CP violation in kaons: BSM B-parameters and precision tests of ε_K

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Contents

FNAL/MILC/SWME Collaboration 2012 — Present

FNAL/MILC/SWME Collaboration I

- Seoul National University (SWME):
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  10 graduate students.

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- Brookhaven National Laboratory (SWME):
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FNAL/MILC/SWME Collaboration II

- Los Alamos National Laboratory (SWME):
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- Fermi National Accelerator Laboratory (FNAL):
  Dr. Andreas S. Kronfeld.
- University of Utah (MILC):
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  Prof. Mehmet B. Oktay.
- KISTI Supercomputing Center (SWME):
  Dr. Jangho Kim (Postdoc)

Lattice Gauge Theory Research Center (SNU)

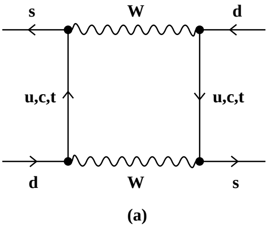
- Center Leader: Prof. Weonjong Lee.
- Research Assistant Prof.: Dr. Jon Bailey
- Research Assitant Prof.: Dr. Nigel Cundy
- 10+1 graduate students
- Secretary: Mrs. Sora Park.
- more details on `http://lgt.snu.ac.kr/`.

Group Photo (2014)

CP Violation and B_K

Kaon Eigenstates and ε

- Flavor eigenstates, $K^0 = (\bar{s}d)$ and $\bar{K}^0 = (s\bar{d})$ mix via box diagrams.



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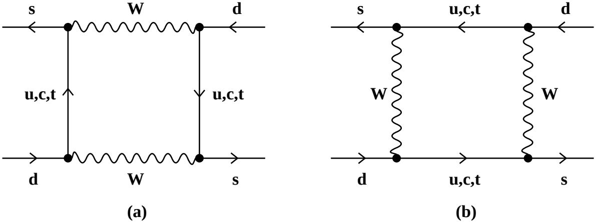

- CP eigenstates K_1(even) and K_2(odd).

$$K_1 = \frac{1}{\sqrt{2}}(K^0 - \bar{K}^0) \qquad K_2 = \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0)$$

Kaon Eigenstates and ε

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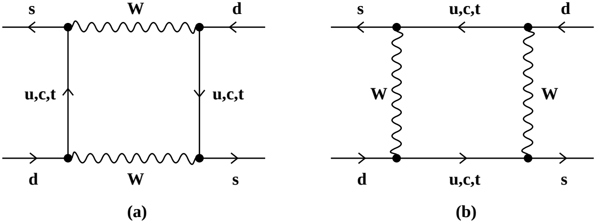


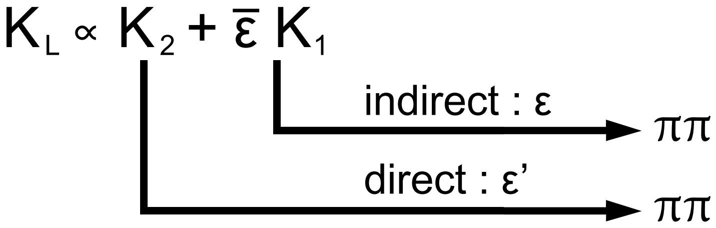
- CP eigenstates K_1(even) and K_2(odd).

$$K_1 = \frac{1}{\sqrt{2}}(K^0 - \bar{K}^0) \qquad K_2 = \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0)$$

- Neutral Kaon eigenstates K_S and K_L.

$$K_S = \frac{1}{\sqrt{1+|\bar{\varepsilon}|^2}}(K_1 + \bar{\varepsilon} K_2) \qquad K_L = \frac{1}{\sqrt{1+|\bar{\varepsilon}|^2}}(K_2 + \bar{\varepsilon} K_1)$$

Indirect CP violation and direct CP violation



ε_K and $\hat{B}_K$, V_{cb} I

- Definition of ε_K

$$\varepsilon_K = \frac{A[K_L \to (\pi\pi)_{I=0}]}{A[K_S \to (\pi\pi)_{I=0}]}$$

- Master formula for ε_K in the Standard Model.

$$\varepsilon_K = \exp(i\theta)\, \sqrt{2}\, \sin(\theta) \left(C_\varepsilon\, X_{\mathsf{SD}}\, \hat{B}_K + \frac{\xi_0}{\sqrt{2}} + \xi_{\mathsf{LD}} \right) + \mathcal{O}(\omega\varepsilon') + \mathcal{O}(\xi_0 \Gamma_2/\Gamma_1)$$

$$X_{\mathsf{SD}} = \mathrm{Im}\lambda_t \Big[\mathrm{Re}\,\lambda_c\, \eta_{cc}\, S_0(x_c) - \mathrm{Re}\,\lambda_t\, \eta_{tt}\, S_0(x_t) - (\mathrm{Re}\,\lambda_c - \mathrm{Re}\,\lambda_t)\, \eta_{ct}\, S_0(x_c, x_t) \Big]$$

ε_K and $\hat{B}_K$, V_{cb} II

$$\lambda_i = V_{is}^* V_{id}, \qquad x_i = m_i^2/M_W^2, \qquad C_\varepsilon = \frac{G_F^2 F_K^2 m_K M_W^2}{6\sqrt{2}\,\pi^2 \Delta M_K}$$

$$\frac{\xi_0}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{\mathrm{Im} A_0}{\mathrm{Re} A_0} \approx -5\%$$

$$\xi_{\text{LD}} = \text{Long Distance Effect} \approx 2\% \qquad \longrightarrow \text{systematic error}$$

- Inami-Lim functions:

$$S_0(x_i) = x_i \left[\frac{1}{4} + \frac{9}{4(1-x_i)} - \frac{3}{2(1-x_i)^2} - \frac{3x_i^2 \ln x_i}{(1-x_i)^3} \right],$$

$$S_0(x_i, x_j) = \left\{ \frac{x_i x_j}{x_i - x_j} \left[\frac{1}{4} + \frac{3}{2(1-x_i)} - \frac{3}{4(1-x_i)^2} \right] \ln x_i - (i \leftrightarrow j) \right\} - \frac{3 x_i x_j}{4(1-x_i)(1-x_j)}$$

ε_K and $\hat{B}_K$, V_{cb} III

$$
\begin{aligned}
S_0(x_t) &\longrightarrow +70\% \\
S_0(x_c, x_t) &\longrightarrow +44\% \\
S_0(x_c) &\longrightarrow -14\%
\end{aligned}
$$

- Dominant contribution (≈70%) comes with $|V_{cb}|^4$.

$$
\begin{aligned}
\mathrm{Im}\lambda_t \cdot \mathrm{Re}\lambda_t &= \bar{\eta}\lambda^2 |V_{cb}|^4 (1-\bar{\rho}) \\
\mathrm{Re}\lambda_c &= -\lambda(1-\frac{\lambda^2}{2}) + \mathcal{O}(\lambda^5) \\
\mathrm{Re}\lambda_t &= -(1-\frac{\lambda^2}{2})A^2\lambda^5(1-\bar{\rho}) + \mathcal{O}(\lambda^7) \\
\mathrm{Im}\lambda_t &= \eta A^2 \lambda^5 + \mathcal{O}(\lambda^7)
\end{aligned}
$$

ε_K and $\hat{B}_K$, V_{cb} IV

- Definition of $\hat{B}_K$ in standard model.

$$B_K = \frac{\langle \bar{K}_0 | [\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{s}\gamma_\mu(1-\gamma_5)d] | K_0 \rangle}{\frac{8}{3}\langle \bar{K}_0 | \bar{s}\gamma_\mu\gamma_5 d | 0 \rangle \langle 0 | \bar{s}\gamma_\mu\gamma_5 d | K_0 \rangle}$$

$$\hat{B}_K = C(\mu) B_K(\mu), \qquad C(\mu) = \alpha_s(\mu)^{-\frac{\gamma_0}{2b_0}} [1 + \alpha_s(\mu) J_3]$$

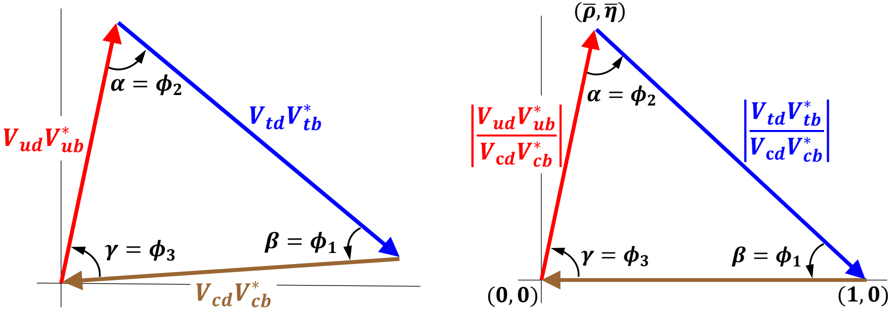
- Experiment:

$$\varepsilon_K = (2.228 \pm 0.011) \times 10^{-3} \times e^{i\phi_\varepsilon}$$

$$\phi_\varepsilon = 43.52(5)^\circ$$

ε_K on the lattice

Unitarity Triangle → $(\bar{\rho}, \bar{\eta})$



(a) (b)

Global UT Fit and Angle-Only-Fit (AOF)

Global UT Fit

- Input: $|V_{ub}|/|V_{cb}|$, Δm_d, $\Delta m_s/\Delta m_d$, ε_K, and $\sin(2\beta)$.
- Determine the UT apex $(\bar{\rho}, \bar{\eta})$.
- Take λ from

$$|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$$

  which comes from K_{l3} and $K_{\mu 2}$.
- Disadvantage: unwanted correlation between $(\bar{\rho}, \bar{\eta})$ and ε_K.

AOF

- Input: $\sin(2\beta)$, $\cos(2\beta)$, $\sin(\gamma)$, $\cos(\gamma)$, $\sin(2\beta+\gamma)$, $\cos(2\beta+\gamma)$, and $\sin(2\alpha)$.
- Determine the UT apex $(\bar{\rho}, \bar{\eta})$.
- Take λ from $|V_{us}| = \lambda + \mathcal{O}(\lambda^7)$, which comes from K_{l3} and $K_{\mu 2}$.
- Use $|V_{cb}|$ to determine A.

$$|V_{cb}| = A\lambda^2 + \mathcal{O}(\lambda^7)$$

- Advantage: NO correlation between $(\bar{\rho}, \bar{\eta})$ and ε_K.

Inputs of Angle-Only-Fit (AOF)

- $A_{\mathsf{CP}}(J/\psi\ K_s) \to S_{\psi K_s} = \sin(2\beta)$ with assumption of $S_{\psi K_s} \ggg C_{\psi K_s}$.
- $(B \to DK) + (B \to [K\pi]_D K) +$ (Dalitz method) give $\sin(\gamma)$ and $\cos(\gamma)$.
- $S(D^-\pi^+)$ and $S(D^+\pi^-)$ give $\sin(2\beta+\gamma)$ and $\cos(2\beta+\gamma)$.
- $(B^0 \to \pi^+\pi^-) + (B^0 \to \rho^+\rho^-) + (B^0 \to (\rho\pi)^0)$ give $\sin(2\alpha)$.
- Combining all of these gives β, γ, and α, which leads to the UT apex $(\bar\rho,\ \bar\eta)$.

Wolfenstein Parameters

Input Parameters for Angle-Only-Fit (AOF)

- ϵ_K, $\hat{B}_K$, and $|V_{cb}|$ are used as inputs to determine the UT angles in the global fit of `UTfit` and `CKMfitter`.
- Instead, we can use angle-only-fit result for the UT apex $(\bar{\rho}, \bar{\eta})$.
- Then, we can take λ independently from

$$|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$$

which comes from K_{l3} and $K_{\mu2}$.

- Use $|V_{cb}|$ instead of A.

$$|V_{cb}| = A\lambda^2 + \mathcal{O}(\lambda^7)$$

λ	0.22537(61)	[1] CKMfitter
	0.2255(6)	[1] UTfit
	0.2253(8)	[1] $|V_{us}|$ (AOF)
$\bar{\rho}$	0.117(21)	[1] CKMfitter
	0.124(24)	[1] UTfit
	0.139(29)	[2] UTfit (AOF)
$\bar{\eta}$	0.353(13)	[1] CKMfitter
	0.354(15)	[1] UTfit
	0.337(16)	[2] UTfit (AOF)

Input Parameters of B_K, V_{cb} and others

B_K

$\hat{B}_K$	$0.7661(99)$	[3] FLAG
	$0.7379(47)(365)$	[4] SWME

V_{cb}

V_{cb} $\times 10^{-3}$	$42.21(78)$	[5] Incl.
	$39.04(49)(53)(19)$	[6] Excl.

Others

G_F	$1.1663787(6) \times 10^{-5}$ GeV^{-2}	[1]
M_W	$80.385(15)$ GeV	[1]
$m_c(m_c)$	$1.275(25)$ GeV	[1]
$m_t(m_t)$	$163.3(2.7)$ GeV	[7]
η_{cc}	$1.72(27)$	[8]
η_{tt}	$0.5765(65)$	[9]
η_{ct}	$0.496(47)$	[10]
θ	$43.52(5)^\circ$	[1]
m_{K^0}	$497.614(24)$ MeV	[1]
ΔM_K	$3.484(6) \times 10^{-12}$ MeV	[1]
F_K	$156.2(7)$ MeV	[1]

ξ_0
Input Parameters

$$\xi_0 = \frac{\mathrm{Re}A_0}{\mathrm{Im}A_0}$$

ξ_0	$-1.63(19)(20) \times 10^{-4}$	[11]

- RBC-UKQCD collaboration performs lattice calculation of $\mathrm{Im}A_2$. From this result, ξ_0 can be obtained by the relation

$$\mathrm{Re}\left(\frac{\epsilon'_K}{\epsilon_K}\right) = \frac{1}{\sqrt{2}|\epsilon_K|}\omega\left(\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \xi_0\right).$$

  Other inputs ω, ϵ_K and ϵ'_K/ϵ_K are taken from the experimental values.
- Here, we choose an approximation of $\cos(\phi_{\epsilon'} - \phi_\epsilon) \approx 1$.
- $\phi_\epsilon = 43.52(5)$, $\phi_{\epsilon'} = 42.3(1.5)$

ϵ_K: FLAG $\hat{B}_K$, AOF of $(\bar\rho, \bar\eta)$, V_{us}

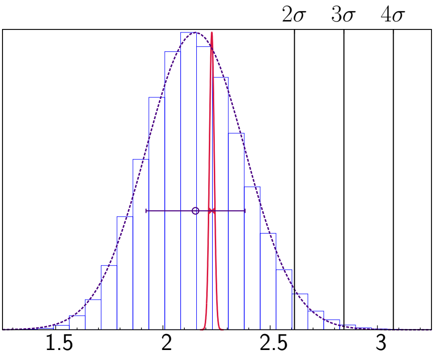


Figure: Inclusive V_{cb}

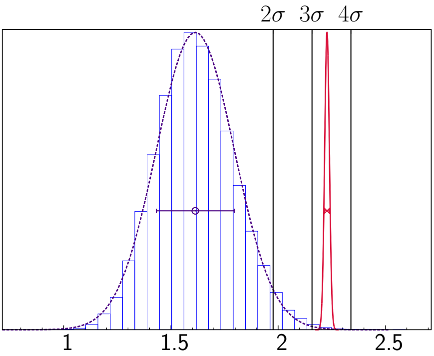


Figure: Exclusive V_{cb}

- With exclusive V_{cb}, it shows 3.4σ tension.

$$\epsilon_K^{Exp} = 2.228(11) \times 10^{-3}$$
$$\epsilon_K^{SM} = 1.61(18) \times 10^{-3}$$

ϵ_K: SWME $\hat{B}_K$, AOF of $(\bar{\rho}, \bar{\eta})$, V_{us}

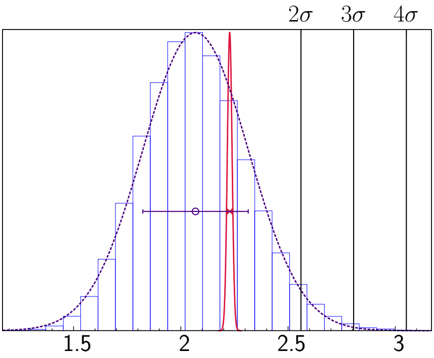


Figure: Inclusive V_{cb}

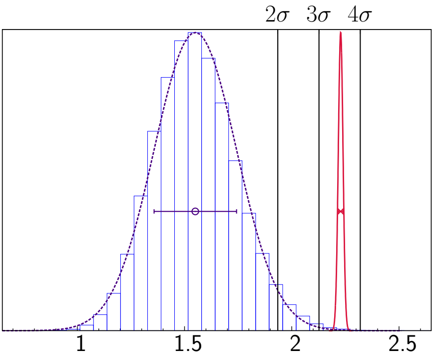


Figure: Exclusive V_{cb}

- With exclusive V_{cb}, it shows 3.5σ tension.

$$\epsilon_K^{Exp} = 2.228(11) \times 10^{-3}$$
$$\epsilon_K^{SM} = 1.55(19) \times 10^{-3}$$

Current Status of ε_K

- FLAG 2014: (in units of 1.0×10^{-3}, AOF)

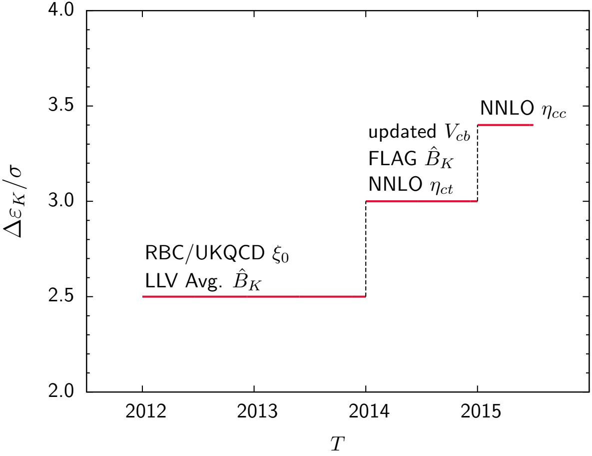
$$\varepsilon_K = 1.61 \pm 0.18 \qquad \text{for Exclusive } V_{cb} \text{ (Lattice QCD)}$$
$$\varepsilon_K = 2.15 \pm 0.23 \qquad \text{for Inclusive } V_{cb} \text{ (QCD Sum Rule)}$$

- Experiments:

$$\varepsilon_K = 2.228 \pm 0.011$$

- Hence, we observe 3.4 σ difference between the SM theory (Lattice QCD) and experiments.
- What does this mean? $\longrightarrow$ Breakdown of SM ?

Time Evolution of $\Delta\varepsilon_K$ on the Lattice



- $\Delta\varepsilon_K \equiv \varepsilon_K^{\text{exp}} - \varepsilon_K^{\text{SM}}$

Error Budget of Exclusive ε_K

source	error (%)	memo
V_{cb}	39.3	Exclusive (FNAL/MILC)
$\bar{\eta}$	20.4	AOF
η_{ct}	16.9	$c-t$ Box
η_{cc}	7.1	$c-c$ Box
$\bar{\rho}$	5.4	AOF
m_t	2.4	
ξ_0	2.2	$\text{Im}(A_0)/\text{Re}(A_0)$
ξ_{LD}	2.0	Long-distance
$\hat{B}_K$	1.5	FLAG
m_c	1.0	Charm quark mass
⋮	⋮	

BSM Corrections to ε_K

BSM Four Fermion Operators I

- New $\Delta S = 2$ four-fermion operators that contribute to Kaon Mixing

$$
\begin{aligned}
Q_1 &= [\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{s}\gamma_\mu(1-\gamma_5)d] \quad \rightarrow \quad B_K \\
Q_2 &= [\bar{s}^a(1-\gamma_5)d^a][\bar{s}^b(1-\gamma_5)d^b] \\
Q_3 &= [\bar{s}^a\sigma_{\mu\nu}(1-\gamma_5)d^a][\bar{s}^b\sigma_{\mu\nu}(1-\gamma_5)d^b] \\
Q_4 &= [\bar{s}^a(1-\gamma_5)d^a][\bar{s}^b(1+\gamma_5)d^b] \\
Q_5 &= [\bar{s}^a\gamma_\mu(1-\gamma_5)d^a][\bar{s}^b\gamma_\mu(1+\gamma_5)d^b]
\end{aligned}
$$

BSM Four Fermion Operators II

- In general, there are additional operators that can be obtained from $Q_{1,2,3}$ by changing $L \to R$, but we do not consider. (Matrix elements for left- and right-handed operators are the same for Kaon mixing.)

- $$\mathcal{H}_{eff}^{\Delta S=2} = \sum_{i=1}^{5} C_i(\mu) Q_i(\mu)$$

- With the constraint from experiment, calculating corresponding hadronic matrix elements

$$\langle \bar{K}_0 | Q_i | K_0 \rangle$$

can impose strong constraints on BSM models.

Lattice Calculation : BSM B-parameters

- **B-parameters**

$$B_K = \frac{\langle \bar{K}_0 | Q_1 | K_0 \rangle}{8/3 \langle \bar{K}_0 | \bar{s}\gamma_0\gamma_5 d | 0 \rangle \langle 0 | \bar{s}\gamma_0\gamma_5 d | K_0 \rangle} \qquad \text{SM, BSM}$$

$$B_i = \frac{\langle \bar{K}_0 | Q_i | K_0 \rangle}{N_i \langle \bar{K}_0 | \bar{s}\gamma_5 d | 0 \rangle \langle 0 | \bar{s}\gamma_5 d | K_0 \rangle} \qquad \text{BSM}$$

Where, $i = 2, 3, 4, 5$ and $(N_2, N_3, N_4, N_5) = (5/3, 4, -2, 4/3)$

- **Golden Combinations :** G_i

$$G_{23} \equiv \frac{B_2}{B_3} \qquad G_{45} \equiv \frac{B_4}{B_5}$$

$$G_{24} \equiv B_2 \cdot B_4 \qquad G_{21} \equiv \frac{B_2}{B_K}$$

1. Advantage: no SU(2) chiral logs at NLO order in G_i (Golden Combinations)

$N_f = 2+1$ QCD: MILC asqtad lattices

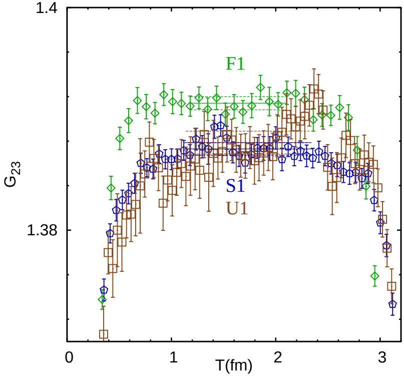
a (fm)	am_l/am_s	geometry	ens×meas	ID	Status
0.09	0.0062/0.0310	$28^3 \times 96$	995×9	F1	
0.09	0.0031/0.0310	$40^3 \times 96$	959×9	F2	
0.09	0.0093/0.0310	$28^3 \times 96$	949×9	F3	
0.09	0.0124/0.0310	$28^3 \times 96$	1995×9	F4	
0.09	0.00465/0.0310	$32^3 \times 96$	651×9	F5	
0.09	0.0062/0.0186	$28^3 \times 96$	950×9	F6	New
0.09	0.0031/0.0186	$40^3 \times 96$	701×9	F7	New
0.09	0.00155/0.0310	$64^3 \times 96$	790×9	F9	New
0.06	0.0036/0.018	$48^3 \times 144$	749×9	S1	
0.06	0.0025/0.018	$56^3 \times 144$	799×9	S2	
0.06	0.0072/0.018	$48^3 \times 144$	593×9	S3	
0.06	0.0054/0.018	$48^3 \times 144$	582×9	S4	
0.06	0.0018/0.018	$64^3 \times 144$	572×9	S5	New
0.045	0.0030/0.015	$64^3 \times 192$	747×1	U1	

Data Analysis

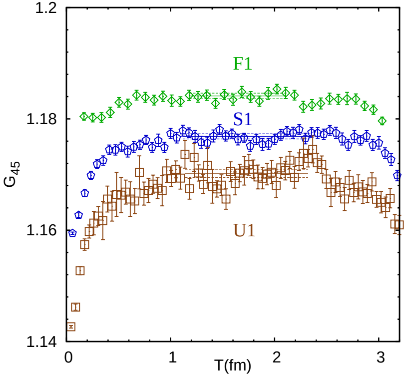
- **Calculate raw data**
  Calculate B_K and G_i for different valence quark mass combinations for each gauge ensemble. ($\overline{\text{MS}}$ scheme with NDR.)
- **Chiral fitting (valence quarks)**
  X-fit: Fix valence strange quark mass and extrapolate the light quark mass m_x to physical down quark mass.
  Y-fit: Extrapolate m_y to physical strange quark mass.
- **RG Evolution**
  Obtain results at μ_f =2GeV or 3GeV by running from $\mu_i = 1/a$.
- **Continuum-chiral extrapolation (sea quarks)**
  Perform [1–3] for different lattices and extrapolate to $a = 0$ and to physical sea quark masses.

Raw Data of G_{23} and G_{45}

- We compare three ensembles which have the same ratio of sea quark mass $m_\ell/m_s = 1/5$.



(a)



(b)

SchPT X-fit and Y-fit of B_K

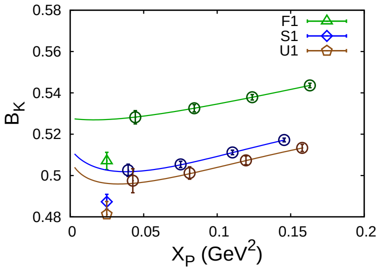
- NNNLO X-fit

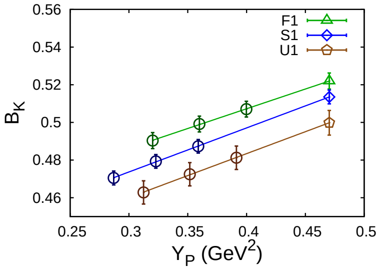
$$\begin{aligned} B_K & \quad \text{(NNNLO)} \\ = & \quad c_1 F_0 + c_2 X + c_3 X^2 \\ + & \quad c_4 X^2 (\ln(X))^2 \\ + & \quad c_5 X^2 \ln(X) + c_6 X^3 \end{aligned}$$

Here, F_0 contains the chiral logs. Bayesian constraints on $c_{4-6} = 0 \pm 1$.

- Y-fit(U1 ensemble)

$$B_K(\text{Y-fit}) = b_1 + b_2 Y_P$$





SchPT X-fit and Y-fit of G_{23}

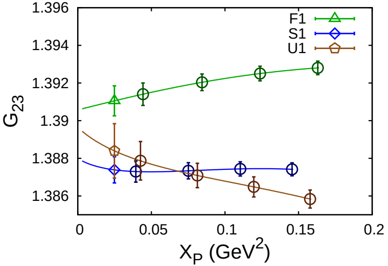
- NNNLO X-fit

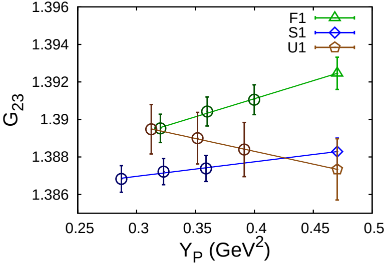
$$\begin{aligned} G_{23} & \quad (\text{NNNLO}) \\ = & \quad c_1 + c_2 X + c_3 X^2 \\ + & \quad c_4 X^2 (\ln(X))^2 \\ + & \quad c_5 X^2 \ln(X) + c_6 X^3 \end{aligned}$$

Bayesian constraints on $c_{4-6} = 0 \pm 1$.

- Y-fit(U1 ensemble)

$$G_{23}(\text{Y-fit}) = b_1 + b_2 Y_P$$





SchPT X-fit and Y-fit of G_{45}

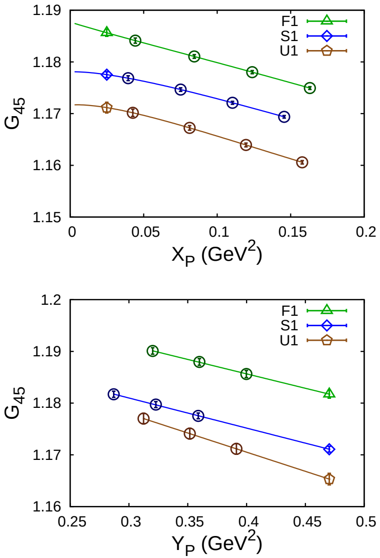
- NNNLO X-fit

$$
\begin{aligned}
G_{45} & \quad \text{(NNNLO)} \\
= & \quad c_1 + c_2 X + c_3 X^2 \\
+ & \quad c_4 X^2 (\ln(X))^2 \\
+ & \quad c_5 X^2 \ln(X) + c_6 X^3
\end{aligned}
$$

Bayesian constraints on $c_{4-6} = 0 \pm 1$.

- Y-fit(U1 ensemble)

$$G_{45}(\text{Y-fit}) = b_1 + b_2 Y_P$$



Chiral-Continuum Fit

- We use 14 data points from 14 MILC ensembles in the fitting. We extrapolate the results to physical point $a = 0$, $L_P = m^2_{\pi_0}$, and $S_P = m^2_{s\bar{s},\text{phys}}$.
- Fitting functional forms come from the SU(2) SChPT theory.
- $\Delta S_P \equiv S_P - m^2_{s\bar{s},\text{phys}}$

fit type	fitting functional form	Bayesian Constraints
F_B^1	$d_1 + d_2 \frac{L_P}{\Lambda_\chi^2} + d_3 \frac{\Delta S_P}{\Lambda_\chi^2} + d_4 (a\Lambda_Q)^2$	$d_2 \cdots d_4 = 0 \pm 2$
F_B^2	$F_B^1 + d_5 (a\Lambda_Q)^2 \frac{L_P}{\Lambda_\chi^2} + d_6 (a\Lambda_Q)^2 \frac{\Delta S_P}{\Lambda_\chi^2}$	$d_2 \cdots d_6 = 0 \pm 2$
F_B^4	$F_B^2 + d_7 (a\Lambda_Q)^2 \alpha_s + d_8 \alpha_s^2 + d_9 (a\Lambda_Q)^4$	$d_2 \cdots d_9 = 0 \pm 2$

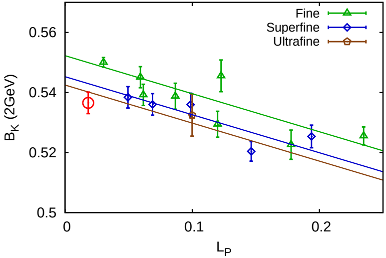
Chiral-Continuum Fit : Fitting quality

- We can see that the χ^2 values for fitting functional forms get saturated as we add higher order terms in the fitting functional forms. We choose F_B^1-fit results as central values for B_K, G_{24}, and G_{21}. For G_{23} and G_{45}, we choose those of F_B^4 as the central values.

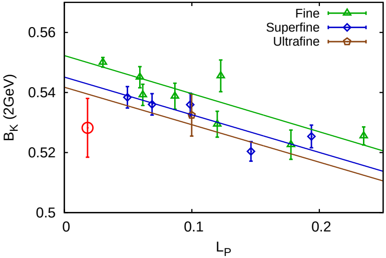
fit type	B_K	G_{23}	G_{45}	G_{24}	G_{21}
F_B^1	1.49	2.01	4.06	1.08	1.25
F_B^2	1.49	1.86	3.75	1.02	1.22
F_B^3	1.48	1.42	1.53	0.93	1.19
F_B^4	1.48	1.32	1.38	0.91	1.18
F_B^5	1.48	1.30	1.33	0.90	1.17
F_B^6	1.48	1.22	1.15	0.88	1.13

Chiral-Continuum Fit of B_K

- The result of Chiral-Continuum fit. The straight line in the plots represents the value of fitting function at fixed S_P and a^2 for fine ($a \approx 0.09$fm), superfine ($a \approx 0.06$fm), and ultrafine ($a \approx 0.045$fm) gauge ensembles.

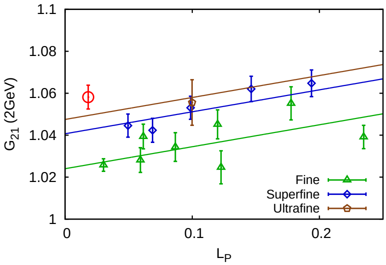


(c) F_B^1, $\chi^2/\text{dof} = 1.49$

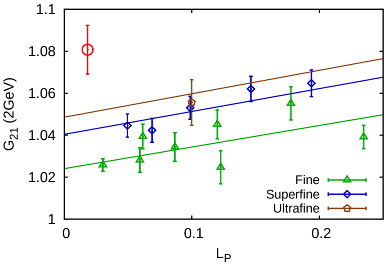


(d) F_B^4, $\chi^2/\text{dof} = 1.48$

Chiral-Continuum Fit of $G_{21} = B_2/B_K$

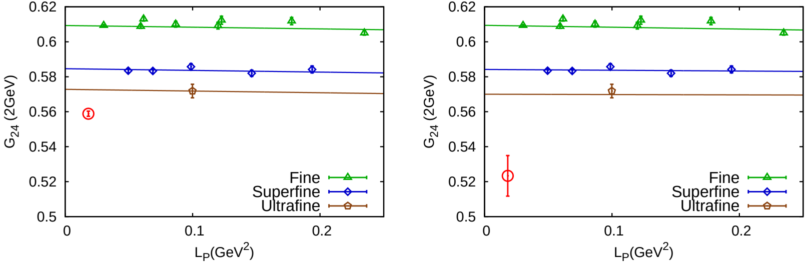


(e) F_B^1 , χ^2/dof $= 1.25$



(f) F_B^4 , χ^2/dof $= 1.18$

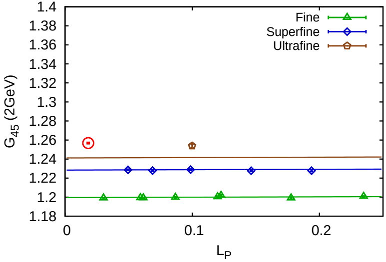
Chiral-Continuum Fit of $G_{24} = B_2 \cdot B_4$



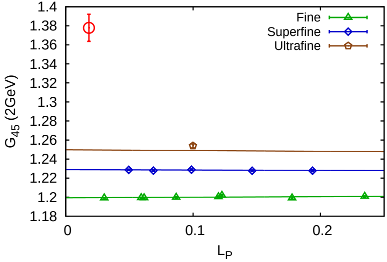
(g) F_B^1 , $\chi^2/\text{dof} = 1.08$ (h) F_B^4 , $\chi^2/\text{dof} = 0.91$

Chiral-Continuum Fit of $G_{45} = B_4/B_5$

- The fitting quality for F_B^1 fit is terribly poor.
- Hence, we have to choose the F_B^4 fit as our central value.



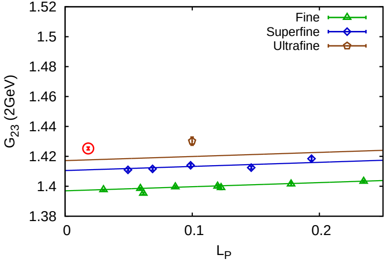
(i) F_B^1 , χ^2/dof $= 4.07$



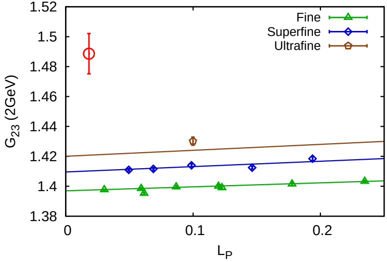
(j) F_B^4 , χ^2/dof $= 1.39$

Chiral-Continuum Fit of $G_{23} = B_2/B_3$

- The fitting quality for F_B^1 fit is poor.
- Hence, we have to choose the F_B^4 fit as our central value.



(k) F_B^1 , $\chi^2/\text{dof} = 2.01$



(l) F_B^4 , $\chi^2/\text{dof} = 1.33$

Historical Progress

- 2013(PRD)[1] $\rightarrow$ 2014(ens) : Add more gauge ensembles.
- 2014(ens) $\rightarrow$ 2014(A.D.) : Correct two-loop contribution to pseudoscalar anomalous dimension.
- 2014(A.D.) $\rightarrow$ 2014(final) : Change fit type from F_B^1 to F_B^4 for G_{23} and G_{45}.

$\mu = 3$GeV	2013(PRD)[1]	2014(ens)	2014(A.D.)	2014(final)
B_K	0.519(7)(23)	0.518(3)	0.518(4)	0.518(4)(24)
B_2	0.549(3)(28)	0.547(1)	0.525(1)	0.525(1)(25)
B_3^{Buras}	0.390(2)(17)	0.390(1)	0.375(1)	0.358(4)(23)
B_3^{SUSY}	0.790(30)	0.783(2)	0.750(2)	0.774(6)(34)
B_4	1.033(6)(46)	1.024(1)	0.981(3)	0.981(3)(71)
B_5	0.855(6)(43)	0.853(3)	0.817(2)	0.748(9)(76)

[1] SWME Collaboration, Phys.ReV. **D88**,071503(2013)

Comparison in 2014

- We obtain B_i from results of G_i and B_K.

$$\text{Dominant error} \begin{cases} \text{Perturbative matching : } 4.4\% \\ \text{Chiral-continuum extrap : } 1.3 \sim 10.1\% \end{cases}$$

- RBC-UKQCD and ETM use RI-MOM for matching.

	SWME14	RBC-UK12	ETM12
B_K	0.518(4)(24)	0.53(2)	0.51(2)
B_2	0.525(1)(25)	0.43(5)	0.47(2)
B_3^{Buras}	0.358(4)(23)	N.A.	N.A.
B_3^{SUSY}	0.774(6)(34)	0.75(9)	0.78(4)
B_4	0.981(3)(71)	0.69(7)	0.75(3)
B_5	0.748(9)(76)	0.47(6)	0.60(3)

Comparison in 2015

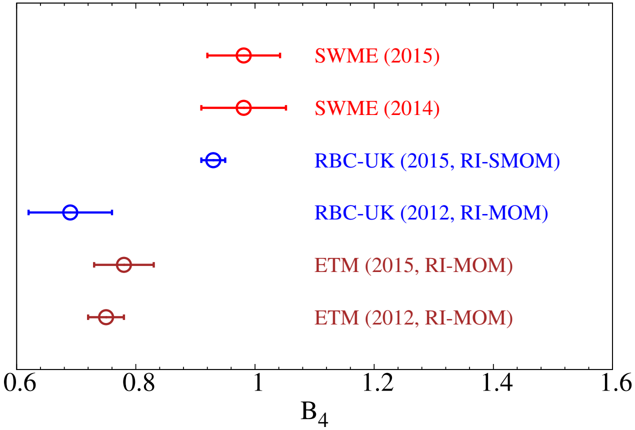
- We obtain B_i from G_i and B_K using the perturbative matching.
- RBC-UKQCD: RI-MOM (2012) $\rightarrow$ RI-SMOM (2015)
- ETM still uses RI-MOM for matching.

	SWME14	RBC-UK12	RBC-UK15	ETM12	ETM15
B_K	0.518(4)(24)	0.53(2)	0.53(2)	0.51(2)	0.51(2)
B_2	0.525(1)(25)	0.43(5)	0.54(3)	0.47(2)	0.46(3)
B_3^{Buras}	0.358(4)(23)	N.A.	N.A.	N.A.	N.A.
B_3^{SUSY}	0.774(6)(34)	0.75(9)	0.79(7)	0.78(4)	0.79(5)
B_4	0.981(3)(71)	0.69(7)	0.93(2)	0.75(3)	0.78(5)
B_5	0.748(9)(76)	0.47(6)	0.68(5)	0.60(3)	0.49(4)

※ RBC-UK15: the errors are preliminary (the error budget is incomplete).

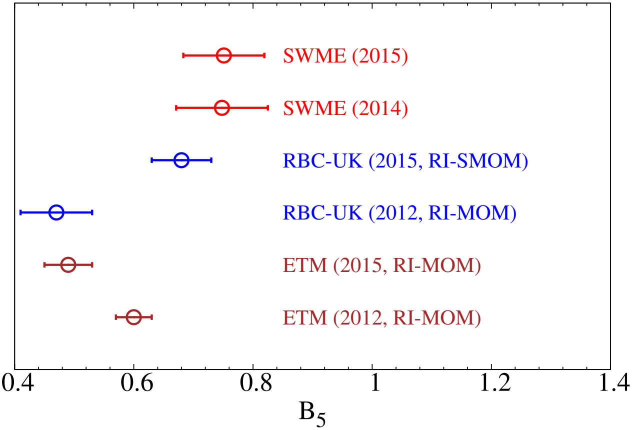
Comparison of B_4

- SWME agree with RBC-UKQCD (2015, RI-SMOM)
- SWME do NOT agree with RBC-UKQCD (2012, RI-MOM) and ETM.



Comparison of B_5

- SWME agree with RBC-UKQCD (2015, RI-SMOM)
- SWME do NOT agree with RBC-UKQCD (2012, RI-MOM) and ETM.



Conclusion

- Our results (SWME) agrees with those of RBC-UKQCD (2015, RI-SMOM).
- Our results of B_4 and B_5 do NOT agree with the results of RBC-UKQCD (2012, RI-MOM) and ETM collaborations.
- The large difference between RBC-UKQCD (2015, RI-SMOM) and RBC-UKQCD (2012, RI-MOM) indicates that the systematic uncertainty in non-perturbative renormalization might be underestimated for BSM B-parameters.
- We plan to use RI-MOM and RI-SMOM to obtain the matching factors in near future.

V_{cb} on the lattice

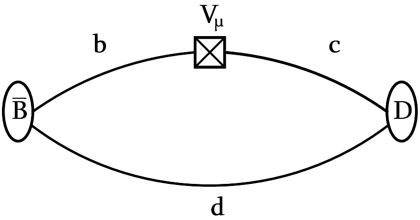
How to obtain V_{cb}

- Exclusive V_{cb} determination.
- $\bar{B} \to D + \ell + \bar{\nu}_\ell$
- $\bar{B} \to D^* + \ell + \bar{\nu}_\ell$

What to calculate on the lattice.

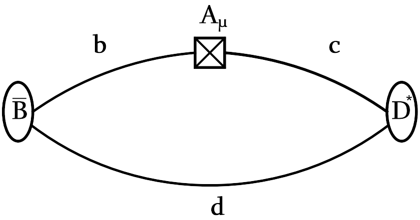
- $\langle D|Q_1|\bar{B}\rangle$ with $Q_1 = V_\mu,\ S$.

$$V_\mu = \bar{b}\gamma_\mu c$$
$$S = \bar{b}c$$



- $\langle D^*|Q_2|\bar{B}\rangle$ with $Q_2 = A_\mu,\ P$.

$$A_\mu = \bar{b}\gamma_\mu\gamma_5 c$$
$$P = \bar{b}\gamma_5 c$$



B_s meson mass

Motivation

- In heavy flavor physics, V_{cb} is of enormous interest.
- The dominant error in ϵ_K comes from V_{cb}.

$$\begin{cases} 39.3\% & \leftarrow V_{cb} \\ 1.5\% & \leftarrow \hat{B}_K \end{cases}$$

- 3.4σ tension is observed using most up to date input parameters.

$$|\epsilon_K|^{\text{exp}} = 2.228(11) \times 10^{-3} \qquad \text{(PDG)}$$

$$|\epsilon_K|^{\text{SM}} = 1.61(18) \times 10^{-3} \qquad \text{(FLAG } \hat{B}_K, \text{FNAL/MILC } V_{cb}\text{)}$$

- More precise determination of V_{cb} might lead to larger tension.
- Because the dominant error in V_{cb} comes from heavy quark discretization effect, we plan to use the OK action for the form factor calculation of the semi-leptonic decays

$$\bar{B} \to D^* l \nu_l \,, \qquad \bar{B} \to D l \nu_l .$$

- Here, we will verify the improvement in B meson spectrum.

OK Action (mass form)

$$S_{\rm OK} = S_{\rm Fermilab} + S_{\rm new}\,, \qquad S_{\rm Fermilab} = S_0 + S_B + S_E$$

$$S_0 = m_0 \sum_x \bar\psi(x)\psi(x) + \sum_x \bar\psi(x)\gamma_4 D_4 \psi(x) - \tfrac{1}{2} a \sum_x \bar\psi(x) \triangle_4 \psi(x)$$

$$+ \zeta \sum_x \bar\psi(x) \vec\gamma \cdot \vec D \psi(x) - \tfrac{1}{2} r_s \zeta a \sum_x \bar\psi(x) \triangle^{(3)} \psi(x)$$

$$= \mathcal{O}(1) + \mathcal{O}(\lambda) \qquad [\lambda \sim a\Lambda,\ \Lambda/m_Q]$$

$$S_B = -\tfrac{1}{2} c_B \zeta a \sum_x \bar\psi(x) i \vec\Sigma \cdot \vec B \psi(x) \to \mathcal{O}(\lambda)$$

$$S_E = -\tfrac{1}{2} c_E \zeta a \sum_x \bar\psi(x) \vec\alpha \cdot \vec E \psi(x) \to \mathcal{O}(\lambda^2) \qquad (c_E \neq c_B : \text{ OK action})$$

$$m_0 = \frac{1}{2\kappa_t} - (1 + 3 r_s \zeta + 18 c_4)$$

[M. B. Oktay and A. S. Kronfeld, PRD **78**, 014504 (2008)]

[A. El-Khadra, A. S. Kronfeld and P. B. Mackenzie, PRD **55**, 3933 (1997)]

OK Action (mass form)

$$
\begin{aligned}
S_{\text{new}} = \mathcal{O}(\lambda^3) = & \; c_1 a^2 \sum_x \bar{\psi}(x) \sum_i \gamma_i D_i \triangle_i \psi(x) \\
& + c_2 a^2 \sum_x \bar{\psi}(x) \{ \vec{\gamma} \cdot \vec{D}, \triangle^{(3)} \} \psi(x) \\
& + c_3 a^2 \sum_x \bar{\psi}(x) \{ \vec{\gamma} \cdot \vec{D}, i \vec{\Sigma} \cdot \vec{B} \} \psi(x) \\
& + c_{EE} a^2 \sum_x \bar{\psi}(x) \{ \gamma_4 D_4, \vec{\alpha} \cdot \vec{E} \} \psi(x) \\
& + c_4 a^3 \sum_x \bar{\psi}(x) \sum_i \triangle_i{}^2 \psi(x) \\
& + c_5 a^3 \sum_x \bar{\psi}(x) \sum_i \sum_{j \neq i} \{ \mathrm{i} \Sigma_i B_i, \triangle_j \} \psi(x)
\end{aligned}
$$

OK Action: Tadpole Improvement (hopping form)

$$c_5 a^3 \bar{\psi}(x) \sum_i \sum_{j\neq i} \{\mathrm{i}\Sigma_i B_{i\,\mathrm{lat}}, \triangle_{j\,\mathrm{lat}}\}\psi(x)$$

$$= \mathrm{i}\frac{2\tilde{c}_5\tilde{\kappa}_t}{4u_0^2}\bar{\psi}_x \sum_i \Sigma_i T_i^{(3)}\psi_x - \mathrm{i}\frac{32\tilde{c}_5\tilde{\kappa}_t}{2u_0^3}\bar{\psi}_x \vec{\Sigma}\cdot\vec{B}\psi_x$$

$$+ \mathrm{i}\frac{2\tilde{c}_5\tilde{\kappa}_t}{u_0^4}\bar{\psi}_x \sum_i \Big(-\frac{1}{4}\Sigma_i T_i^{(3)} + \sum_{j\neq i}\{\Sigma_i B_i, (T_j + T_{-j})\}\Big)\psi_x$$

$$T_i^{(3)} \equiv \sum_{j,k=1}^{3} \epsilon_{ijk}\Big(T_{-k}(T_j - T_{-j})T_k - T_k(T_j - T_{-j})T_{-k}\Big)$$

Measurement

Gauge Ensemble, Heavy Quark κ, Meson Momentum

- MILC asqtad $N_f = 2 + 1$

a(fm)	$N_L^3 \times N_T$	β	am_l'	am_s'	u_0	a^{-1}(GeV)	N_{conf}	$N_{t_{\text{src}}}$
0.12	$20^3 \times 64$	6.79	0.02	0.05	0.8688	1.683^{+43}_{-16}	500	6

- 11 momenta $|\boldsymbol{p}a| = 0, 0.099, \cdots, 1.26$

Measurement: Interpolating Operator

- Meson correlator

$$C(t,\boldsymbol{p}) = \sum_{\boldsymbol{x}} e^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \langle \mathcal{O}^\dagger(t,\boldsymbol{x})\mathcal{O}(0,\mathbf{0})\rangle$$

- Heavy-light meson interpolating operator

$$\mathcal{O}_{\mathsf{t}}(x) = \bar{\psi}_\alpha(x)\Gamma_{\alpha\beta}\Omega_{\beta\mathsf{t}}(x)\chi(x)$$

$$\Gamma = \begin{cases} \gamma_5 & \text{(Pseudo-scalar)} \\ \gamma_\mu & \text{(Vector)} \end{cases} \quad , \ \Omega(x) \equiv \gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\gamma_4^{x_4}$$

- Quarkonium interpolating operator

$$\mathcal{O}(x) = \bar{\psi}_\alpha(x)\Gamma_{\alpha\beta}\psi_\beta(x)$$

[Wingate *et al.*, PRD **67**, 054505 (2003) , C. Bernard *et al.*, PRD **83**, 034503 (2011)]

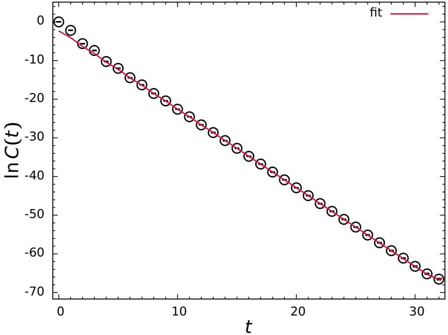
Correlator Fit

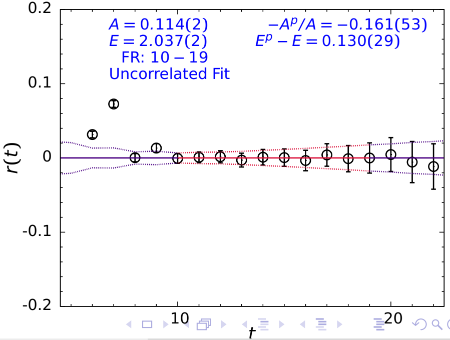
- fit function

$$f(t) = A\{e^{-Et} + e^{-E(T-t)}\} + (-1)^t A^p\{e^{-E^p t} + e^{-E^p(T-t)}\}$$

- fit residual

$$r(t) = \frac{C(t) - f(t)}{|C(t)|}, \text{where } C(t) \text{ is data.}$$





Correlator Fit: Effective Mass

$$m_{\text{eff}}(t) = \frac{1}{2} \ln\left(\frac{C(t)}{C(t+2)}\right)$$

For small t,

$$\begin{aligned} C(t) \cong& A(e^{-Et} + \beta e^{-(E+\Delta E)t}) \\ =& A e^{-Et}(1 + \beta e^{-(\Delta E)t}) \ , \end{aligned}$$

$$\begin{cases} \beta > 0 & \text{(excited state)} \\ \beta \sim -(-1)^t & \text{(time parity state)} \end{cases}$$

$$m_{\text{eff}} \approx E + \beta(\Delta E) e^{-(\Delta E)t}$$

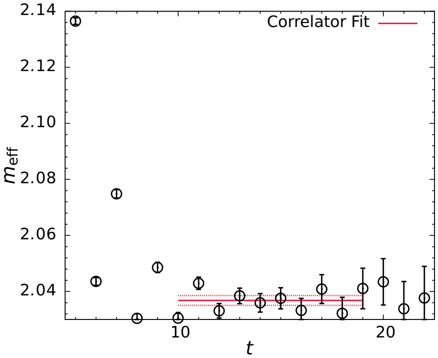


Figure: $[\overline{Q}q$, PS, $\kappa = 0.041$, $\boldsymbol{p} = \boldsymbol{0}]$

Dispersion Relation

$$E = M_1 + \frac{\boldsymbol{p}^2}{2M_2} - \frac{(\boldsymbol{p}^2)^2}{8M_4^3} - \frac{a^3 W_4}{6} \sum_i p_i^4$$

$$\widetilde{E} = E + \frac{a^3 W_4}{6} \sum_i p_i^4 \,, \quad \boldsymbol{n} = (2,2,1)\,,(3,0,0)$$

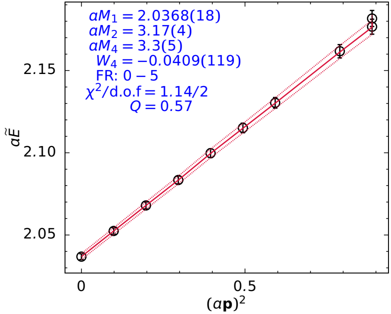


Figure: $[\overline{Q}q$, PS, $\kappa = 0.041]$

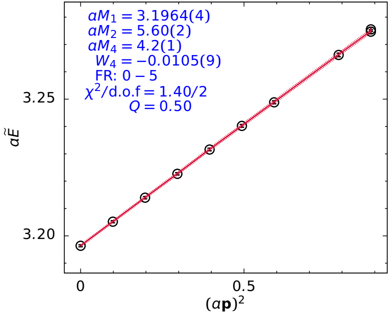


Figure: $[\overline{Q}Q$, PS, $\kappa = 0.041]$

Improvement Test: Inconsistency Parameter

$$I \equiv \frac{2\delta M_{\overline{Q}q} - (\delta M_{\overline{Q}Q} + \delta M_{\overline{q}q})}{2M_{2\overline{Q}q}} = \frac{2\delta B_{\overline{Q}q} - (\delta B_{\overline{Q}Q} + \delta B_{\overline{q}q})}{2M_{2\overline{Q}q}}$$

$$M_{1\overline{Q}q} = m_{1\overline{Q}} + m_{1q} + B_{1\overline{Q}q} \qquad \delta M_{\overline{Q}q} = M_{2\overline{Q}q} - M_{1\overline{Q}q}$$

$$M_{2\overline{Q}q} = m_{2\overline{Q}} + m_{2q} + B_{2\overline{Q}q} \qquad \delta B_{\overline{Q}q} = B_{2\overline{Q}q} - B_{1\overline{Q}q}$$

[S. Collins *et al.*, NPB **47**, 455 (1996) , A. S. Kronfeld, NPB **53**, 401 (1997)]

- Inconsistency parameter I can be used to examine the improvements by $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action. OK action is designed to improve these terms and matched at tree-level.
- Binding energies B_1 and B_2 are of order $\mathcal{O}(\boldsymbol{p}^2)$. Because the kinetic meson mass M_2 appears with a factor $\boldsymbol{p}^2$, the leading contribution of binding energy B_2 generated by $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action.

$$E = M_1 + \frac{\boldsymbol{p}^2}{2M_2} + \cdots = M_1 + \frac{\boldsymbol{p}^2}{2(m_{2\overline{Q}} + m_{2q})}\left[1 - \frac{B_{2\overline{Q}q}}{(m_{2\overline{Q}} + m_{2q})} + \cdots\right] + \cdots$$

Improvement Test: Inconsistency Parameter

$$I \cong \frac{2\delta M_{\overline{Q}q} - \delta M_{\overline{Q}Q}}{2M_{2\overline{Q}q}} \cong \frac{2\delta B_{\overline{Q}q} - \delta B_{\overline{Q}Q}}{2M_{2\overline{Q}q}}$$

- Considering non-relativistic limit of quark and anti-quark system, for S-wave case ($\mu_2^{-1} = m_{2\overline{Q}}^{-1} + m_{2q}^{-1}$),

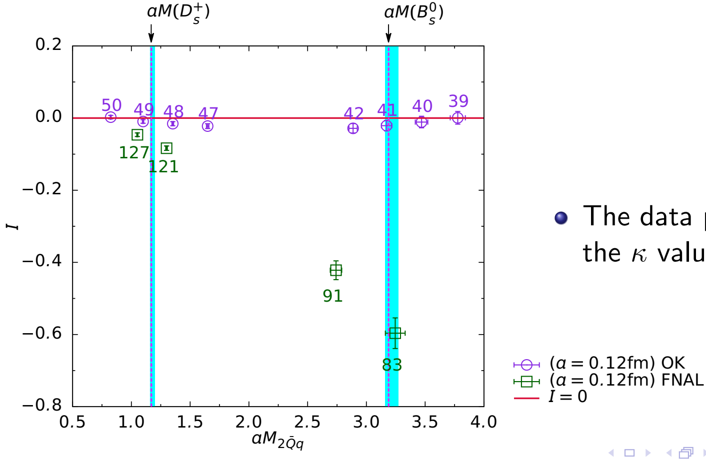
$$\begin{aligned} \delta B_{\overline{Q}q} &= \frac{5}{3}\frac{\langle \boldsymbol{p}^2 \rangle}{2\mu_2}\left[\mu_2\left(\frac{m_{2\overline{Q}}^2}{m_{4\overline{Q}}^3} + \frac{m_{2q}^2}{m_{4q}^3}\right) - 1\right] \qquad (m_4 : c_1, c_3) \\ &+ \frac{4}{3}a^3\frac{\langle \boldsymbol{p}^2 \rangle}{2\mu_2}\mu_2(w_{4\overline{Q}} m_{2\overline{Q}}^2 + w_{4q} m_{2q}^2) \qquad (w_4 : c_2, c_4) \\ &+ \mathcal{O}(p^4) \end{aligned}$$

[A. S. Kronfeld, NPB **53**, 401 (1997) , C. Bernard *et al.*, PRD **83**, 034503 (2011)]

- Leading contribution of $\mathcal{O}(\boldsymbol{p}^2)$ in δB vanishes when $w_4 = 0$, $m_2 = m_4$, not only for S-wave states but also for higher harmonics.

Improvement Test: Inconsistency Parameter

- The coarse ($a = 0.12$fm) ensemble data covers the B_s^0 mass and shows significant improvement compared to the Fermilab action.



- The data point labels denote the κ values.

Improvement Test: Hyperfine Splitting Δ

$$\Delta_1 = M_1^* - M_1\,,\ \Delta_2 = M_2^* - M_2$$

Recall,

$$\begin{aligned} M^{(*)}_{1\overline{Q}q} &= m_{1\overline{Q}} + m_{1q} + B^{(*)}_{1\overline{Q}q} \\ M^{(*)}_{2\overline{Q}q} &= m_{2\overline{Q}} + m_{2q} + B^{(*)}_{2\overline{Q}q} \\ \delta B^{(*)} &= B_2^{(*)} - B_1^{(*)} \end{aligned}$$

Then,

$$\Delta_2 = \Delta_1 + \delta B^* - \delta B$$

- The difference in hyperfine splittings $\Delta_2 - \Delta_1$ also can be used to examine the improvement from $\mathcal{O}(\boldsymbol{p}^4)$ terms in the action.

Improvement Test: Hyperfine Splitting Δ

$$\Delta_2 = \Delta_1 + \delta B^* - \delta B$$

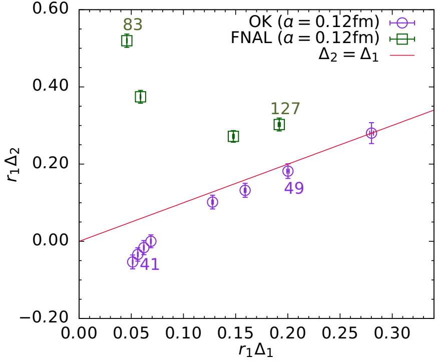


Figure: Quarkonium

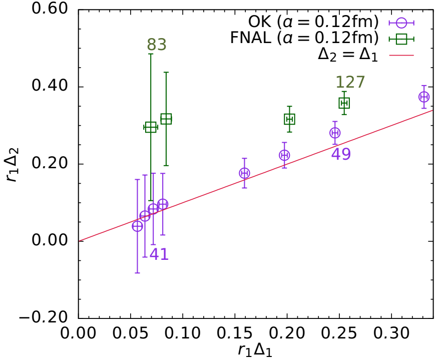


Figure: Heavy-light

Conclusion and Outlook

- Inconsistency parameter shows that the OK action clearly improves $\mathcal{O}(\boldsymbol{p}^4)$ terms.
- Hyperfine splitting shows that the OK action clearly improves the higher dimension magnetic effects for the quarkonium.
- We plan to calculate V_{cb} with the highest precision possible.
- Improved current relevant to the decay $\bar{B} \to D^* l\nu$ at zero recoil is needed. (Jon A. Bailey and J. Leem)
- We plan to calculate the 1-loop coefficients for c_B and c_E in the OK action. (Y.C. Jang)
- Highly optimized inverter using QUDA will be available soon. (Y.C. Jang)

Grand Challenges in the front

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4. In addition, we need to obtain the matching factor using NPR (Jangho Kim) and using the two-loop perturbation theory ($\cdots$).
   ※ matching error $<$ 1.0%

Tentative Goals (2)

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4. Our goal is to determine V_{cb} with its statistical and systematic error $\leq 0.5\%$.

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3. We need $N_f = 2 + 1 + 1$ calculation on the lattice. MILC provides HISQ ensembles with $N_f = 2 + 1 + 1$.
4. As a by-product, a substantial gain is that the charm quark mass dependence might be under control in this way. (Brod and Gorbahn)

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3. As a result, we would like to guide the whole particle physics community into a new world beyond the standard model.

Thank God for your help !!!

References for the Input Parameters I

[1] K.A. Olive et al.
Review of Particle Physics.
Chin.Phys., C38:090001, 2014.

[2] http://www.utfit.org/UTfit/ResultsSummer2014PostMoriondSM.

[3] Sinya Aoki, Yasumichi Aoki, Claude Bernard, Tom Blum, Gilberto Colangelo, et al.
Review of lattice results concerning low energy particle physics.
2013.

[4] Taegil Bae et al.
Improved determination of B_K with staggered quarks.
Phys.Rev., D89:074504, 2014.

[5] Andrea Alberti, Paolo Gambino, Kristopher J. Healey, and Soumitra Nandi.
Precision Determination of the Cabibbo-Kobayashi-Maskawa Element V_{cb}.
Phys.Rev.Lett., 114(6):061802, 2015.

[6] Jon A. Bailey, A. Bazavov, C. Bernard, et al.
Phys.Rev., D89:114504, 2014.

References for the Input Parameters II

[7] S. Alekhin, A. Djouadi, and S. Moch.
The top quark and Higgs boson masses and the stability of the electroweak vacuum.
Phys.Lett., B716:214–219, 2012.

[8] Jon A. Bailey, Yong-Chull Jang, Weonjong Lee, and Sungwoo Park.
Standard Model evaluation of ε_K using lattice QCD inputs for $\hat{B}_K$ and V_{cb}.
hep-lat/1503.05388, 2015.

[9] Andrzej J. Buras and Diego Guadagnoli.
Phys.Rev., D78:033005, 2008.

[10] Joachim Brod and Martin Gorbahn.
ϵ_K at Next-to-Next-to-Leading Order: The Charm-Top-Quark Contribution.
Phys.Rev., D82:094026, 2010.

[11] T. Blum, P.A. Boyle, N.H. Christ, N. Garron, E. Goode, et al.
Phys.Rev.Lett., 108:141601, 2012.