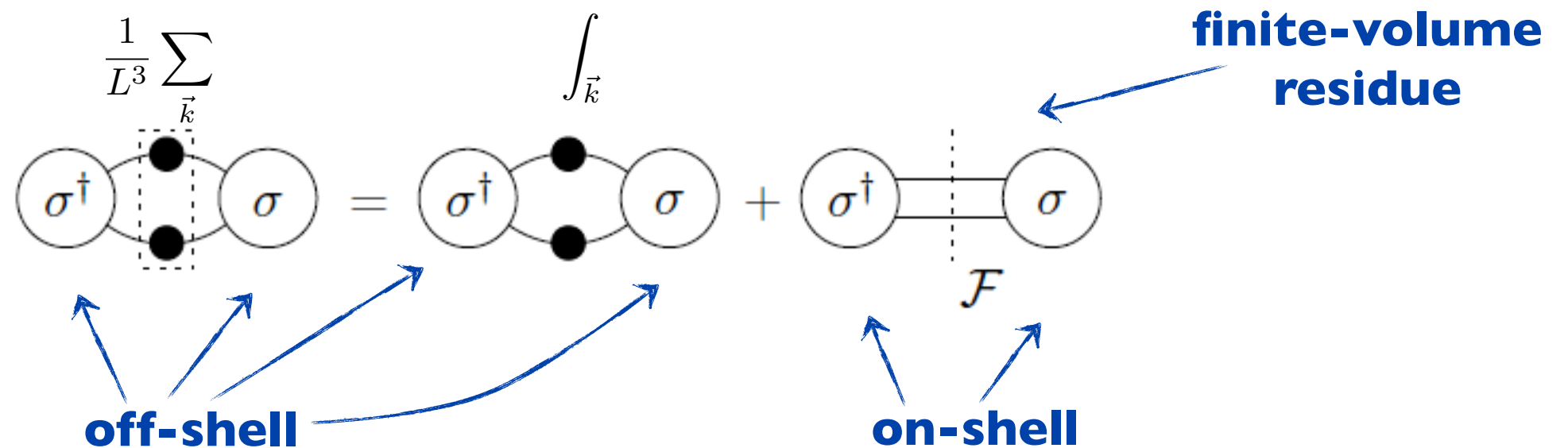
Key step 2

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$$\left(\int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} - \int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(P - k)^2 - m^2 + i\epsilon} g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*'}} f^*(\hat{q}^*) \mathcal{F}(q^*, q^{*'}) g^*(\hat{q}^{*'})$$

- Diagrammatically



Variant of key step 2

- For generalization to 3 particles use (modified) PV prescription instead of $i\epsilon$

$$\left(\int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\vec{k}} \overset{\widetilde{PV}}{-} \int \frac{d^4k}{(2\pi)^4} \right) f(k) \frac{1}{k^2 - m^2 + \cancel{i\epsilon}} \frac{1}{(P - k)^2 - m^2 + \cancel{i\epsilon}} g(k)$$

$$= \int d\Omega_{q^*} d\Omega_{q^{*'}} f^*(\hat{q}^*) \mathcal{F}_{\widetilde{PV}}(q^*, q^{*'}) g^*(\hat{q}^{*'})$$

- Key properties of \mathbf{F}_{PV} (discussed below): real and no unitary cusp at threshold

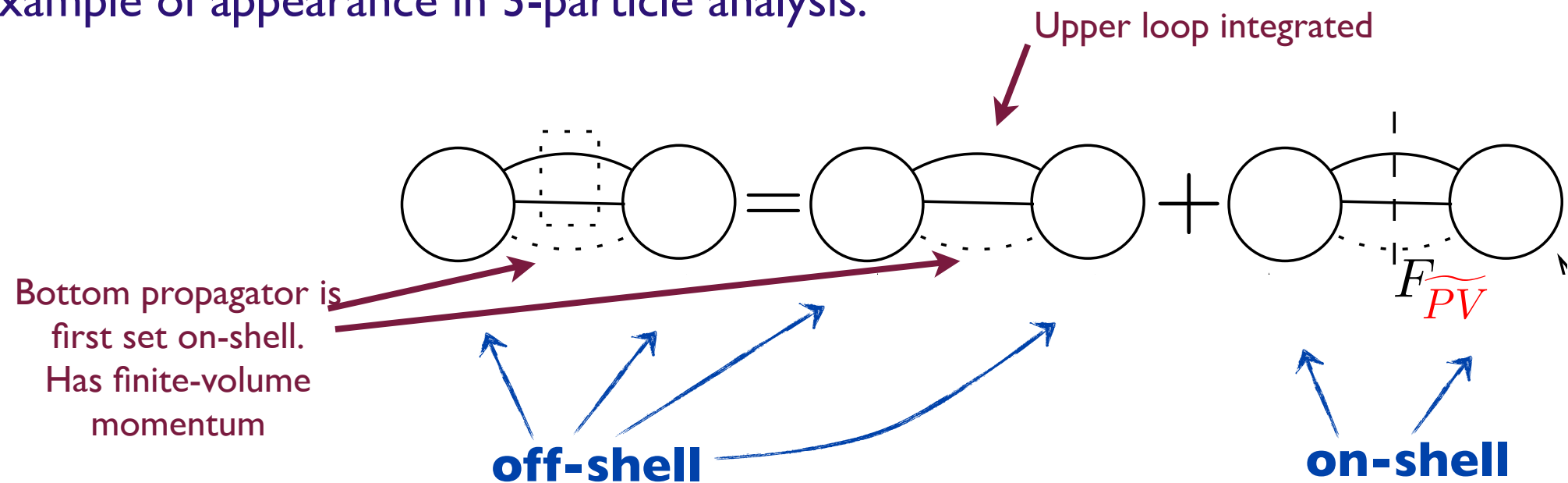
Variant of key step 2

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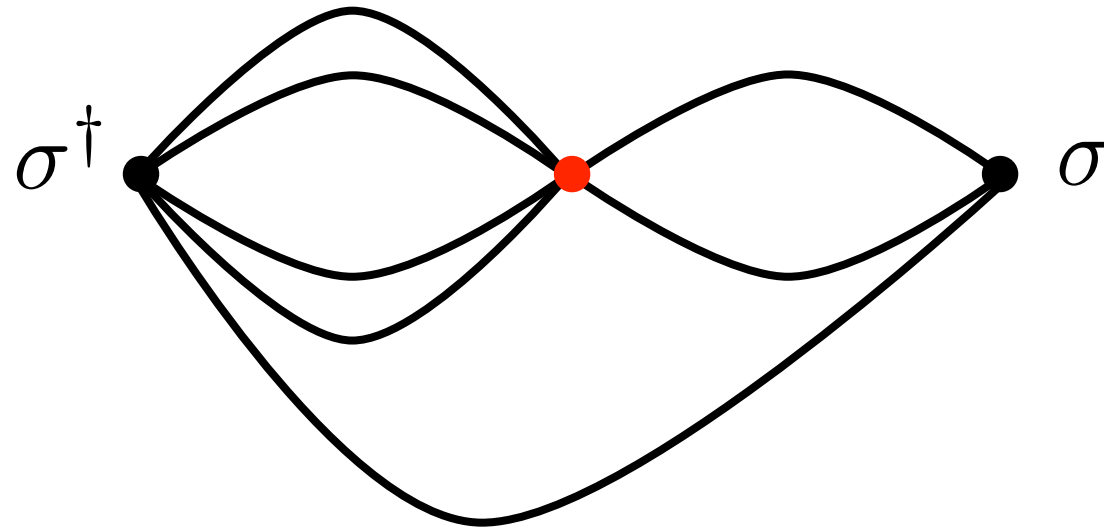
$$= \int d\Omega_{q^*} d\Omega_{q^{*'}} f^*(\hat{q}^*) \mathcal{F}_{\widetilde{PV}}(q^*, q^{*'}) g^*(\hat{q}^{*'})$$

- Key properties of F_{PV} (discussed below): real and no unitary cusp at threshold
- Example of appearance in 3-particle analysis:



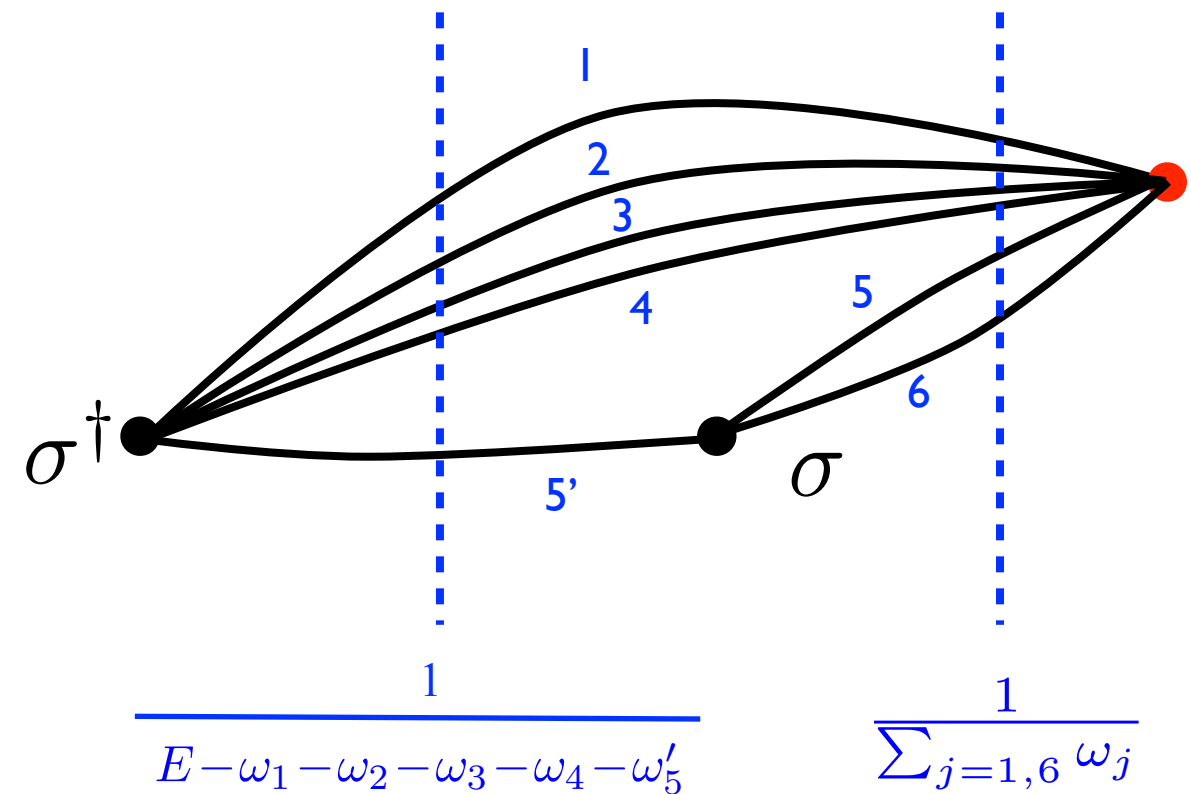
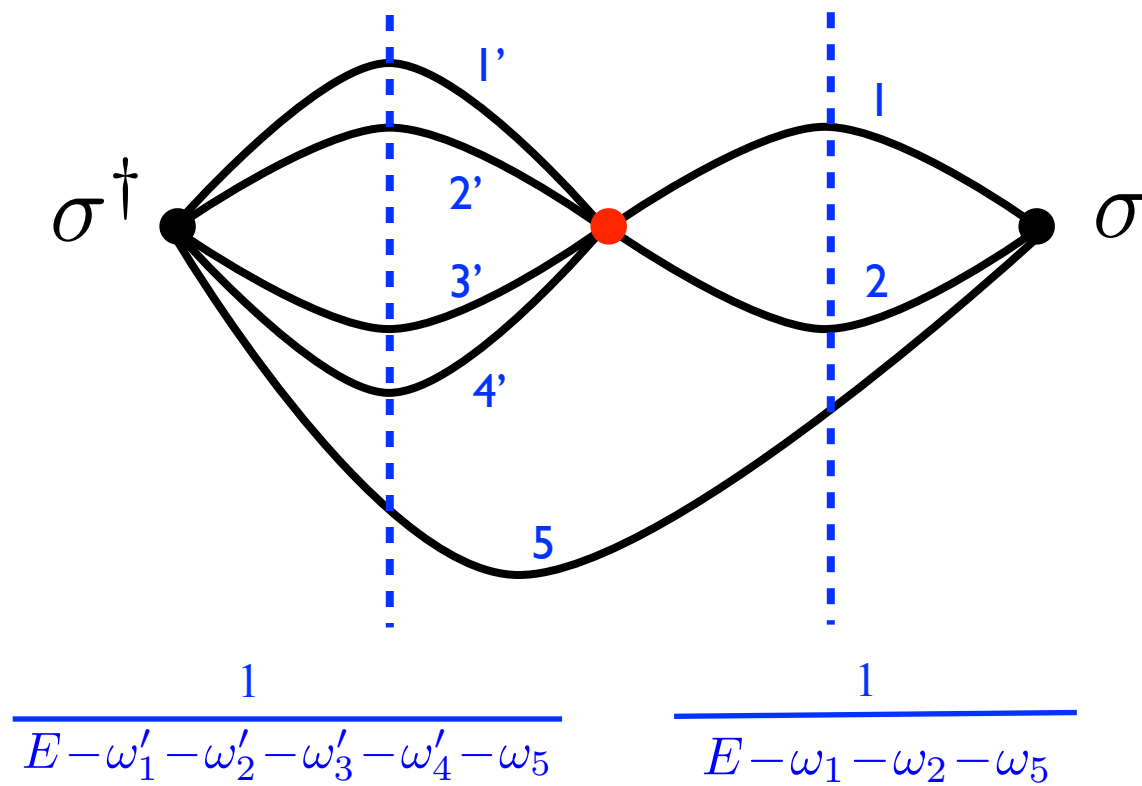
Key step 3

- Identify potential singularities: can use time-ordered PT (i.e. do k_0 integrals)
- Example



Key step 3

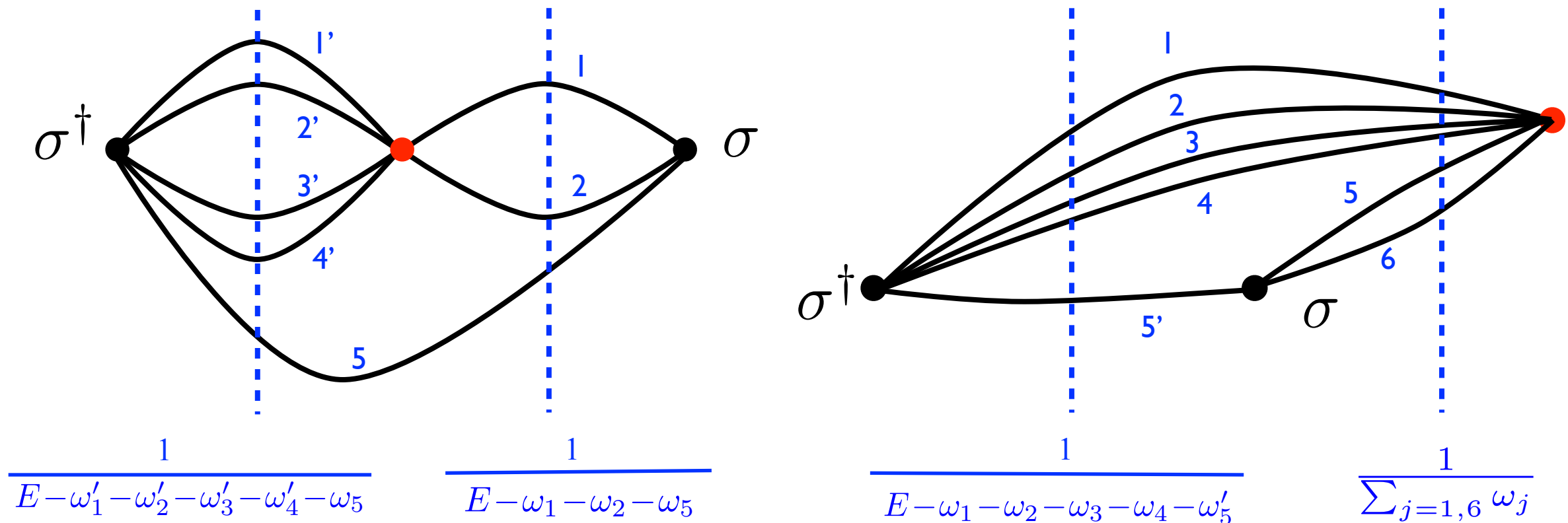
- 2 out of 6 time orderings:



On-shell energy $\omega_j = \sqrt{\vec{k}_j^2 + M^2}$

Key step 3

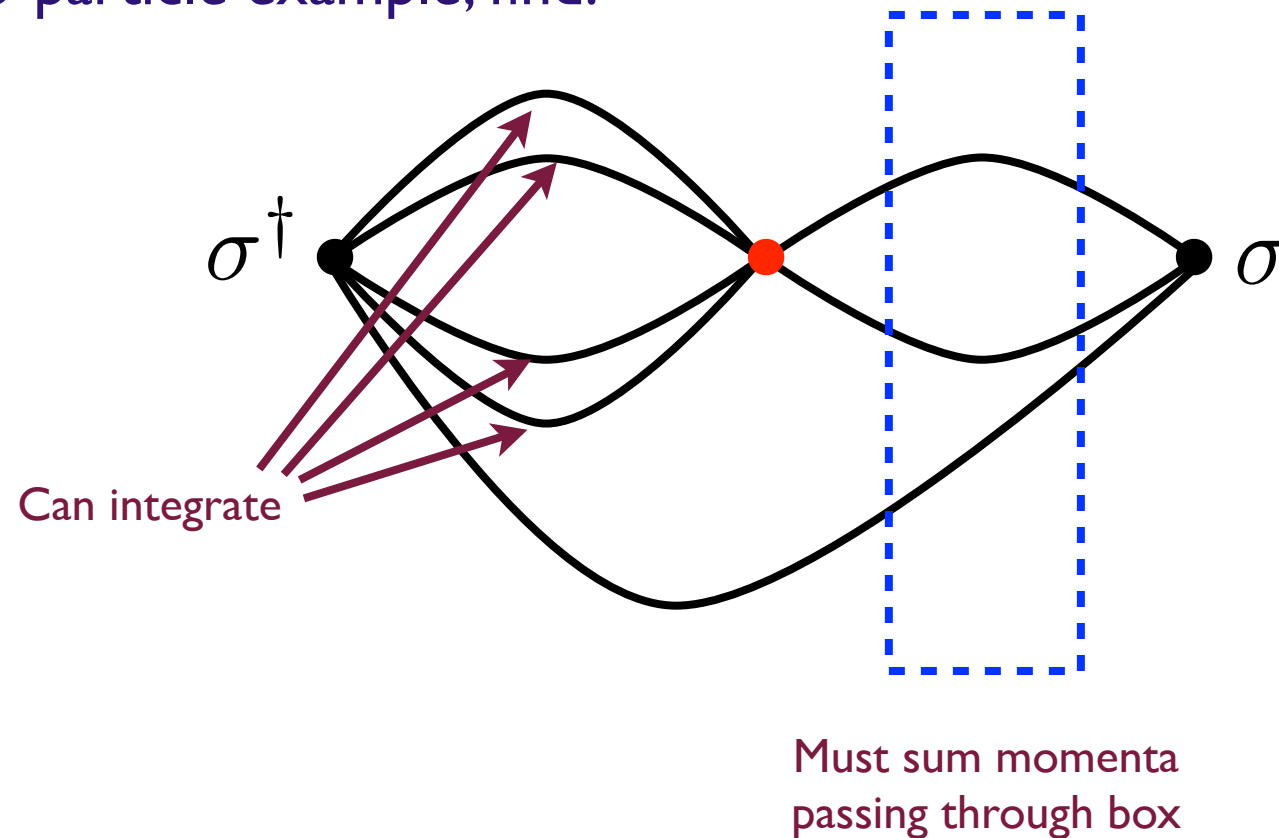
- 2 out of 6 time orderings:



- If restrict $M < E^* < 5M$ then only 3-particle “cuts” have singularities, and these occur only when all three particles to go on-shell

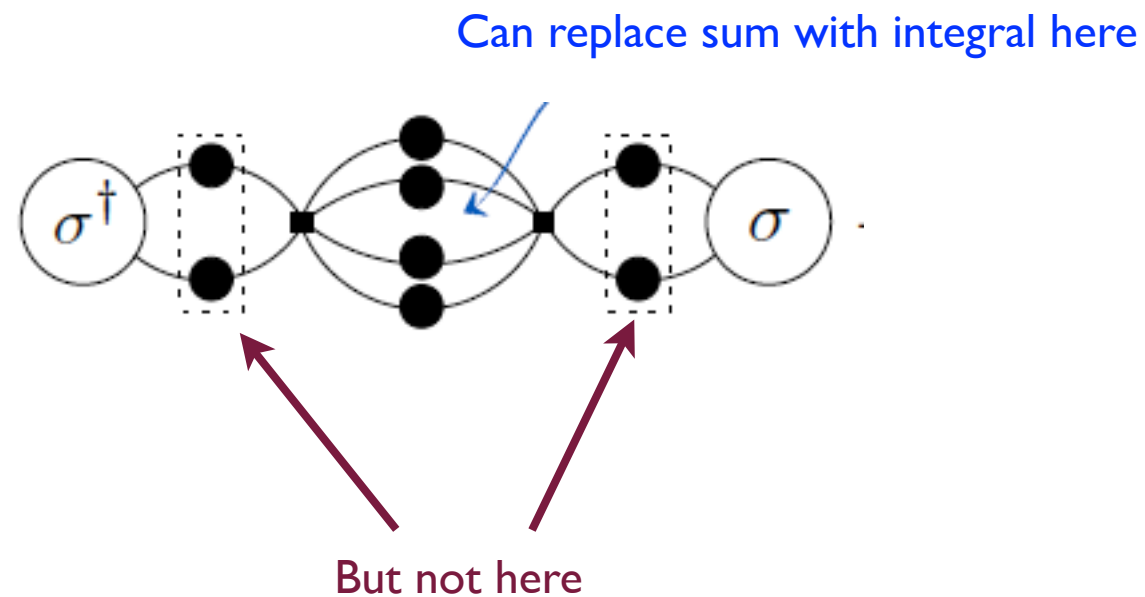
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- In our 3-particle example, find:



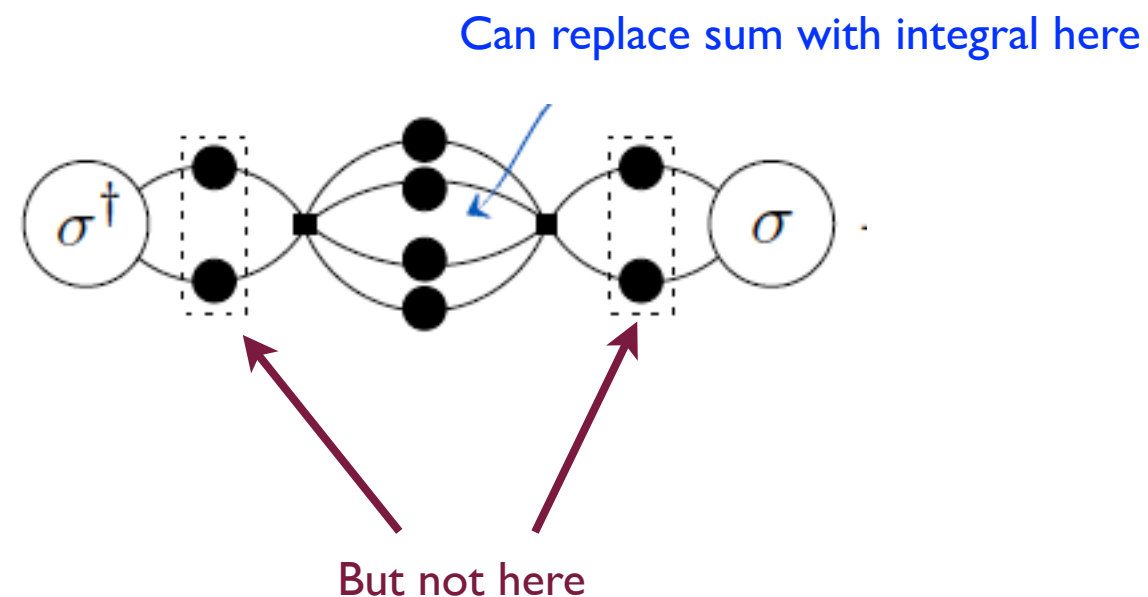
Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- In our 2-particle example, find:



Combining key steps 1-3

- For each diagram, determine which momenta must be summed, and which can be integrated
- In our 2-particle example, find:



- Then repeatedly use **sum=integral** + “sum-integral” to simplify

Key issues 4-6

- Dealing with cusps, avoiding divergences in 3-particle scattering amplitude, and dealing with breaking of particle interchange symmetry
- Discuss later!

2-particle quantization condition

Following method of [Kim, Sachrajda & SS 05]

- Apply previous analysis to 2-particle correlator ($0 < E^* < 4M$)

$$C_L(E, \vec{P}) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \dots$$

these loops are now integrated

- Collect terms into infinite-volume Bethe-Salpeter kernels

$$C_L(E, \vec{P}) = \text{diagram 1} + \text{diagram 2} + \dots$$

- Apply previous analysis to 2-particle correlator
- Collect terms into infinite-volume Bethe-Salpeter kernels

$$C_L(E, \vec{P}) = \text{diagram} + \text{diagram} \left\{ \text{diagram} + \text{diagram} + \text{diagram} + \dots \right\} \text{diagram} + \dots$$

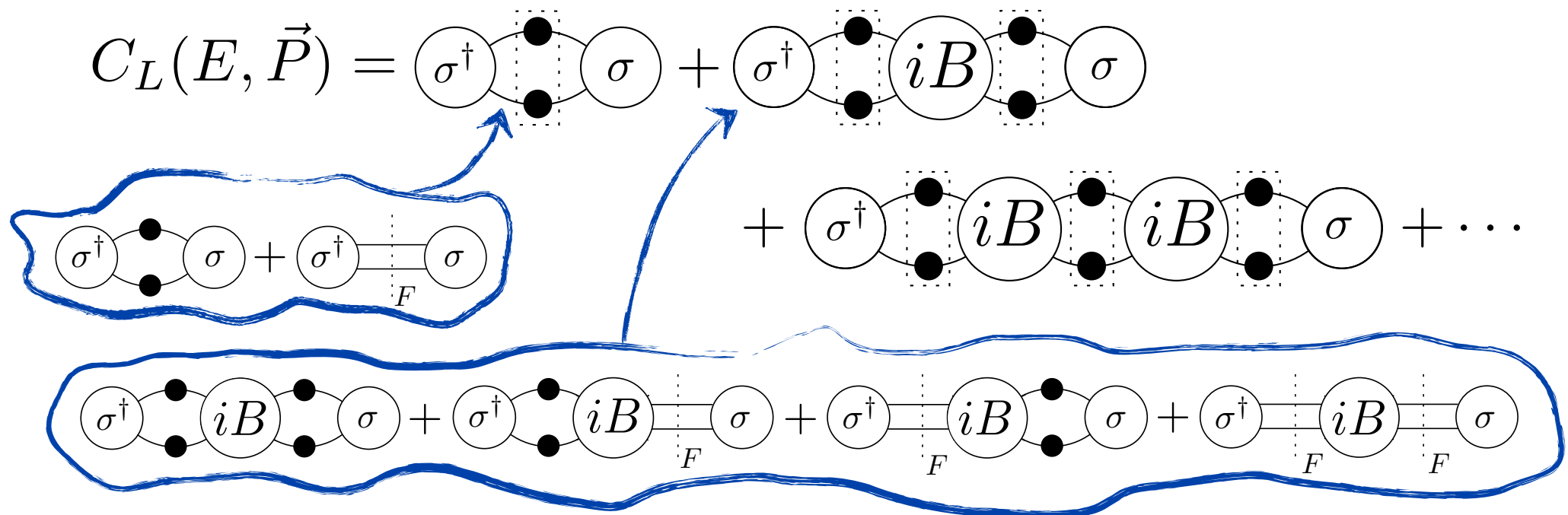
The diagram shows the expansion of the 2-particle correlator $C_L(E, \vec{P})$. The first term is a circle labeled σ^\dagger on the left and σ on the right, with two black dots between them. A dashed box encloses these two dots. This is followed by a plus sign and another similar diagram. A blue bracket groups a series of diagrams: a diagram with two dots, a diagram with four dots and two arcs, and a diagram with two dots and two arcs. A blue arrow points from a cloud labeled iB to the first diagram in this group. The series ends with an ellipsis. This group is followed by a diagram with two dots and a circle labeled σ , and another ellipsis.

- Leading to

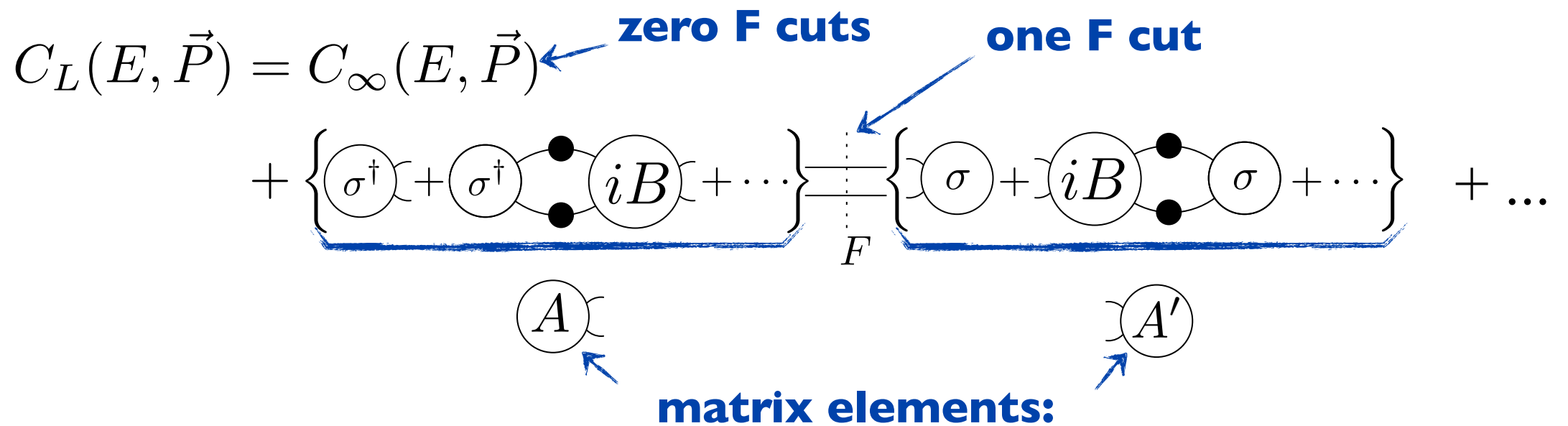
$$C_L(E, \vec{P}) = \text{diagram} + \text{diagram} + \text{diagram} + \dots$$

The diagram shows the resummed series for the 2-particle correlator. It starts with the first diagram from the previous equation, followed by a plus sign and a diagram where the two dots are enclosed in a circle labeled iB . This is followed by another plus sign and a diagram with two iB circles in series, and finally an ellipsis.

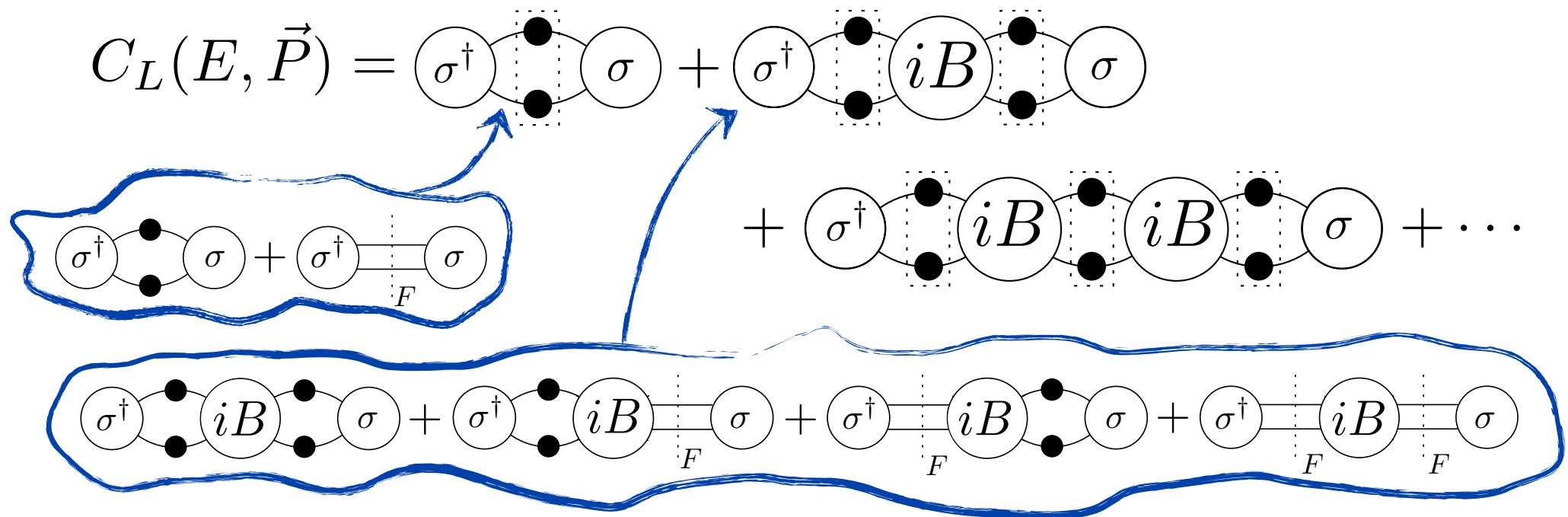
- Next use sum identity



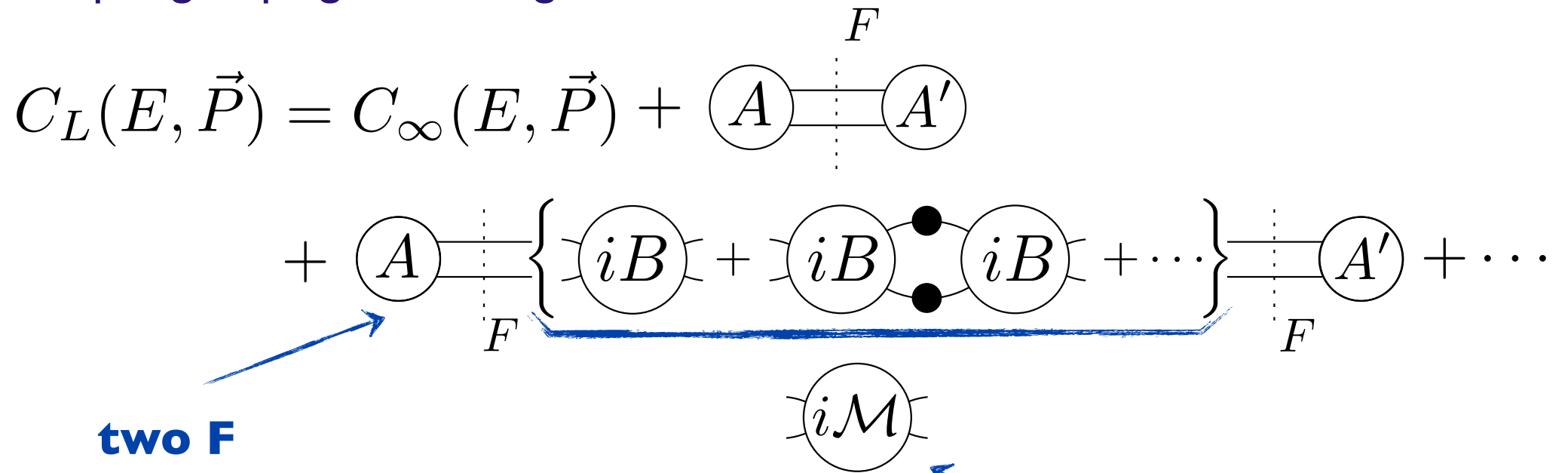
- And regroup according to number of “F cuts”



- Next use sum identity



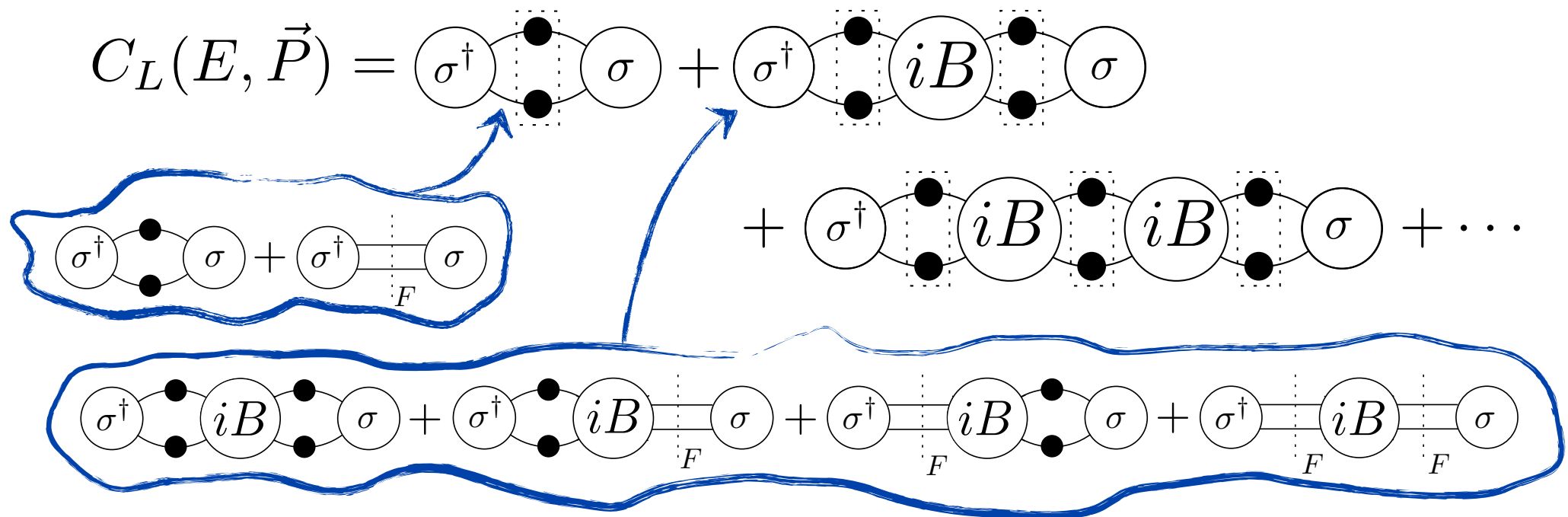
- And keep regrouping according to number of “F cuts”



two F cuts

the infinite-volume, on-shell 2→2 scattering amplitude

- Next use sum identity



- Alternate form if use PV-tilde prescription:

$$C_L(E, \vec{P}) = C_\infty^{\widetilde{PV}}(E, \vec{P}) + \begin{array}{c} F_{\widetilde{PV}} \\ \hline A \hline \hline A' \\ \hline F_{\widetilde{PV}} \end{array} + \begin{array}{c} A \\ \hline F_{\widetilde{PV}} \end{array} \left\{ \begin{array}{c} iB \\ \hline \hline iB \bullet iB \\ \hline \hline \dots \end{array} \right\} \begin{array}{c} \hline A' \\ \hline F_{\widetilde{PV}} \end{array} + \dots$$

**the infinite-volume, on-shell
2→2 K-matrix**

- Final result:

$$\begin{aligned}
 C_L(E, \vec{P}) &= C_\infty(E, \vec{P}) \\
 &+ \text{Diagram 1} + \text{Diagram 2} \\
 &+ \text{Diagram 3} + \dots
 \end{aligned}$$

Diagram 1: A circle labeled A on the left and a circle labeled A' on the right, connected by a horizontal line. A vertical dashed line labeled F is positioned between them.

Diagram 2: A circle labeled A on the left, a circle labeled $i\mathcal{M}$ in the middle, and a circle labeled A' on the right, all connected by a horizontal line. Two vertical dashed lines labeled F are positioned between A and $i\mathcal{M}$, and between $i\mathcal{M}$ and A' .

Diagram 3: A circle labeled A on the left, two circles labeled $i\mathcal{M}$ in the middle, and a circle labeled A' on the right, all connected by a horizontal line. Three vertical dashed lines labeled F are positioned between A and the first $i\mathcal{M}$, between the two $i\mathcal{M}$ circles, and between the second $i\mathcal{M}$ and A' .

- $$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2 \rightarrow 2} iF]^n A$$
- Correlator is expressed in terms of infinite-volume, physical quantities and kinematic functions encoding the finite-volume effects

- Final result:

$$\begin{aligned}
 C_L(E, \vec{P}) &= C_\infty(E, \vec{P}) \\
 &+ \text{Diagram 1} + \text{Diagram 2} \\
 &+ \text{Diagram 3} + \dots
 \end{aligned}$$

Diagram 1: A circle labeled A connected to a circle labeled A' by a horizontal line. A vertical dashed line labeled F is positioned between them.

Diagram 2: A circle labeled A connected to a circle labeled A' by a horizontal line. A vertical dashed line labeled F is between A and a circle labeled $i\mathcal{M}$. Another vertical dashed line labeled F is between $i\mathcal{M}$ and A' .

Diagram 3: A circle labeled A connected to a circle labeled A' by a horizontal line. Two vertical dashed lines labeled F are between A and two circles labeled $i\mathcal{M}$. A third vertical dashed line labeled F is between the second $i\mathcal{M}$ circle and A' .

- $$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2 \rightarrow 2} iF]^n A$$

- $$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2 \rightarrow 2} iF} A$$

↑ no poles, only cuts (pointing to C_∞)
 ↗ (pointing to $A' iF$)
 ↖ matrices in l,m space (pointing to $i\mathcal{M}_{2 \rightarrow 2}$)
 ↘ no poles, only cuts (pointing to the denominator)

- $$C_L(E, \vec{P}) \text{ diverges whenever } iF \frac{1}{1 - i\mathcal{M}_{2 \rightarrow 2} iF} \text{ diverges}$$

- Final result:

$$\begin{aligned}
 C_L(E, \vec{P}) &= C_\infty(E, \vec{P}) \\
 &+ \text{Diagram 1} + \text{Diagram 2} \\
 &+ \text{Diagram 3} + \dots
 \end{aligned}$$

Diagram 1: A circle labeled A connected to a circle labeled A' by a horizontal line. A vertical dashed line labeled F is positioned between them.

Diagram 2: A circle labeled A connected to a circle labeled $i\mathcal{M}$ connected to a circle labeled A' by horizontal lines. Vertical dashed lines labeled F are positioned between A and $i\mathcal{M}$, and between $i\mathcal{M}$ and A' .

Diagram 3: A circle labeled A connected to a circle labeled $i\mathcal{M}$ connected to a circle labeled $i\mathcal{M}$ connected to a circle labeled A' by horizontal lines. Vertical dashed lines labeled F are positioned between A and the first $i\mathcal{M}$, between the two $i\mathcal{M}$ circles, and between the second $i\mathcal{M}$ and A' .

- $$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2 \rightarrow 2} iF]^n A$$

- $$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2 \rightarrow 2} iF} A$$

↑ no poles, only cuts (pointing to C_∞)
 ↗ (pointing to $A' iF$)
 ↖ matrices in l,m space (pointing to $i\mathcal{M}_{2 \rightarrow 2}$)
 ↖ no poles, only cuts (pointing to the denominator)

⇒

$$\Delta_{L, \vec{P}}(E) = \det \left[(iF)^{-1} - i\mathcal{M}_{2 \rightarrow 2} \right] = 0$$

- Final result:

$$\begin{aligned}
 C_L(E, \vec{P}) &= C_\infty(E, \vec{P}) \\
 &+ \text{Diagram 1} + \text{Diagram 2} \\
 &+ \text{Diagram 3} + \dots
 \end{aligned}$$

Diagram 1: A circle labeled A connected to a circle labeled A' by a horizontal line. A vertical dashed line labeled F is positioned between them.

Diagram 2: A circle labeled A connected to a circle labeled $i\mathcal{M}$ connected to a circle labeled A' by horizontal lines. Two vertical dashed lines labeled F are positioned between A and $i\mathcal{M}$, and between $i\mathcal{M}$ and A' .

Diagram 3: A circle labeled A connected to a circle labeled $i\mathcal{M}$ connected to another circle labeled $i\mathcal{M}$ connected to a circle labeled A' by horizontal lines. Three vertical dashed lines labeled F are positioned between A and the first $i\mathcal{M}$, between the two $i\mathcal{M}$ circles, and between the second $i\mathcal{M}$ and A' .

- $$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + \sum_{n=0}^{\infty} A' iF [i\mathcal{M}_{2 \rightarrow 2} iF]^n A$$

- $$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A' iF \frac{1}{1 - i\mathcal{M}_{2 \rightarrow 2} iF} A$$

↑ no poles, only cuts
 ↑ matrices in l,m space
 ← no poles, only cuts

⇒

$$\Delta_{L, \vec{P}}(E) = \det \left[(F_{\widetilde{PV}})^{-1} + \mathcal{K}_2 \right] = 0$$
Alternative form

2-particle quantization condition

- At fixed L & \vec{P} , the finite-volume spectrum E_1, E_2, \dots is given by solutions to

$$\Delta_{L, \vec{P}}(E) = \det \left[(F_{\vec{P}V})^{-1} + \mathcal{K}_2 \right] = 0$$

- $\mathcal{K}_2, F_{\vec{P}V}$ are matrices in l, m space
- \mathcal{K}_2 is diagonal in l, m
- $F_{\vec{P}V}$ is off-diagonal, since the box violates rotation symmetry
- To make useful, truncate by assuming that \mathcal{K}_2 vanishes above l_{max}

$$i\mathcal{K}_{2;00;00}(E_n^*) = \left[iF_{\vec{P}V;00;00}(E_n, \vec{P}, L) \right]^{-1}$$

Equivalent to generalization of s-wave Lüscher equation to moving frame [Rummukainen & Gottlieb]

3-particle quantization condition

Following [Hansen & SS 14]

Recall Final result

$$\Delta_{L,P}(E) = \det [F_3^{-1} + \mathcal{K}_{\text{df},3}] = 0$$

$$F_3 = \frac{F_{\widetilde{\text{PV}}}}{2\omega L^3} \left[-\frac{2}{3} + \frac{1}{1 + (1 + \mathcal{K}_2 G)^{-1} \mathcal{K}_2 F_{\widetilde{\text{PV}}}} \right]$$

- Successfully separated infinite volume quantities from finite volume kinematic factors, but....
 - What is $\mathcal{K}_{\text{df},3}$?
 - How do we obtain this result?
 - How can it be made useful?

Key issue 4: dealing with cusps

- Can sum subdiagrams without 3-particle cuts into Bethe-Salpeter kernels
 \Rightarrow Skeleton expansion in terms of Bethe-Salpeter kernels

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

The diagrams in the expansion represent various particle interactions. The first row shows diagrams with orange circles (representing 3-particle cuts) and dashed boxes. The second and third rows show diagrams with purple circles (representing Bethe-Salpeter kernels) and dashed boxes. The fourth row shows diagrams with both orange and purple circles. The fifth row shows diagrams with purple circles and dashed boxes. The sixth row shows diagrams with orange circles and dashed boxes.

$$i\mathcal{B}_2 \quad \text{Purple circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

$$i\mathcal{B}_3 \quad \text{Orange circle} \equiv \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

Key issue 4: dealing with cusps

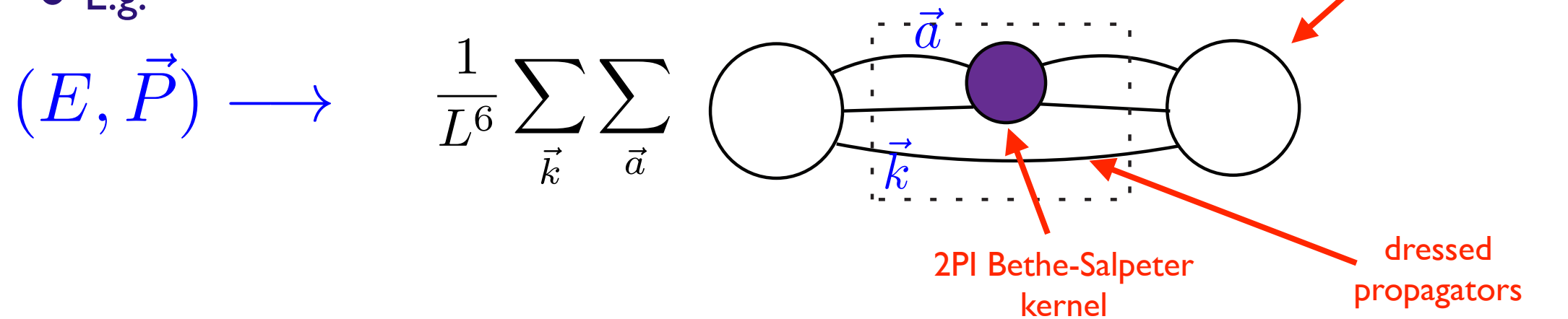
- Want to replace sums with integrals + F-cuts as in 2-particle analysis
- Straightforward implementation fails when have 3 particle intermediate states adjacent to $2 \rightarrow 2$ kernels

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

Cusp analysis (1)

- Aim: replace sums with integrals + finite-volume residue

- E.g.



- Can replace sums with integrals for smooth, non-singular parts of summand
- Singular part of left-hand 3-particle intermediate state

$$\frac{1}{L^6} \sum_{\vec{k}} \sum_{\vec{a}} \frac{A(\vec{k}, \vec{a}) B(\vec{k}, \vec{a})}{E - \omega_k - \omega_a - \omega_{ka}}$$

Diagram illustrating the singular part of the left-hand 3-particle intermediate state. The denominator is $E - \omega_k - \omega_a - \omega_{ka}$. The numerator is $A(\vec{k}, \vec{a}) B(\vec{k}, \vec{a})$. The denominator vanishes on-shell. The numerator is smooth functions.

Labels for the denominator terms:

- ω_k : $\sqrt{\vec{k}^2 + m^2}$
- ω_a : $\sqrt{\vec{a}^2 + m^2}$
- ω_{ka} : $\sqrt{(\vec{P} - \vec{k} - \vec{a})^2 + m^2}$

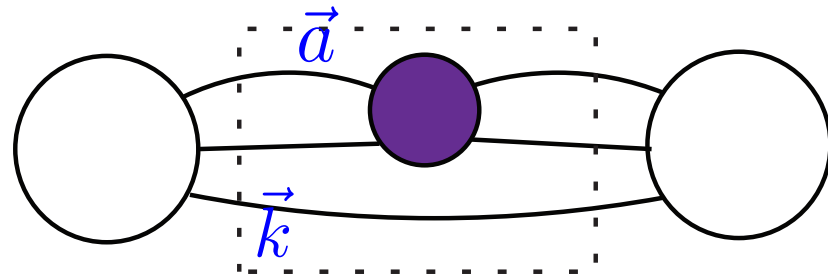
Labels for the numerator:

- $A(\vec{k}, \vec{a}) B(\vec{k}, \vec{a})$: smooth functions

Label for the denominator:

- $E - \omega_k - \omega_a - \omega_{ka}$: denominator vanishes on-shell

Cusp analysis (2)



$$\frac{1}{L^6} \sum_{\vec{k}} \sum_{\vec{a}} \frac{A(\vec{k}, \vec{a}) B(\vec{k}, \vec{a})}{E - \omega_k - \omega_a - \omega_{k_a}}$$

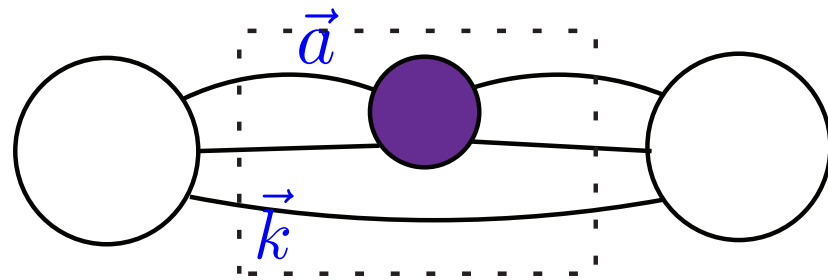
Step I: treat sum over a

$$\frac{1}{L^3} \sum_{\vec{a}} \longrightarrow \int_{\vec{a}} + \left(\frac{1}{L^3} \sum_{\vec{a}} - \int_{\vec{a}} \right)$$

Difference gives zeta-function F with A & B projected on shell [Lüscher,...]

F has multiple singularities,
so leave k summed
for F-term

Cusp analysis (2)



$$\frac{1}{L^6} \sum_{\vec{k}} \sum_{\vec{a}} \frac{A(\vec{k}, \vec{a}) B(\vec{k}, \vec{a})}{E - \omega_k - \omega_a - \omega_{ka}}$$

Step 1: treat sum over a

$$\frac{1}{L^3} \sum_{\vec{a}} \longrightarrow \int_{\vec{a}} + \left(\frac{1}{L^3} \sum_{\vec{a}} - \int_{\vec{a}} \right)$$

Difference gives zeta-function F with A & B projected on shell [Lüscher,...]

Step 2: treat sum over k

- Want to replace sum over k with integral for $\int_{\vec{a}}$ term
- Only possible if integral over a gives smooth function
- $i\epsilon$ prescription and standard principal value (PV) lead to cusps at threshold \Rightarrow sum-integral $\sim 1/L^4$ [Polejaeva & Rusetsky]
- Requires use of modified $\widetilde{\text{PV}}$ prescription

F has multiple singularities, so leave k summed for F-term

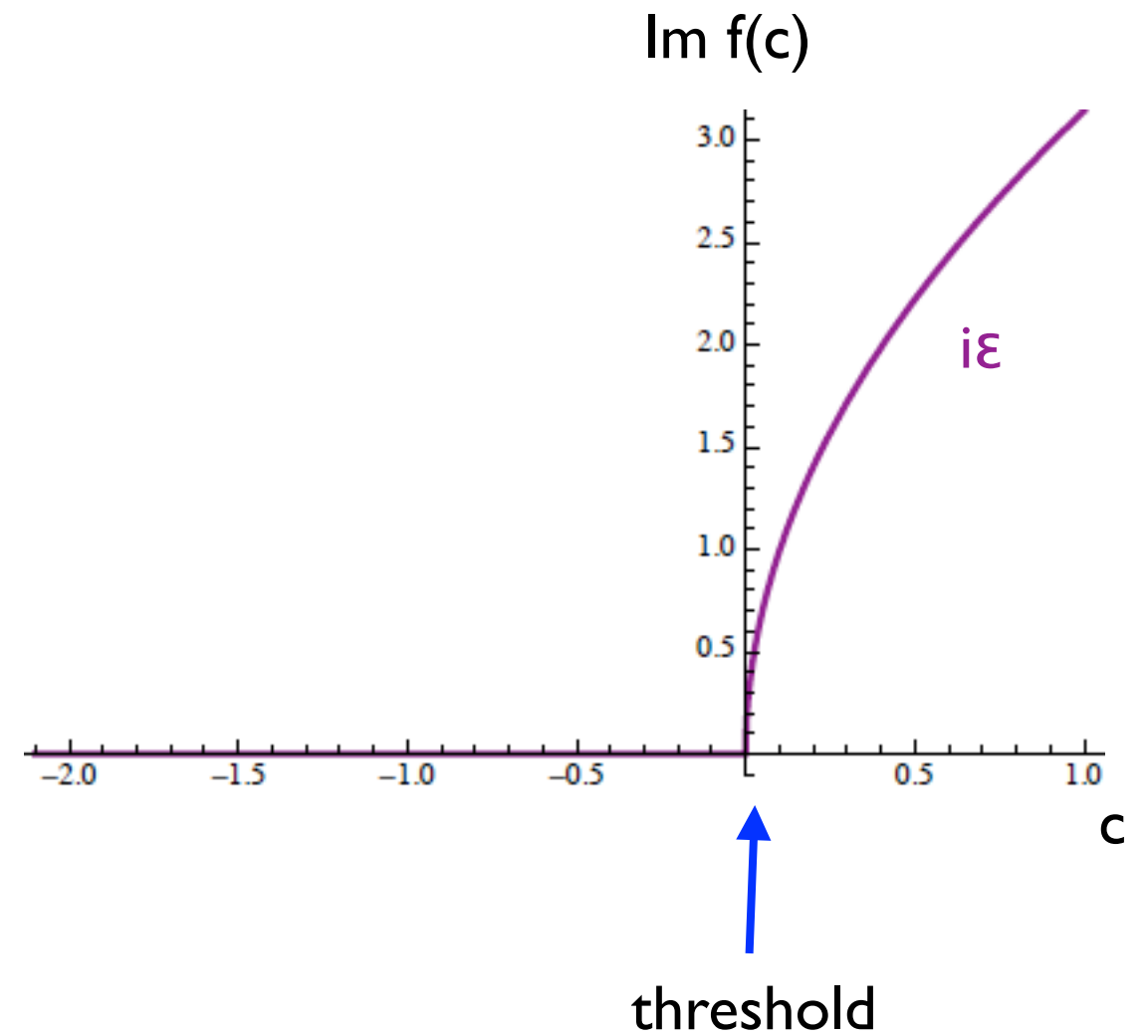
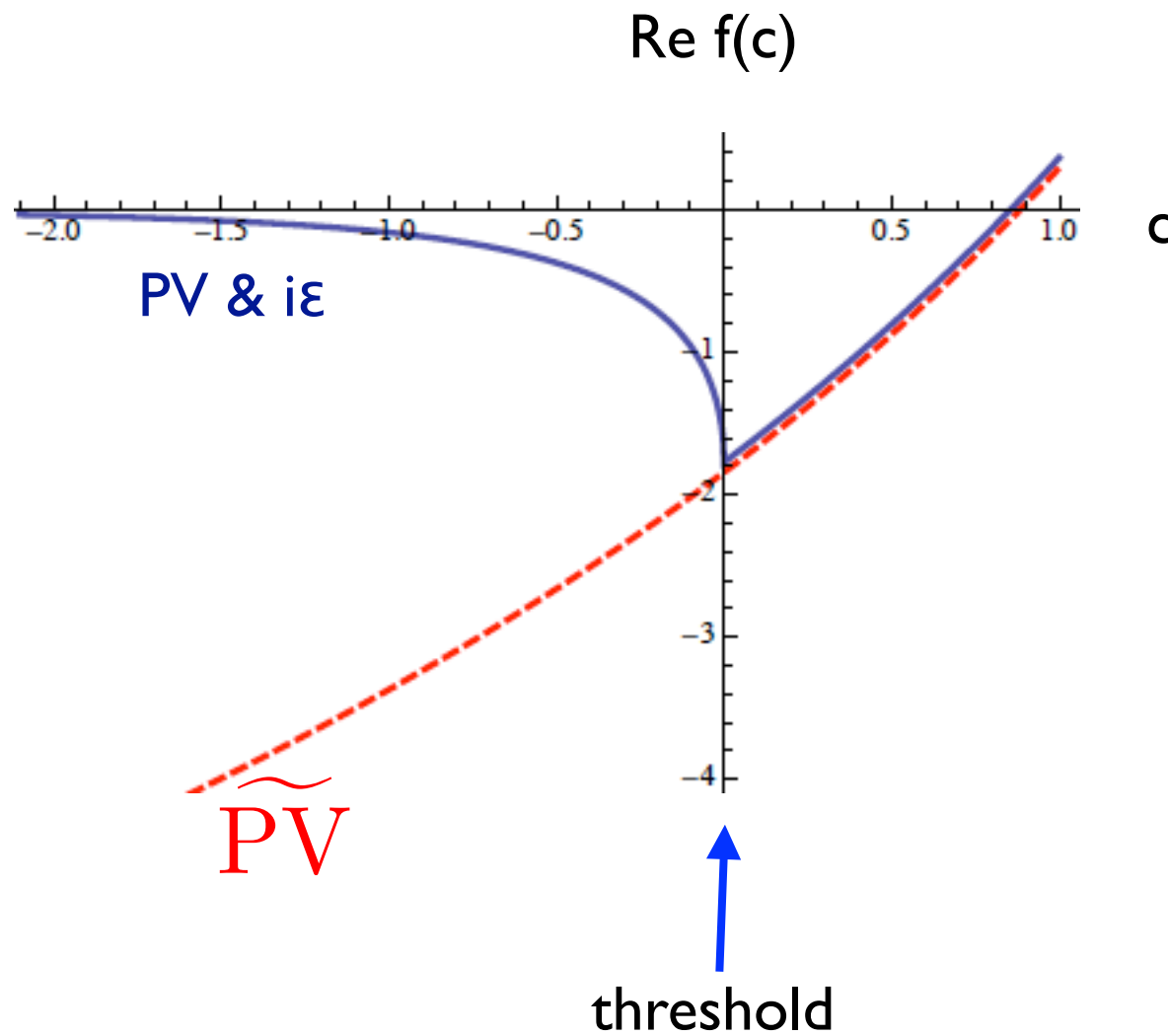
Result:
$$\frac{1}{L^6} \sum_{\vec{k}} \sum_{\vec{a}} = \int_{\vec{k}} \int_{\vec{a}} + \sum_{\vec{k}} \text{“F term”}$$

Cusp analysis (3)

$$x \sim (a^*)^2$$



• Simple example: $\int_{\vec{a}} \frac{A(\vec{k}, \vec{a}) B(\vec{k}, \vec{a})}{E - \omega_k - \omega_a - \omega_{ka}} \longrightarrow f(c) = \int_0^\infty dx \frac{\sqrt{x} e^{-(x-c)}}{c-x}$



- Far below threshold, \widetilde{PV} smoothly turns back into PV

Cusp analysis (4)

- Bottom line: must use $\widetilde{P}\widetilde{V}$ prescription for all loops
- This is why K-matrix \mathcal{K}_2 appears in 2-particle summations
- \mathcal{K}_2 is standard above threshold, and given below by analytic continuation (so there is no cusp)
- This prescription is that used previously when studying finite-volume effects on bound-state energies using two-particle quantization condition [Detmold, Savage,...]
- Far below threshold smoothly turns into \mathcal{M}_2^{ℓ}

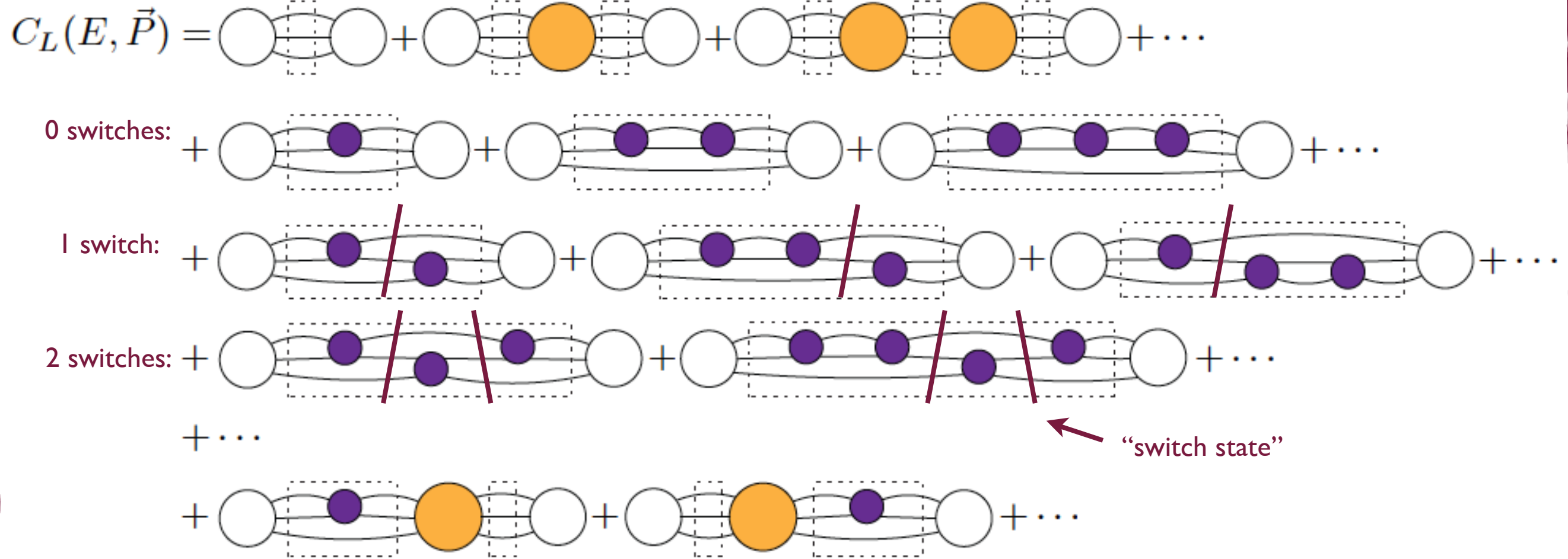
Key issue 5: dealing with “switches”

$$\begin{aligned}
 C_L(E, \vec{P}) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} + \dots \\
 & + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \dots \\
 & + \text{Diagram 10} + \text{Diagram 11} + \dots \\
 & + \dots \\
 & + \text{Diagram 12} + \text{Diagram 13} + \dots
 \end{aligned}$$

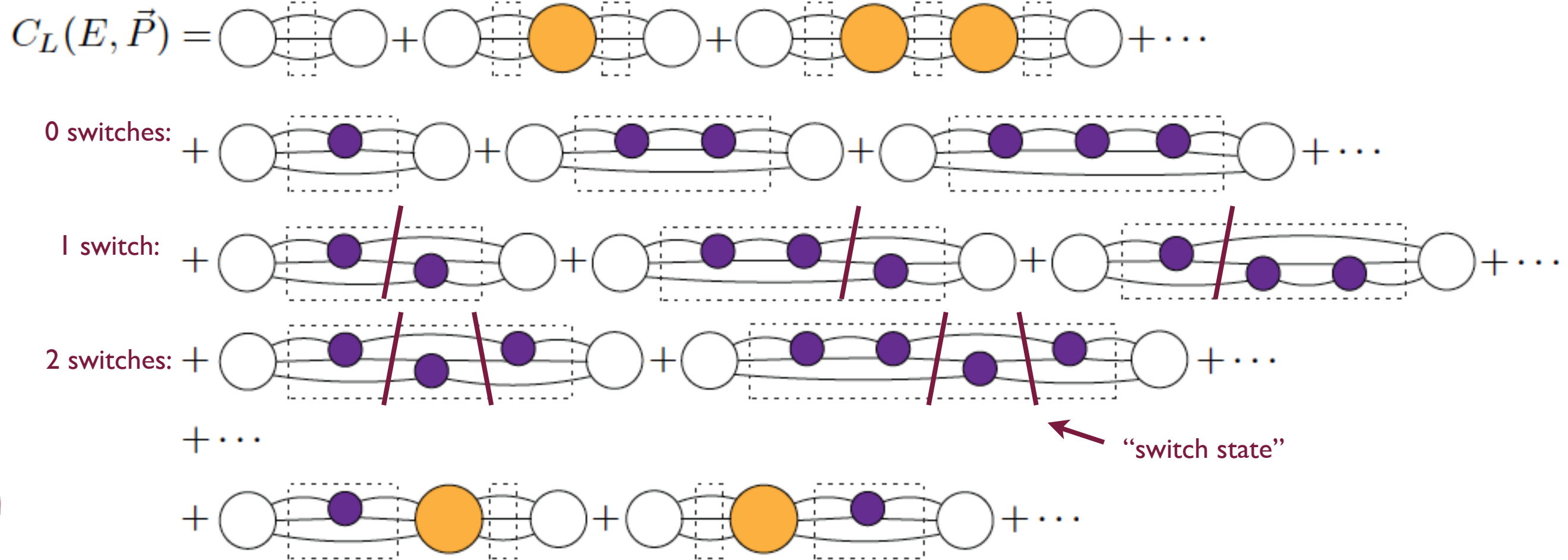
The diagrams represent terms in a series expansion for the longitudinal correlation function $C_L(E, \vec{P})$. Each diagram consists of two external white circles connected by two lines. The internal structure is enclosed in a dashed box and includes:

- Diagram 1: Two white circles.
- Diagram 2: One orange circle (switch).
- Diagram 3: Two orange circles (switches).
- Diagram 4: One purple circle (exciton).
- Diagram 5: Two purple circles (excitons).
- Diagram 6: Three purple circles (excitons).
- Diagram 7: Two purple circles (excitons) with a curved line between them.
- Diagram 8: Three purple circles (excitons) with curved lines between them.
- Diagram 9: Four purple circles (excitons) with curved lines between them.
- Diagram 10: Three purple circles (excitons) with curved lines between them.
- Diagram 11: Four purple circles (excitons) with curved lines between them.
- Diagram 12: One purple circle and one orange circle (switch).
- Diagram 13: One orange circle (switch) and one purple circle (exciton).

Key issue 5: dealing with “switches”



Key issue 5: dealing with “switches”



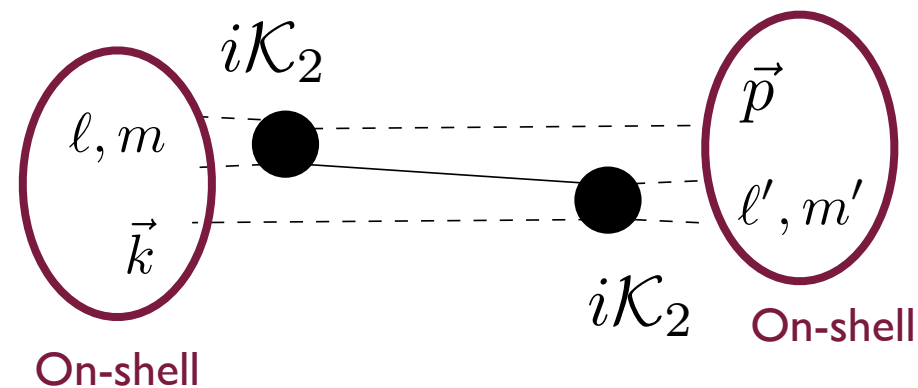
- With cusps removed, no-switch diagrams can be summed as for 2-particle case
- “Switches” present a new challenge

One-switch diagrams

$$C_L^{(2)} = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

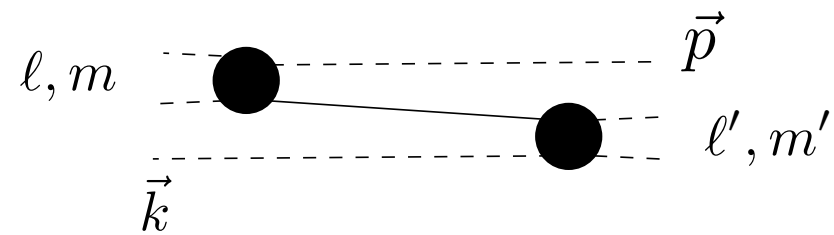
Can treat similarly to 2-particle case
leading to a series of F_{PV} 's and \mathcal{K}_2 's

- End up with L-dependent part of $C^{(2)}$ having at its core:

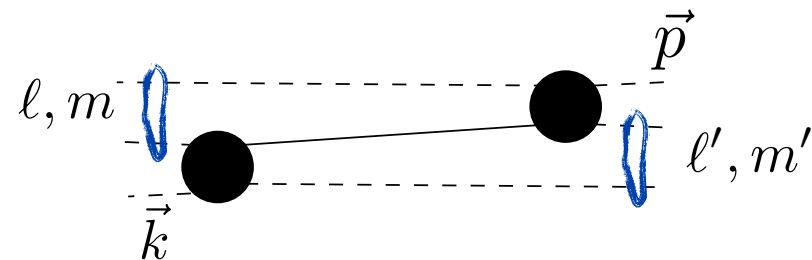


- This is our first contribution to the infinite-volume 3 particle scattering amplitude

One-switch problem



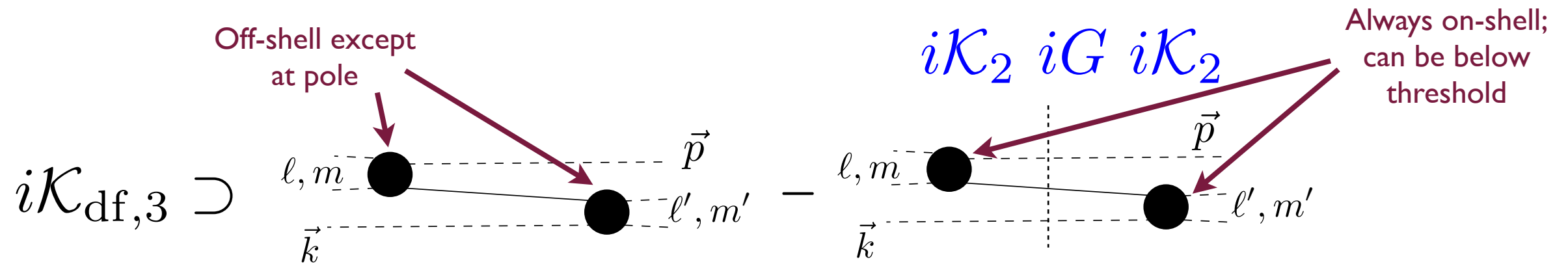
- Amplitude is singular for some choices of \mathbf{k} , \mathbf{p} in physical regime
 - Propagator goes on shell if top two (and thus bottom two) scatter elastically
- Not a problem per se, but leads to difficulties when amplitude is symmetrized
 - Occurs when include three-switch contributions



- Singularity implies that decomposition in $Y_{l,m}$ will not converge uniformly
 - Cannot usefully truncate angular momentum expansion

One-switch solution

- Define divergence-free amplitude by subtracting singular part
 - Utility of subtraction noted in [Rubin, Sugar & Tiktopoulos, '66]

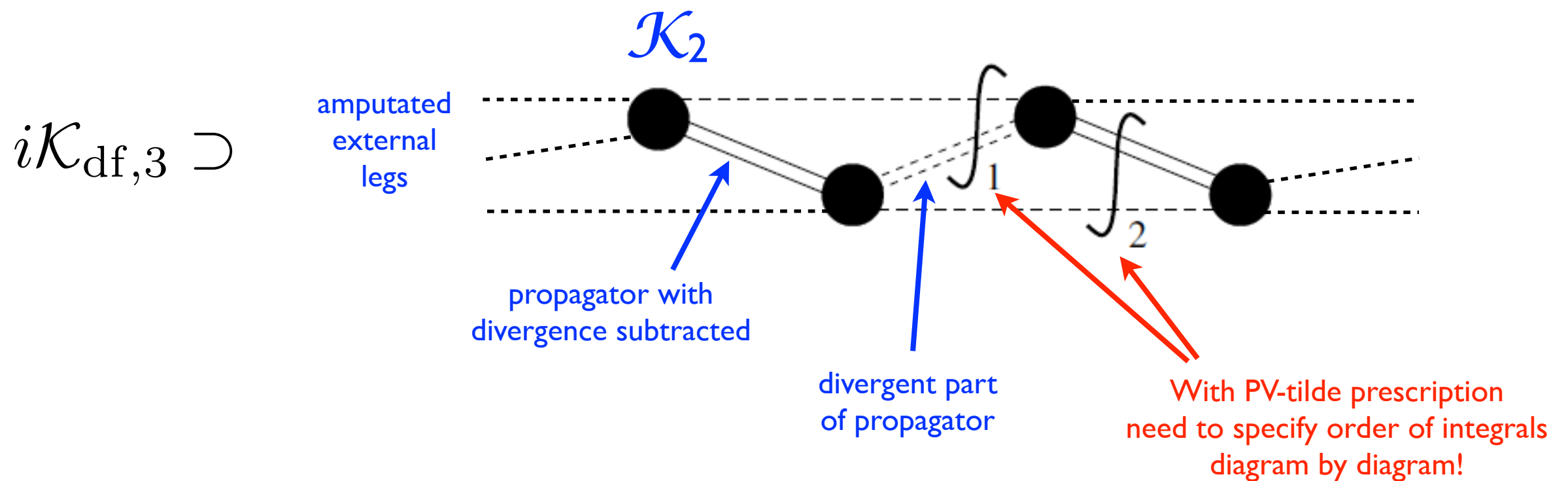


$$G_{p,\ell',m';k,\ell,m} \equiv \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell',m'}(\hat{k}^*) H(\vec{p}) H(\vec{k}) Y_{\ell,m}^*(\hat{p}^*)}{2\omega_{kp}(E - \omega_k - \omega_p - \omega_{kp})} \left(\frac{p^*}{q_k^*}\right)^{\ell} \frac{1}{2\omega_k L^3}$$

- Key point: $\mathcal{K}_{df,3}$ is local and its expansion in harmonics can be truncated
- Subtracted term must be added back---leads to G contributions to F_3
- Can extend divergence-free definition to any number of switches

Key issue 6: symmetry breaking

- Using \widetilde{PV} prescription breaks particle interchange symmetry
 - Top two particles treated differently from spectator
 - Leads to very complicated definition for $\mathcal{K}_{df,3}$, e.g.



- Can extend definition of $\mathcal{K}_{df,3}$ to all orders, in such a way that it is symmetric under interchange of external particles

Key issue 6: symmetry breaking

- Final definition of $\mathcal{K}_{\text{df},3}$ is, crudely speaking:
 - Sum all Feynman diagrams contributing to \mathcal{M}_3
 - Use $\widetilde{\text{PV}}$ prescription, plus a (well-defined) set of rules for ordering integrals
 - Subtract leading divergent parts
 - Apply a set of (completely specified) extra factors (“decorations”) to ensure external symmetrization
- $\mathcal{K}_{\text{df},3}$ is an UGLY infinite-volume quantity related to scattering
- At the time of our initial paper, we did not know the relation between $\mathcal{K}_{\text{df},3}$ and \mathcal{M}_3 & \mathcal{M}_2 , although we had reasons to think that such a relationship exists
- Now we know the relationship

Infinite volume relation between $\mathcal{K}_{df,3}$ & \mathcal{M}_3

[Hansen & SS 15, in preparation]

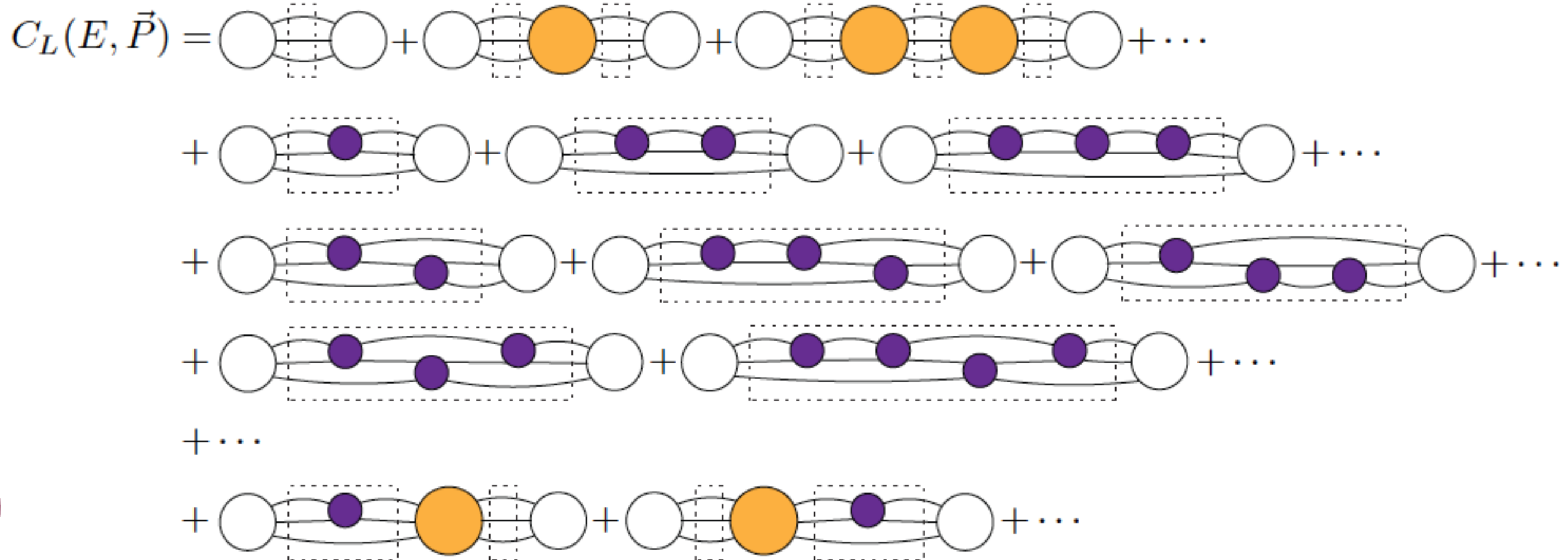
The issue

- Three particle quantization condition depends on $\mathcal{K}_{\text{df},3}$ rather than the three particle scattering amplitude \mathcal{M}_3
- $\mathcal{K}_{\text{df},3}$ is an infinite volume quantity (loops involve integrals) but is not physical
 - Has a very complicated, unwieldy definition
 - Depends on the cut-off function H
 - However, it was forced on us by the analysis, and is some sort of local vertex
- To complete the quantization condition we must relate $\mathcal{K}_{\text{df},3}$ to \mathcal{M}_3

The method

- Define a “finite volume scattering amplitude” $\mathcal{M}_{L,3}$ which goes over to \mathcal{M}_3 in an (appropriately taken) $L \rightarrow \infty$ limit
- Relate $\mathcal{M}_{L,3}$ to $\mathcal{K}_{\text{df},3}$ at finite volume—which turns out to require a small generalization of the methods used to derive the quantization condition
- Take $L \rightarrow \infty$, obtaining nested integral equations

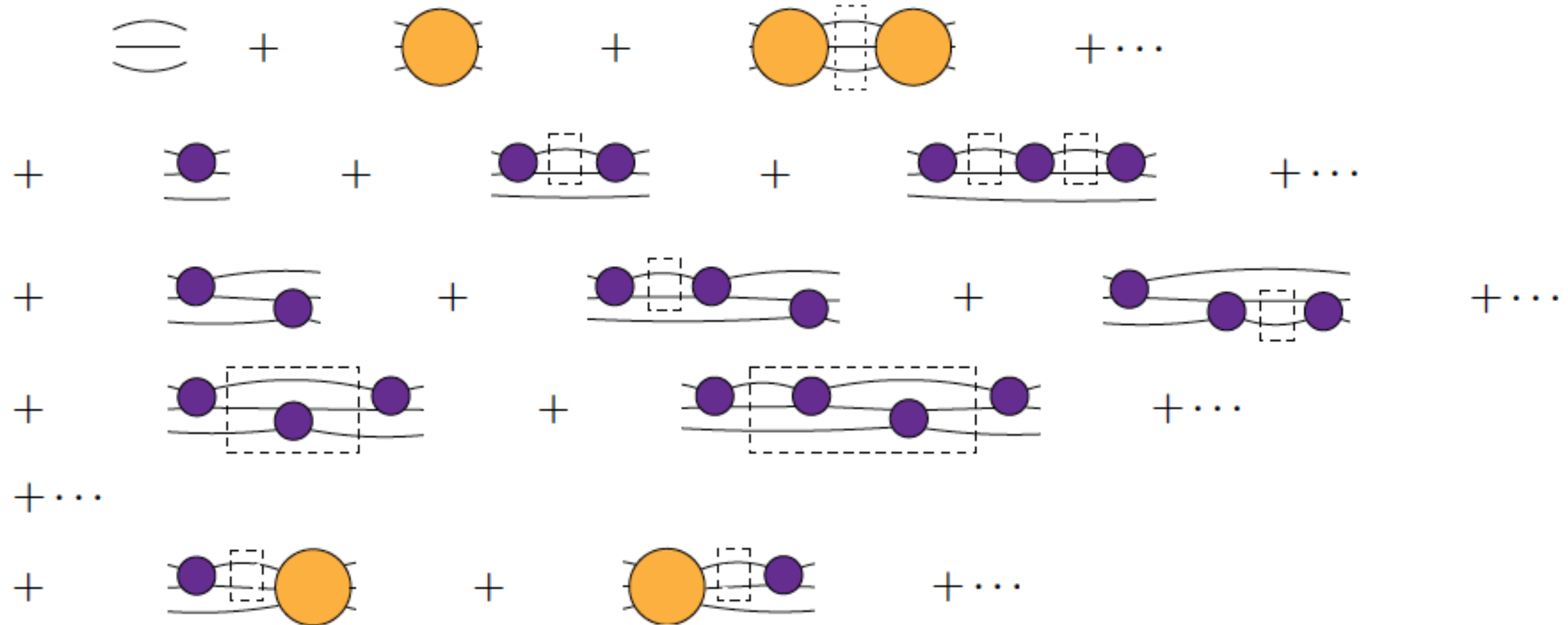
Modifying C_L to obtain $\mathcal{M}_{L,3}$



$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - i\mathcal{K}_{df,3 \rightarrow 3} iF_3} A_3$$

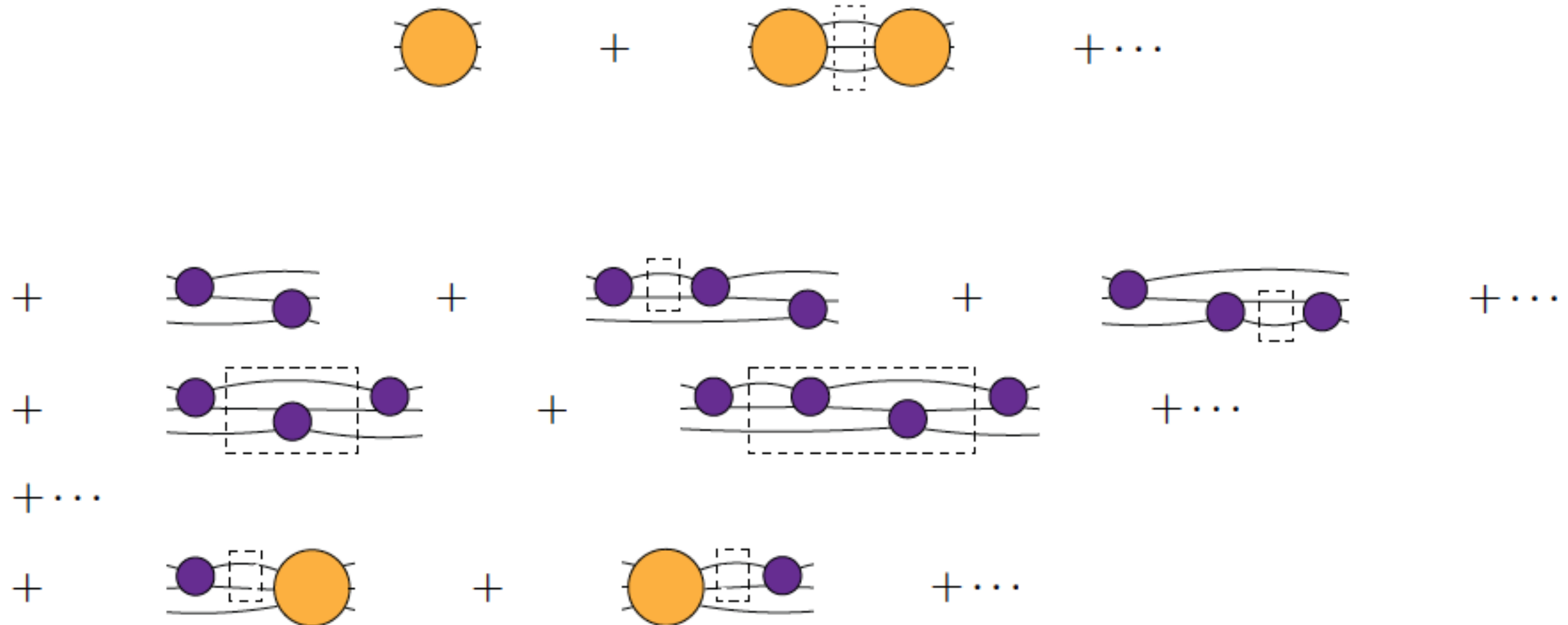
no poles
no poles
no poles

Modifying C_L to obtain $\mathcal{M}_{L,3}$



Step I: “amputate”

Modifying C_L to obtain $\mathcal{M}_{L,3}$



Step 2: Drop disconnected diagrams

Modifying C_L to obtain $\mathcal{M}_{L,3}$

$$i\mathcal{M}_{L,3 \rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{Diagram 1} + \text{Diagram 2} + \dots \\ + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \dots \\ + \text{Diagram 6} + \text{Diagram 7} + \dots \\ + \dots \\ + \text{Diagram 8} + \text{Diagram 9} + \dots \end{array} \right\}$$

Step 3: Symmetrize

$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{\text{df},3}$

$$i\mathcal{M}_{L,3\rightarrow 3} \equiv \mathcal{S} \left\{ \begin{array}{l} \text{Diagram 1} + \text{Diagram 2} + \dots \\ + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \dots \\ + \text{Diagram 6} + \text{Diagram 7} + \dots \\ + \dots \\ + \text{Diagram 8} + \text{Diagram 9} + \dots \end{array} \right\}$$

The diagrams represent various particle interactions. Diagram 1 is a single orange circle with three external lines. Diagram 2 shows two orange circles connected by a dashed box, with three external lines. Diagrams 3-7 show chains of purple circles with wavy lines between them and dashed boxes around some, with three external lines. Diagrams 8-9 show a purple circle connected to an orange circle, with dashed boxes around the connection and three external lines.

$$i\mathcal{M}_{L,3\rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[\mathcal{L}_L \ i\mathcal{K}_{\text{df},3\rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3\rightarrow 3}} \ \mathcal{R}_L \right]$$

$$i\mathcal{D}_L \equiv \mathcal{S} \left[\frac{1}{1 - i\mathcal{M}_{L,2\rightarrow 2} \ iG} \ i\mathcal{M}_{L,2\rightarrow 2} \ iG \ i\mathcal{M}_{L,2\rightarrow 2} [2\omega L^3] \right]$$

$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{\text{df},3}$

$$i\mathcal{M}_{L,3\rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[\mathcal{L}_L \ i\mathcal{K}_{\text{df},3\rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3\rightarrow 3}} \ \mathcal{R}_L \right]$$

$$i\mathcal{D}_L \equiv \mathcal{S} \left[\frac{1}{1 - i\mathcal{M}_{L,2\rightarrow 2} \ iG} \ i\mathcal{M}_{L,2\rightarrow 2} \ iG \ i\mathcal{M}_{L,2\rightarrow 2} [2\omega L^3] \right]$$

- \mathcal{L}_L and \mathcal{R}_L depend only on $\mathcal{M}_{L,2}$, G and F_{PV}
- $\mathcal{M}_{L,2}$ is “finite volume two particle scattering amplitude”

$$i\mathcal{M}_{L,2\rightarrow 2} \equiv i\mathcal{K}_{2\rightarrow 2} \frac{1}{1 - iF_{\text{PV}} i\mathcal{K}_{2\rightarrow 2}}$$

$\mathcal{M}_{L,3}$ in terms of $\mathcal{K}_{\text{df},3}$

$$i\mathcal{M}_{L,3\rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[\mathcal{L}_L \ i\mathcal{K}_{\text{df},3\rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3\rightarrow 3}} \ \mathcal{R}_L \right]$$

$$i\mathcal{D}_L \equiv \mathcal{S} \left[\frac{1}{1 - i\mathcal{M}_{L,2\rightarrow 2} \ iG} \ i\mathcal{M}_{L,2\rightarrow 2} \ iG \ i\mathcal{M}_{L,2\rightarrow 2} [2\omega L^3] \right]$$

- Key point: the *same (ugly)* $\mathcal{K}_{\text{df},3}$ appears in $\mathcal{M}_{L,3}$ as in \mathcal{C}_L

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A'_3 iF_3 \frac{1}{1 - i\mathcal{K}_{\text{df},3\rightarrow 3} \ iF_3} A_3$$

- Can use $\mathcal{M}_{L,3}$ to derive quantization condition

Final step: taking $L \rightarrow \infty$

$$i\mathcal{M}_{L,3 \rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[\mathcal{L}_L \ i\mathcal{K}_{\text{df},3 \rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3 \rightarrow 3}} \ \mathcal{R}_L \right]$$

$$i\mathcal{D}_L \equiv \mathcal{S} \left[\frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} \ iG} \ i\mathcal{M}_{L,2 \rightarrow 2} \ iG \ i\mathcal{M}_{L,2 \rightarrow 2} [2\omega L^3] \right]$$

$$iF_3 \equiv \frac{iF_{\text{PV}}}{2\omega L^3} \left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} \ iG} \ i\mathcal{M}_{L,2 \rightarrow 2} \ iF_{\text{PV}} \right]$$

- All equations involve matrices with indices k, l, m

Spectator momentum
 $\mathbf{k} = 2 \mathbf{n} \pi / L$
 Summed over \mathbf{n}

Already in infinite
 volume variables

Final step: taking $L \rightarrow \infty$

$$i\mathcal{M}_{L,3 \rightarrow 3} = i\mathcal{D}_L + \mathcal{S} \left[\mathcal{L}_L \ i\mathcal{K}_{\text{df},3 \rightarrow 3} \frac{1}{1 - iF_3 \ i\mathcal{K}_{\text{df},3 \rightarrow 3}} \ \mathcal{R}_L \right]$$

$$i\mathcal{D}_L \equiv \mathcal{S} \left[\frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} \ iG} \ i\mathcal{M}_{L,2 \rightarrow 2} \ iG \ i\mathcal{M}_{L,2 \rightarrow 2} [2\omega L^3] \right]$$

$$iF_3 \equiv \frac{iF_{\text{PV}}}{2\omega L^3} \left[\frac{1}{3} + \frac{1}{1 - i\mathcal{M}_{L,2 \rightarrow 2} \ iG} \ i\mathcal{M}_{L,2 \rightarrow 2} \ iF_{\text{PV}} \right]$$

- Sums over momenta \rightarrow integrals (+ now irrelevant $1/L$ terms!)
- Must introduce pole prescription for sums to avoid singularities

$$i\mathcal{M}_{3 \rightarrow 3} = \lim_{L \rightarrow \infty} \left|_{i\epsilon} \ i\mathcal{M}_{L,3 \rightarrow 3} \right.$$

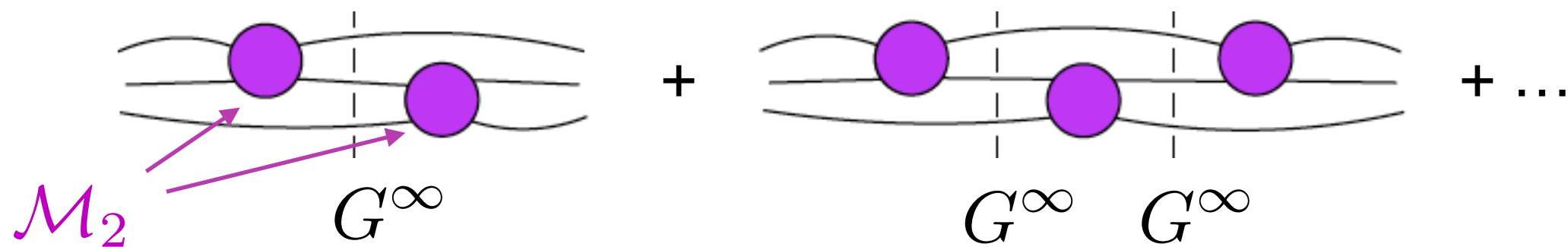
Final result: nested integral equations

(1) Obtain $L \rightarrow \infty$ limit of \mathcal{D}_L

$$i\mathcal{D}^{(u,u)}(\vec{p}, \vec{k}) = i\mathcal{M}_2(\vec{p})iG^\infty(\vec{p}, \vec{k})i\mathcal{M}_2(\vec{k}) + \int_{\vec{s}} \frac{1}{2\omega_s} i\mathcal{M}_2(\vec{p})iG^\infty(\vec{p}, \vec{s})i\mathcal{D}^{(u,u)}(\vec{s}, \vec{k})$$

$$G_{\ell'm'; \ell m}^\infty(\vec{p}, \vec{k}) \equiv \left(\frac{k^*}{q_p^*}\right)^{\ell'} \frac{4\pi Y_{\ell'm'}(\hat{k}^*) H(\vec{p}) H(\vec{k}) Y_{\ell m}^*(\hat{p}^*)}{2\omega_{kp}(E - \omega_k - \omega_p - \omega_{kp} + i\epsilon)} \left(\frac{p^*}{q_k^*}\right)^\ell$$

- Quantities are still matrices in l, m space
- Presence of cut-off function means that integrals have finite range
- $\mathcal{D}^{(u,u)}$ sums geometric series which gives physical divergences in \mathcal{M}_3



Final result: nested integral equations

(2) Sum geometric series involving $\mathcal{K}_{\text{df},3}$

$$i\mathcal{T}(\vec{p}, \vec{k}) = i\mathcal{K}_{\text{df},3}(\vec{p}, \vec{k}) + \int_{\vec{s}} \int_{\vec{r}} i\mathcal{K}_{\text{df},3}(\vec{p}, \vec{s}) \frac{i\rho(\vec{s})}{2\omega_s} i\mathcal{L}^{(u,u)}(\vec{s}, \vec{r}) i\mathcal{T}(\vec{r}, \vec{k}),$$

$$\mathcal{L}^{(u,u)}(\vec{p}, \vec{k}) = \left(\frac{1}{3} + i\mathcal{M}_2(\vec{p})i\rho(\vec{p}) \right) (2\pi)^3 \delta^3(\vec{p} - \vec{k}) + i\mathcal{D}^{(u,u)}(\vec{p}, \vec{k}) \frac{i\rho(\vec{k})}{2\omega_k},$$

- $\rho(\mathbf{k})$ is a phase space factor (analytically continued when below threshold)
- Requires $\mathcal{D}^{(u,u)}$ and \mathcal{M}_2
- Corresponds to summing the core geometric series, i.e.

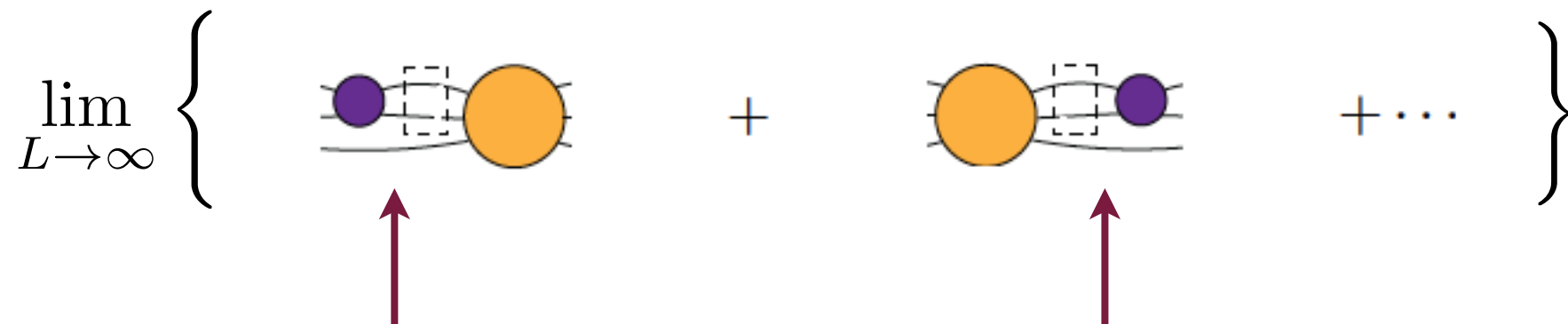
$$i\mathcal{K}_{\text{df},3 \rightarrow 3} \frac{1}{1 - iF_3 i\mathcal{K}_{\text{df},3 \rightarrow 3}}$$

Final result: nested integral equations

(3) Add in effects of external 2→2 scattering:

$$\underbrace{\mathcal{M}_3(\vec{p}, \vec{k}) - \mathcal{S} \left\{ \mathcal{D}^{(u,u)}(\vec{p}, \vec{k}) \right\}}_{\mathcal{M}_{df,3}} = -\mathcal{S} \left\{ \int_s \int_r \mathcal{L}^{(u,u)}(\vec{p}, \vec{s}) \mathcal{T}(\vec{s}, \vec{r}) \mathcal{R}^{(u,u)}(\vec{r}, \vec{k}) \right\}$$

- Sums geometric series on “outside” of $\mathcal{K}_{df,3}$'s



- Can also invert and determine $\mathcal{K}_{df,3}$ given \mathcal{M}_3 and \mathcal{M}_2