Hadronic light-by-light scattering in the muon $g-2$: a dispersive approach

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Institute for Nuclear Theory
University of Washington

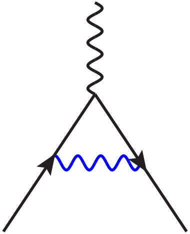
QCD for New Physics Searches at the Precision Frontier

Seattle, September 29, 2015

G. Colangelo, MH, M. Procura, P. Stoffer, JHEP 09 (2014) 091, JHEP 09 (2015) 074
G. Colangelo, MH, B. Kubis, M. Procura, P. Stoffer, PLB 738 (2014) 6
MH, B. Kubis, S. Leupold, F. Niecknig, S. Schneider, EPJC 74 (2014) 3180

Overview of SM prediction

	a_μ [10^{-11}]	Δa_μ [10^{-11}]
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.



Schwinger 1948

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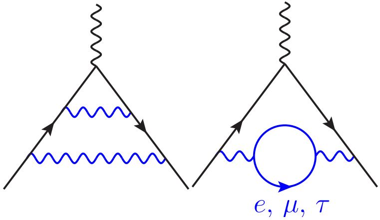

Sommerfield, Petermann 1957

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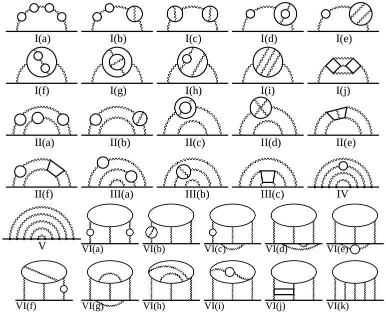

Kinoshita et al. 2012

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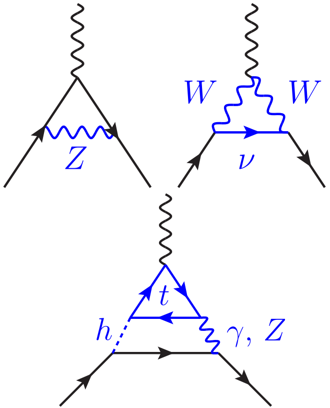

1-loop: Jackiw, Weinberg and others 1972

2-loop: Kukhto et al. 1992, Czarnecki, Krause, Marciano 1995, Degrassi, Giudice 1998, Knecht, Peris, Perrottet, de Rafael 2002, Vainshtein 2003, Heinemeyer, Stöckinger, Weiglein 2004, Gribouk, Czarnecki 2005

Update after Higgs discovery: Gnendiger et al. 2013

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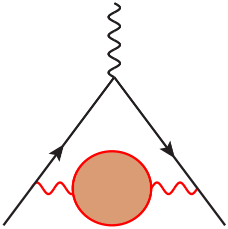

Hagiwara et al. 2011

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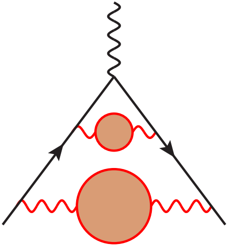

Calmet et al. 1976, Hagiwara et al. 2011

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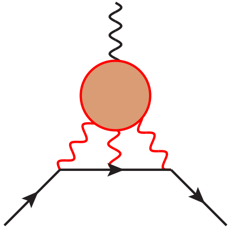

Hayakawa, Kinoshita, Sanda 1995

Bijnens, Pallante, Prades 1995

Knecht, Nyffeler 2001

Jegerlehner, Nyffeler 2009

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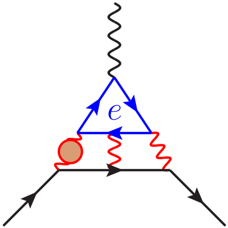

Kurz, Liu, Marquard, Steinhauser 2014

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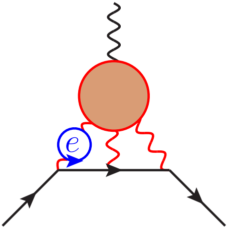

Colangelo, MH, Nyffeler, Passera, Stoffer 2014

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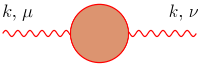
$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (234 \pm 86) \cdot 10^{-11} [2.7\sigma]$$

⇒**Theory error** comes almost exclusively from **hadronic part**

Outline

Hadronic vacuum polarization

- General principles yield **direct connection with experiment**
  - **Gauge invariance**



$$= -i\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\Pi\left(k^2\right)$$

  - **Analyticity**

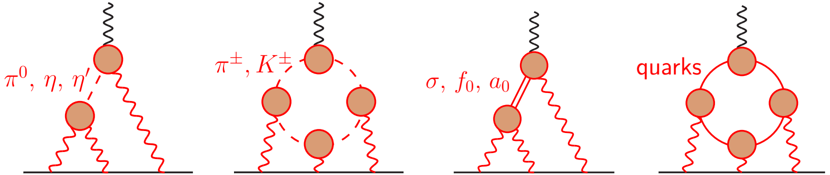
$$\Pi_{\text{ren}} = \Pi\left(k^2\right) - \Pi(0) = \frac{k^2}{\pi}\int_{4M_\pi^2}^{\infty} ds \frac{\operatorname{Im}\Pi(s)}{s(s-k^2)}$$

  - **Unitarity**

$$\operatorname{Im}\Pi(s) = \frac{s}{4\pi\alpha}\sigma_{\text{tot}}\left(e^+e^- \to \text{hadrons}\right) = \frac{\alpha}{3}R(s)$$

- 1 Lorentz structure, 1 kinematic variable, parameter-free
- **Dedicated e^+e^- program** under way: BaBar, Belle, BESIII, CMD3, KLOE2, SND (still hard to go much below 1%)

HLbL: irreducible uncertainty?



Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	–	–	–	0 ± 10	–	–	–
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	–	–	–	$2.3\pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

Jegerlehner, Nyffeler 2009

- HVP systematically improvable
- HLbL more challenging
  - **4-point function** of EM currents
  - Up to now: model calculations
  - 5 kinematic variables, many more Lorentz structures (but only 7 master structures)
  - Folk theorem: "*it cannot be expressed in terms of measurable quantities*"
- Our suggestion: adapt methods from HVP, stay as **data-driven** as possible

Approaches to HLbL

- **Model calculations**
  - ENJL — Bijnens, Pallante, Prades 1995-96
  - NJL and hidden gauge — Hayakawa, Kinoshita, Sanda 1995-96
  - Nonlocal χQM — Dorokhov, Broniowski 2008
  - AdS/CFT — Cappiello, Cata, D'Ambrosio 2010
  - Dyson–Schwinger — Goecke, Fischer, Williams 2011
  - Constituent χQM — Greynat, de Rafael 2012
  - Resonances in narrow-width limit — Pauk, Vanderhaeghen 2014
- **Rigorous constraints from QCD**
  - High-energy constraints taken into account in several models above, addressed specifically by — Knecht, Nyffeler 2001
  - ChPT for a_μ — Knecht, Nyffeler, Perrottet, de Rafael 2002, Ramsey-Musolf, Wise 2002
  - High-energy constraints related to the axial anomaly — Melnikov, Vainshtein 2004, Nyffeler 2009
  - Sum rules for $\gamma^*\gamma \to X$ — Pascalutsa, Pauk, Vanderhaeghen 2012
  - Low-energy constraints from pion polarizabilities — Engel, Ramsey-Musolf 2013
- **Lattice** — Blum et al. 2005, 2012-15, Green et al. 2015

  $\hookrightarrow$ see talk by T. Blum

Why dispersive approach?

- Analytic structure: poles and cuts

  $\hookrightarrow$ **Residues** and **imaginary parts** $\Rightarrow$ by definition **on-shell** quantities

  $\hookrightarrow$ **form factors** and **scattering amplitudes** from experiment

  $\hookrightarrow$ **model-independent** definition of all contributions!

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  $\hookrightarrow$ **form factors** and **scattering amplitudes** from experiment

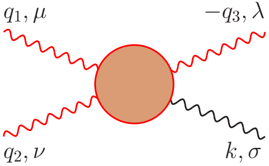
  $\hookrightarrow$ **model-independent** definition of all contributions!
- Challenges
  - Find suitable quantities for dispersive analysis: **Bardeen–Tung–Tarrach basis**
  - Large number of amplitudes and invariants: no closed formula as for HVP

    $\hookrightarrow$ Expansion in mass of intermediate states and partial waves
- Pseudoscalar poles most important, next $\pi\pi$ cuts
- Decompose the tensor according to

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

  $\hookrightarrow$ accounts for **one-** and **two-pion** intermediate states
- Generalizes immediately to η, η', $K\bar{K}$, but e.g. 3π more difficult

A suitable basis for HLbL



- Need a decomposition of the HLbL tensor

$$\Pi^{\mu\nu\lambda\sigma} = \sum_{i=1}^{N} T_i^{\mu\nu\lambda\sigma} \Pi_i$$

in such a way that the Π_i fulfill dispersion relations

$\hookrightarrow$ avoid **kinematic zeros and singularities**

- General procedure known: BTT basis Bardeen, Tung 1968, Tarrach 1975
- Start with 138 (136 Eichmann, Fischer, Heupel, Williams 2014) independent Lorentz structures
- $N = 54$ remain, but only 7 completely independent thanks to **crossing symmetry**

$$T_1^{\mu\nu\lambda\sigma} = \epsilon^{\mu\nu\alpha\beta}\epsilon^{\lambda\sigma\gamma\delta} q_{1\,\alpha} q_{2\,\beta} q_{3\,\gamma} q_{4\,\delta} \qquad T_4^{\mu\nu\lambda\sigma} = \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_4^\lambda q_3^\sigma - q_3\cdot q_4 g^{\lambda\sigma}\right)$$

$$T_7^{\mu\nu\lambda\sigma} = \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_1\cdot q_4 \left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_1^\sigma q_1\cdot q_3 - q_1^\lambda q_1^\sigma q_3\cdot q_4\right)$$

$$T_{19}^{\mu\nu\lambda\sigma} = \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2\cdot q_4 \left(q_1^\lambda q_3^\sigma - q_1\cdot q_3 g^{\lambda\sigma}\right) + q_4^\lambda q_2^\sigma q_1\cdot q_3 - q_1^\lambda q_2^\sigma q_3\cdot q_4\right)$$

$$T_{31}^{\mu\nu\lambda\sigma} = \left(q_2^\mu q_1^\nu - q_1\cdot q_2 g^{\mu\nu}\right)\left(q_2^\lambda q_1\cdot q_3 - q_1^\lambda q_2\cdot q_3\right)\left(q_2^\sigma q_1\cdot q_4 - q_1^\sigma q_2\cdot q_4\right)$$

$$T_{37}^{\mu\nu\lambda\sigma} = \left(q_3^\mu q_1\cdot q_4 - q_4^\mu q_1\cdot q_3\right)\Big(q_3^\nu q_4^\lambda q_2^\sigma - q_4^\nu q_2^\lambda q_3^\sigma + g^{\lambda\sigma}\left(q_4^\nu q_2\cdot q_3 - q_3^\nu q_2\cdot q_4\right)$$
$$+ g^{\nu\sigma}\left(q_2^\lambda q_3\cdot q_4 - q_4^\lambda q_2\cdot q_3\right) + g^{\lambda\nu}\left(q_3^\sigma q_2\cdot q_4 - q_2^\sigma q_3\cdot q_4\right)\Big)$$

$$T_{49}^{\mu\nu\lambda\sigma} = \dots$$

Master formula for a_μ

Colangelo, MH, Procura, Stoffer 2015

Master formula for a_μ

$$a_\mu^{\text{HLbL}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_{i=1}^{12} \hat{T}_i(q_1, q_2; p)\bar{\Pi}_i(q_1, q_2, -q_1 - q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 \left((p+q_1)^2 - m_\mu^2\right)\left((p-q_2)^2 - m_\mu^2\right)}$$

- $\hat{T}_i$: known kernel functions
- $\bar{\Pi}_i$: linear combinations of Π_i
- Can perform five integrations with Gegenbauer polynomials

Colangelo, MH, Procura, Stoffer 2015

Master formula for a_μ

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\,Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1,Q_2,\tau)\bar{\Pi}_i(Q_1,Q_2,\tau)$$

- T_i: known kernel functions
- $\bar{\Pi}_i$: linear combinations of Π_i
- Can perform five integrations with Gegenbauer polynomials
- Wick rotation: all input quantities at **space-like kinematics**
- Decomposition completely general, now dispersion relations for $\bar{\Pi}_i$

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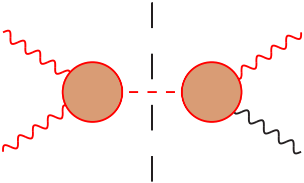
- Can perform five integrations with Gegenbauer polynomials
- Wick rotation: all input quantities at **space-like kinematics**
- Decomposition completely general, now dispersion relations for $\bar{\Pi}_i$
- Alternative: dispersion relations for **Pauli form factor** $F_2(t)$ Pauk, Vanderhaeghen 2014
  - a_μ^{HLbL} from $a_\mu = F_2(0)$
  - Do the 2-loop integral dispersively, known result for pseudoscalar pole reproduced
  - Large number of cuts for higher intermediate states

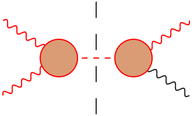
Setting up the dispersive calculation: pion pole

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



- Pion pole: known
- Projection onto BTT basis: done
- Master formula reproduces explicit expressions in the literature
- To be done: incorporation of pQCD constraints

Digression: why we disagree with Arkady



$$= \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2, q_4^2)}{s - M_\pi^2}$$

- **Separation into subproblems**:
  1. Dispersive reconstruction of the full HLbL tensor ⇒ Mandelstam variables s, t, u and general, fixed virtualities q_i^2
  2. Perform limit $q_4 \to 0$, then momentum integrals in $g - 2$

↪ pion pole completely **unambiguous** in this framework

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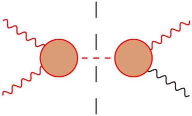

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  1. Dispersive reconstruction of the full HLbL tensor $\Rightarrow$ Mandelstam variables s, t, u and general, fixed virtualities q_i^2
  2. Perform limit $q_4 \to 0$, then momentum integrals in $g-2$

$\hookrightarrow$ pion pole completely **unambiguous** in this framework

- **Triangle argument**: reduced HLbL amplitude scales as $1/q_3^2$ and $F_{\pi^0\gamma^*\gamma^*}(q_3^2, 0) \to F_{\pi^0\gamma^*\gamma^*}(M_\pi^2, 0)$ provides interpolation Melnikov, Vainshtein 2004
- However:
  - **Ad-hoc procedure**: taking residue in q_3^2 would require dispersion relations on the level of the three-point function
  - **Higher intermediate states** can remedy the asymptotic behavior Melnikov, Vainshtein 2004

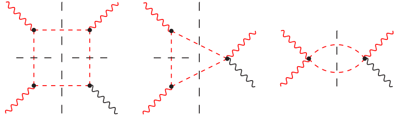
For a unified approach need to match to pQCD/OPE on the level of the BTT functions!

Setting up the dispersive calculation: $\pi\pi$ intermediate states

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

In JHEP 2014 paper

$$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)\,F^V_\pi(q_2^2)\,F^V_\pi(q_3^2) \times \Big[\quad \Big]$$



Separate contribution with two simultaneous cuts

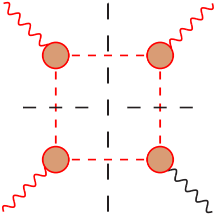
- Analytic properties like the box diagram in sQED
- Triangle and bulb required by gauge invariance
- Multiplication with vector form factor F^V_π gives correct q^2-dependence $\Rightarrow$ FsQED

Claim: **FsQED is not an approximation** $\Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} = \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma}$

Setting up the dispersive calculation: $\pi\pi$ intermediate states

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

Now with BTT basis



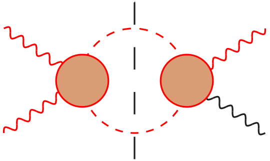
- Constructed a Mandelstam representation for $\pi\pi$ intermediate states with pion-pole left-hand cut
- Checked explicitly that this agrees with FsQED

Proven: **FsQED is not an approximation** $\Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} = \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma}$

Uniquely defines the notion of a “pion loop”

Setting up the dispersive calculation: $\pi\pi$ intermediate states

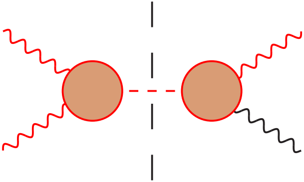
$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



- Remainder $\bar{\Pi}_{\mu\nu\lambda\sigma}$ has cuts only in one channel
- Physics: $\pi\pi$ rescattering
- Calculated with a **partial-wave expansion**
- Similar for η, η' poles and $K\bar{K}$ intermediate states

Pion pole

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\text{-box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



$$\Pi_i^{\pi^0\text{-pole}}(s,t,u) = \frac{\rho_{i;s}}{s-M_\pi^2} + \frac{\rho_{i;t}}{t-M_\pi^2} + \frac{\rho_{i;u}}{u-M_\pi^2}$$

$$\rho_{i,s} = \delta_{i1} F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) F_{\pi^0\gamma^*\gamma^*}(q_3^2,q_4^2)$$

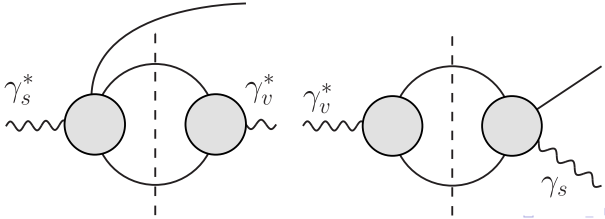
$$\rho_{i,t} = \delta_{i2} F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_3^2) F_{\pi^0\gamma^*\gamma^*}(q_2^2,q_4^2)$$

$$\rho_{i,u} = \delta_{i3} F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_4^2) F_{\pi^0\gamma^*\gamma^*}(q_2^2,q_3^2)$$

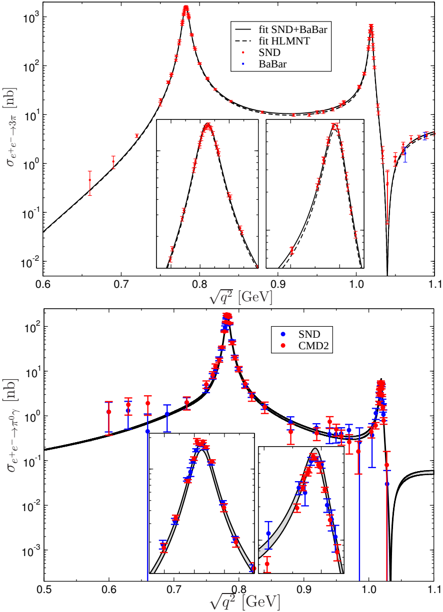
- Crucial ingredient: **pion transition form factor** $F_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2)$
- Dispersive approach: pion on-shell → data input

Dispersive analysis of the pion transition form factor

- In principle, the doubly-virtual form factor $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ can be measured
- Absent data, and/or to improve accuracy: **dispersive reconstruction**
- Required input
  - Pion vector form factor
  - $\gamma^* \to 3\pi$ amplitude
  - $\pi\pi$ scattering amplitude
- Done for the singly-virtual case MH, Kubis, Leupold, Niecknig, Schneider 2014, doubly-virtual in progress
- Transition form factors $\omega, \phi \to \pi^0\gamma^*$ probe a particular doubly-virtual configuration



Predicting $\sigma(e^+e^- \to \pi^0\gamma)$ from $\sigma(e^+e^- \to 3\pi)$



1. Fit dispersive representation to $e^+e^- \to 3\pi$
2. Determines singly-virtual form factor in **time-like region**
3. Predict $e^+e^- \to \pi^0\gamma$ as check on the formalism

Extraction of slope and space-like continuation

- For HLbL need the form factor in the

  space-like region

  $\hookrightarrow$ another dispersion relation

$$F_{\pi^0\gamma^*\gamma}(q^2,0) = F_{\pi\gamma\gamma} + \frac{q^2}{\pi}\int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im}\, F_{\pi^0\gamma^*\gamma}(s',0)}{s'(s'-q^2)}$$

- Sum rules for $F_{\pi\gamma\gamma}$ and **slope parameters**

$$a_\pi = \frac{M_{\pi^0}^2}{F_{\pi\gamma\gamma}} \frac{1}{\pi}\int_{s_{\text{thr}}}^{\infty} ds' \frac{\text{Im}\, F_{\pi^0\gamma^*\gamma}(s',0)}{s'^2}$$

$$= (30.7 \pm 0.6) \times 10^{-3}$$

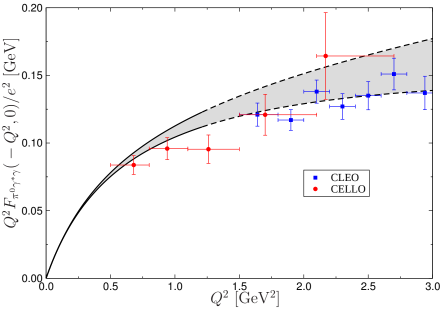
$$b_\pi = (1.10 \pm 0.02) \times 10^{-3}$$

- Soon to be tested at BESIII

- Similar program for η, η'

  Hanhart, Kupść, Meißner, Stollenwerk, Wirzba 2013

  Kubis, Plenter 2015, Xiao et al. 2015



Pion box: projection onto BTT

- Very compact expressions in BTT basis

$$\Pi_i^{\pi\text{-box}}(q_1^2,q_2^2,q_3^2) = F_\pi^V(q_1^2)F_\pi^V(q_2^2)F_\pi^V(q_3^2)\frac{1}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy\, I_i(x,y)$$

$$I_1(x,y) = -\frac{2}{3}\frac{(1-2y)(1-2x-2y)(1-6x(1-x))}{\Delta^2} \qquad I_7(x,y) = -\frac{4}{3}\frac{(1-2x)^2(1-2y)^2y(1-y)}{\Delta^3}$$

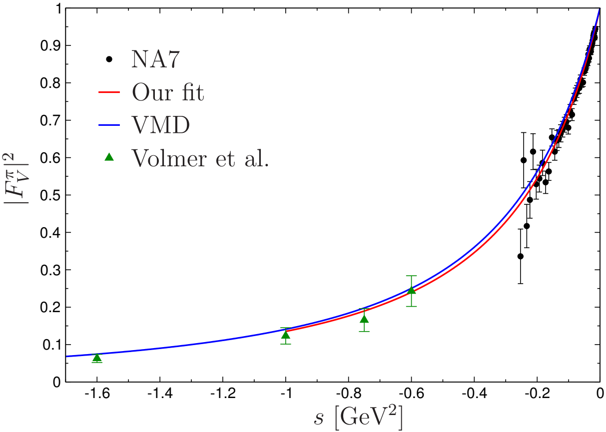
$$I_4(x,y) = -\frac{2}{3}\frac{(1-2x)\big(1+2x(1-3x(1-2y)-6y(1-y))\big)}{\Delta^2} \qquad \dots$$

$$\Delta = M_\pi^2 - xyq_1^2 - x(1-x-y)q_2^2 - y(1-x-y)q_3^2$$

- Manifestly free of kinematic singularities
- Only **9 independent functions** due to remaining crossing symmetries, e.g.

$$\Pi_2 = \mathcal{C}_{23}[\Pi_1] \qquad \Pi_5 = \mathcal{C}_{23}[\Pi_4] \qquad \Pi_9 = \mathcal{C}_{13}[\mathcal{C}_{23}[\Pi_7]] \qquad \Pi_{10} = \mathcal{C}_{23}[\Pi_7]$$

Pion box: numerics



- Only input space-like pion vector form factor
- Preliminary numbers: $a_\mu^{\pi\text{-box}} = -15.9 \times 10^{-11}$, $a_\mu^{\pi\text{-box,VMD}} = -16.4 \times 10^{-11}$
- Error estimate in progress, but uncertainties will be tiny

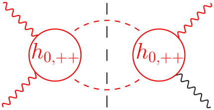
$\pi\pi$ intermediate states: rescattering

- Dispersion relations for Π_i, e.g. fixed-u at $u = u_b = q_1^2$

$$\Pi_1(q_1^2, q_2^2, q_3^2) = \lim_{q_4^2 \to 0} \left(\frac{1}{\pi} \int_{4M_\pi^2}^{\infty} ds' \frac{D_1^{s;u}(s'; u_b)}{s' - q_3^2} + \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} dt' \frac{D_1^{t;u}(t'; u_b)}{t' - q_2^2} \right)$$

- Discontinuities from **unitarity**: diagonal in helicity basis for partial waves, e.g.

$$\text{Im}\, h^J_{++,++}(s; q_1^2, q_2^2; q_3^2, 0) = \frac{\sigma(s)}{16\pi} h^*_{J,++}(s; q_1^2, q_2^2) h_{J,++}(s; q_3^2, 0)$$



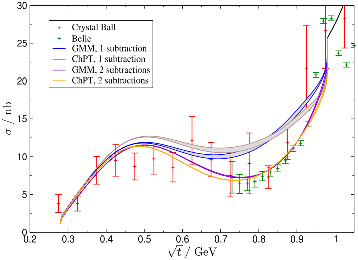
$\hookrightarrow$ need to project onto BTT basis

- Study of pion box in progress: partial-wave expansion, numerics of $g - 2$ integral
- Next: $\pi\pi$ rescattering $\Rightarrow$ **partial waves for $\gamma^*\gamma^* \to \pi\pi$**

$\gamma^*\gamma^* \to \pi\pi$ partial waves

Roy(–Steiner) equations $=$ Dispersion relations $+$ partial-wave expansion $+$ crossing symmetry $+$ unitarity $+$ gauge invariance

- **On-shell case** $\gamma\gamma \to \pi\pi$ García-Martín, Moussallam 2010, MH, Phillips, Schat 2011, partial-wave analysis Dai, Pennington 2014
- **Singly-virtual** $\gamma^*\gamma \to \pi\pi$ Moussallam 2013
- **Doubly-virtual** $\gamma^*\gamma^* \to \pi\pi$: **anomalous thresholds** Colangelo, MH, Procura, Stoffer arXiv:1309.6877



$\gamma^*\gamma^* \to \pi\pi$ partial waves

Roy(–Steiner) equations $=$ **Dispersion relations** $+$ **partial-wave expansion**

$+$ **crossing symmetry** $+$ **unitarity** $+$ **gauge invariance**

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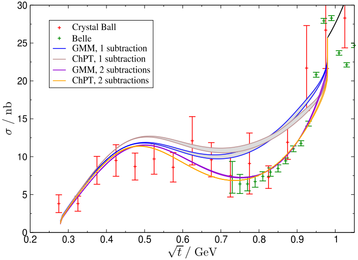
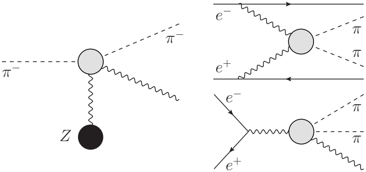
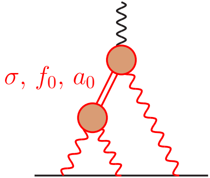


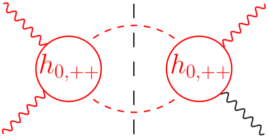
- Constraints
  - **Low energies**: pion polarizabilities, ChPT
  - **Primakoff**: $\gamma\pi \to \gamma\pi$ (COMPASS), $\gamma\gamma \to \pi\pi$ (JLab)
  - **Scattering**: $e^+e^- \to e^+e^-\pi\pi$, $e^+e^- \to \pi\pi\gamma$
  - **(Transition) Form factors**: F_V^π, $\omega, \phi \to \pi^0\gamma^*$



Physics of $\gamma^*\gamma^* \to \pi\pi$

- **$\pi\pi$ rescattering** includes dofs corresponding to **resonances**, e.g. $f_2(1270)$
- *S*-wave provides **model-independent** implementation of the σ



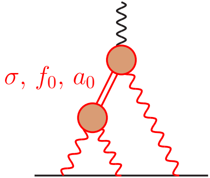


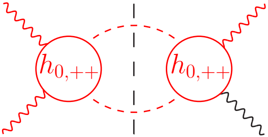
Physics of $\gamma^*\gamma^* \to \pi\pi$

- **$\pi\pi$ rescattering** includes dofs corresponding to **resonances**, e.g. $f_2(1270)$
- S-wave provides **model-independent** implementation of the σ
- **Analytic continuation** with dispersion theory: resonance properties
  - Precise determination of σ-pole parameters from $\pi\pi$ scattering Caprini, Colangelo, Leutwyler 2006

$$M_\sigma = 441^{+16}_{-8}\,\text{MeV} \qquad \Gamma_\sigma = 544^{+18}_{-25}\,\text{MeV}$$

  - Coupling $\sigma \to \gamma\gamma$ from $\gamma\gamma \to \pi\pi$ MH, Phillips, Schat 2011





$f_0(500)$ or σ
was $f_0(600)$

$I^G(J^{PC}) = 0^+(0^{++})$

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$f_0(500)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\,\text{Im}(\sqrt{s_{\text{pole}}})$.

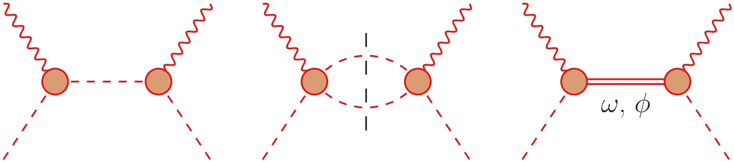
VALUE (MeV)	*DOCUMENT ID*	*TECN*	*COMMENT*
(400–550)−i(200–350) OUR ESTIMATE			
• • • We do not use the following data for averages, fits, limits, etc. • • •			
$(445 \pm 25)-i(278^{+22}_{-18})$	[1,2] GARCIA-MAR..11	RVUE	Compilation
$(457^{+14}_{-13})-i(279^{+11}_{-7})$	[1,3] GARCIA-MAR..11	RVUE	Compilation
$(442^{+5}_{-8})-i(274^{+6}_{-5})$	[4] MOUSSALLAM11	RVUE	Compilation

$f_0(500)$ PARTIAL WIDTHS

$\Gamma(\gamma\gamma)$ **Γ_2**

VALUE (keV)	*DOCUMENT ID*	*TECN*	*COMMENT*
• • • We do not use the following data for averages, fits, limits, etc. • • •			
1.7 ± 0.4	[54] HOFERICHTER11	RVUE	Compilation
3.08 ± 0.82	[55] MENNESSIER 11	RVUE	Compilation
$2.08 \pm 0.2\ ^{+0.07}$	[56] MOUSSALLAM11	RVUE	Compilation

Left-hand cut



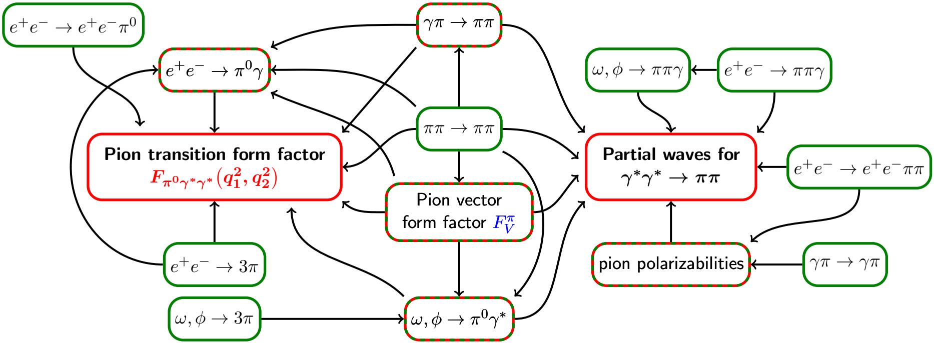
- **Pion pole**: coupling determined by F_V^π as before
- **Multi-pion intermediate states**: approximate in terms of **resonances**
  - $2\pi \sim \rho$: can even be done **exactly** using $\gamma^* \to 3\pi$ amplitude

    $\hookrightarrow$ cf. pion transition form factor MH, Kubis, Sakkas 2012, MH, Kubis, Leupold, Niecknig, Schneider 2014
  - $3\pi \sim \omega, \phi$: narrow-width approximation

    $\hookrightarrow$ **transition form factors** for $\omega, \phi \to \pi^0\gamma^*$ Schneider, Kubis, Niecknig 2012
  - Higher intermediate states also potentially relevant: **axials**, **tensors**

    $\hookrightarrow$ **sum rules** to constrain their transition form factors Pauk, Vanderhaeghen 2014

Towards a data-driven analysis of HLbL



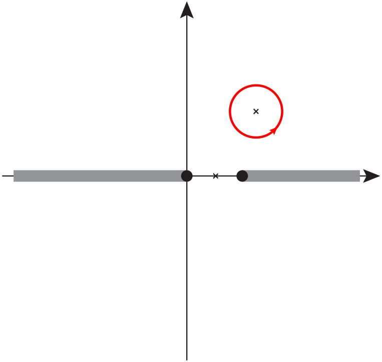
- Reconstruction of $\gamma^*\gamma^* \to \pi\pi, \pi^0$: combine experiment and theory constraints
- Beyond: η, η', $K\bar{K}$, multi-pion channels (resonances), pQCD constraints, ...

Summary

- **Dispersive framework** for the calculation of the HLbL contribution to a_μ
- Includes **one- and two-pion** intermediate states, can be extended to other pseudoscalar poles and two-meson states
- General **master formula** in terms of BTT function
- Next steps
  - Doubly-virtual pion transition form factor
  - $\pi\pi$ rescattering in partial-wave framework
  - Construction of $\gamma^*\gamma^* \to \pi\pi$ input
  - Implementation of pQCD constraints
  - Error analysis: which input quantity has the biggest impact on a_μ?

- **Cauchy's theorem**

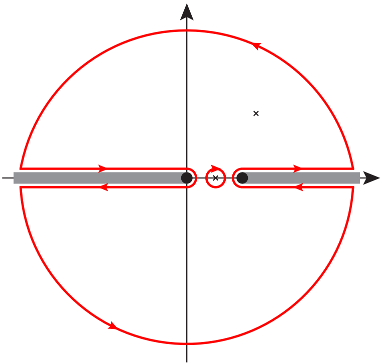
$$f(s) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\mathrm{d}s' \, f(s')}{s' - s}$$



From Cauchy's theorem to dispersion relations

- **Cauchy's theorem**

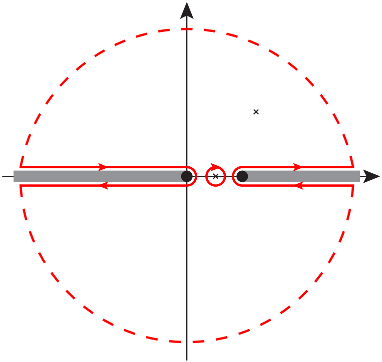
$$f(s) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{ds'\, f(s')}{s'-s}$$



- **Dispersion relation**

$$f(s) = \frac{g}{s - M^2} + \frac{1}{\pi}\int_{\text{cuts}} \frac{ds' \,\text{Im}\, f(s')}{s' - s}$$

$\hookrightarrow$ **analyticity**



From Cauchy's theorem to dispersion relations

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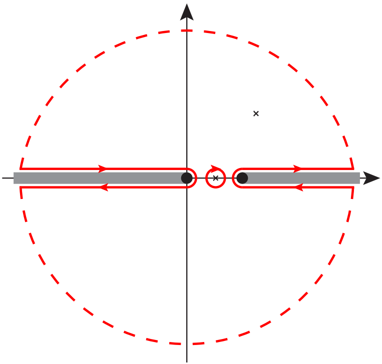
$\hookrightarrow$ **analyticity**

- Subtractions

$$f(s) = \frac{g}{s - M^2} + \underbrace{C}_{f(0)+\frac{g}{M^2}} + \frac{s}{\pi}\int_{\text{cuts}} \frac{ds' \, \text{Im}\, f(s')}{s'(s' - s)}$$



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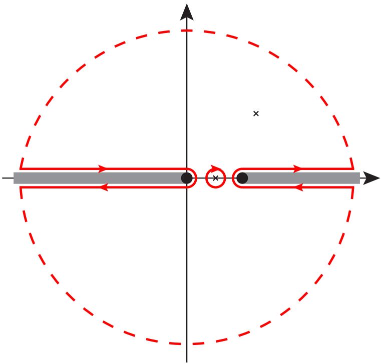

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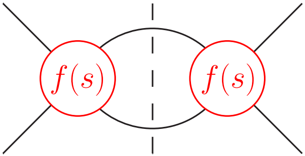
$$f(s) = \frac{g}{s - M^2} + \underbrace{C}_{f(0)+\frac{g}{M^2}} + \frac{s}{\pi}\int_{\text{cuts}} \frac{ds' \,\text{Im}\, f(s')}{s'(s' - s)}$$

- Imaginary part from **Cutkosky rules**

  $\hookrightarrow$ forward direction: **optical theorem**

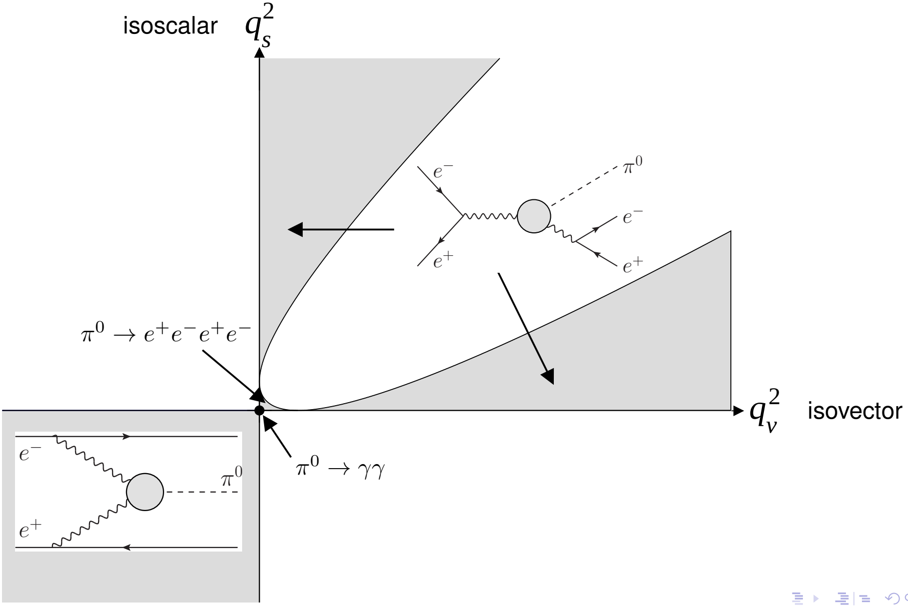
  see HVP and $\sigma(e^+e^- \to \text{hadrons})$



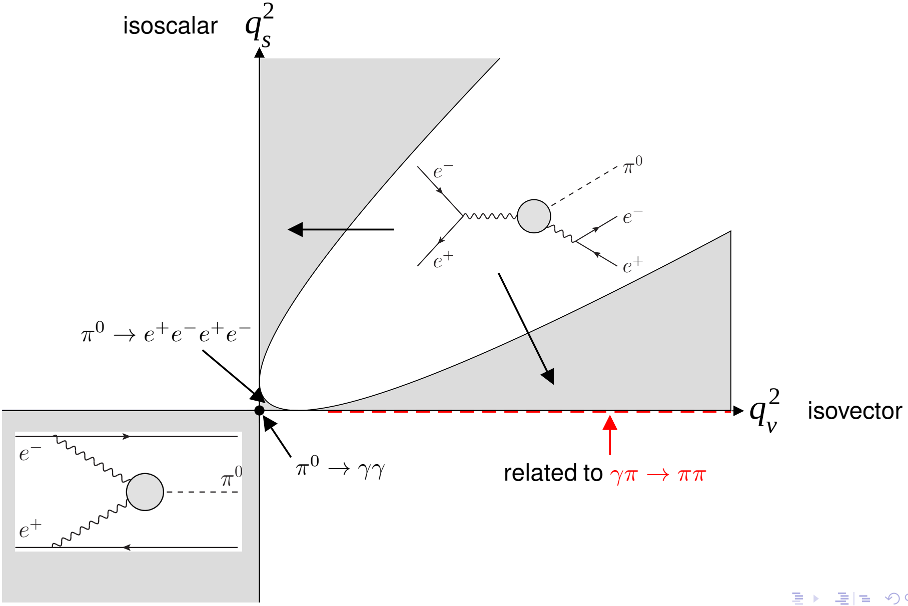
- **Unitarity** for partial waves: $\text{Im}\, f(s) = \rho(s)|f(s)|^2$

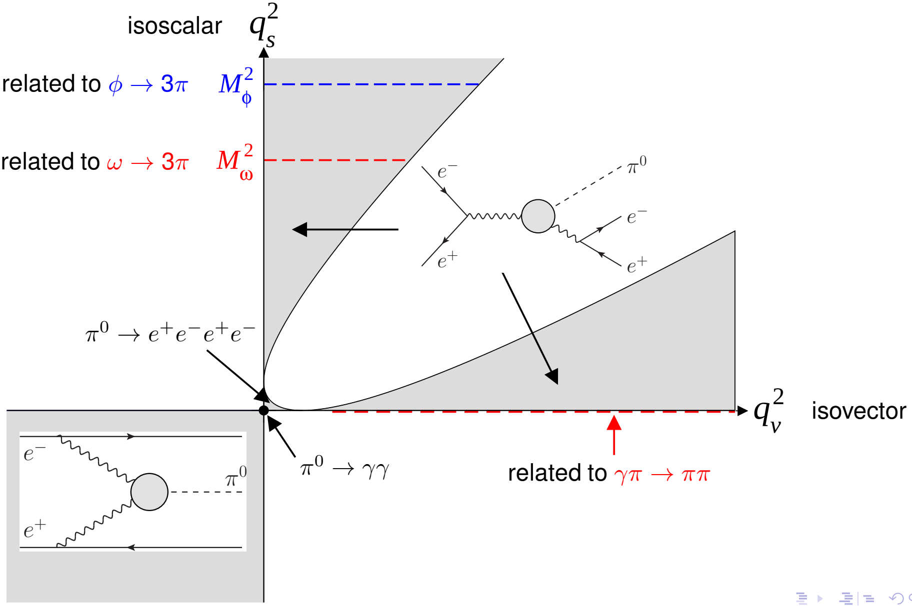
- Residue g **reaction-independent**

Pion transition form factor: physical regions



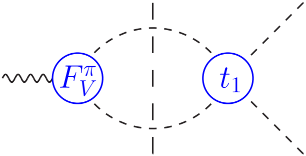
Pion transition form factor: physical regions




isoscalar
q_s^2
related to $\phi \to 3\pi$
M_ϕ^2
related to $\omega \to 3\pi$
M_ω^2
e^-
π^0
e^-
e^+
e^+
$\pi^0 \to e^+e^-e^+e^-$
q_v^2
isovector
e^-
π^0
e^+
$\pi^0 \to \gamma\gamma$
related to $\gamma\pi \to \pi\pi$

- **Unitarity** for **pion vector form factor**

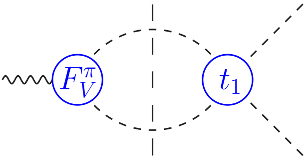
$$\text{Im}\, F_V^\pi(s) = \theta\big(s - 4M_\pi^2\big)\, F_V^\pi(s) e^{-i\delta_1(s)} \sin\delta_1(s)$$



$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ *P*-wave phase δ_1 Watson 1954

- **Unitarity** for **pion vector form factor**

$$\text{Im}\, F_V^\pi(s) = \theta\big(s - 4M_\pi^2\big) F_V^\pi(s) e^{-i\delta_1(s)} \sin\delta_1(s)$$



$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

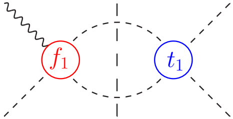
- Solution in terms of **Omnès function** Omnès 1958

$$F_V^\pi(s) = P(s)\Omega_1(s) \qquad \Omega_1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1(s')}{s'(s'-s)}\right\}$$

- Asymptotics + normalization $\Rightarrow$ $P(s) = 1$

- **Unitarity**

$$\text{Im}\, f_1(s) = \theta(s - 4M_\pi^2)\, f_1(s) e^{-i\delta_1(s)} \sin\delta_1(s)$$



$\hookrightarrow$ again Watson's theorem, but now **left-hand cut** in $f_1(s)$

- **Unitarity**

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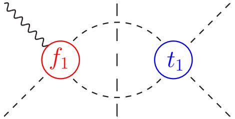

$\hookrightarrow$ again Watson's theorem, but now **left-hand cut** in $f_1(s)$

- Including the left-hand cut

$$\text{Im}\, f_1(s) = \text{Im}\,\mathcal{F}(s) = \big(\underbrace{\mathcal{F}(s)}_{\text{RHC}} + \underbrace{\hat{\mathcal{F}}(s)}_{\text{LHC}}\big)\theta(s - 4M_\pi^2) \sin\delta_1(s) e^{-i\delta_1(s)}$$

$$f_1(s) = \mathcal{F}(s) + \hat{\mathcal{F}}(s) \qquad \hat{\mathcal{F}}(s) = 3\big\langle (1 - z^2)\mathcal{F} \big\rangle \qquad \langle z^n \mathcal{F}\rangle = \frac{1}{2}\int_{-1}^{1} \text{d}z\, z^n \mathcal{F}(t)$$

Omnès solution for $\mathcal{F}(s)$

$$\mathcal{F}(s) = \Omega_1(s)\left\{ \frac{C_1}{3}\big(1 - \dot{\Omega}_1(0)s\big) + \frac{C_2}{3}s + \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} \text{d}s' \frac{\hat{\mathcal{F}}(s')\sin\delta_1(s')}{s'^2(s'-s)|\Omega_1(s')|} \right\}$$

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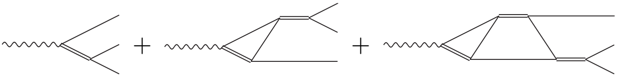
- Solve for $\mathcal{F}(s)$ by iteration
- $\hat{\mathcal{F}}(s)$ corresponds to crossed-channel $\pi\pi$ **rescattering**

$\mathcal{F}(s) =$  $+ \cdots$

- Important observation: $\mathcal{F}(s)$ **linear** in C_i

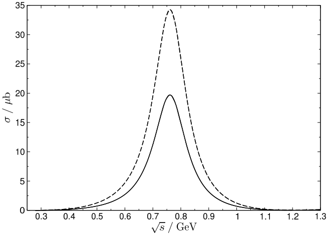
$$\mathcal{F}(s) = C_1\mathcal{F}_1(s) + C_2\mathcal{F}_2(s)$$

$\hookrightarrow$ **basis functions** $\mathcal{F}_i(s)$ can be calculated once and for all

$\gamma\pi \to \pi\pi$: from cross-section data to the transition form factor

- Representation of the cross section in terms of **two parameters** $\hookrightarrow$ fit C_i to data MH, Kubis, Sakkas 2012
  - Test of **chiral anomaly** $F_{3\pi} = e/(4\pi^2 F_\pi^3)$
  - Precise description of f_1
- Looking forward to **COMPASS** result

  $\hookrightarrow$ currently: use chiral prediction



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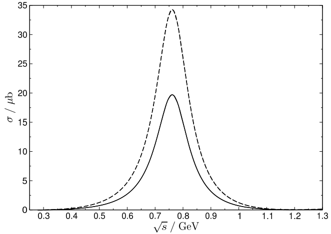
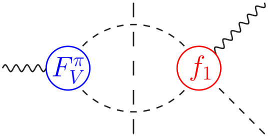


- Dispersion relation for $f_{\pi^0\gamma}(s) = F_{vs}(s,0)$

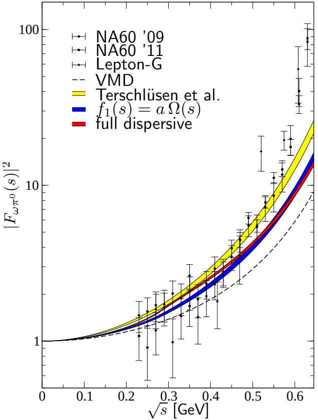
$$f_{\pi^0\gamma}(s) = f_{\pi^0\gamma}(0) + \frac{s}{12\pi^2}\int_{4M_\pi^2}^{\infty} ds' \frac{q_\pi^3(s')\big(F_V^\pi(s')\big)^* f_1(s')}{s'^{3/2}(s'-s)}$$

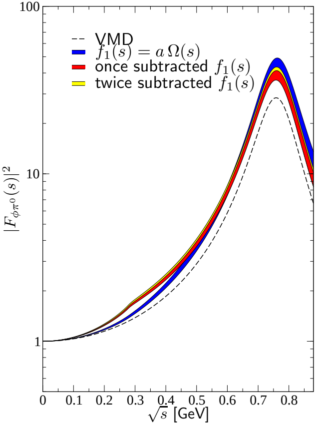


$$q_\pi(s) = \sqrt{s/4 - M_\pi^2}$$

- Subtraction constant: $f_{\pi^0\gamma}(0) = \frac{F_{\pi\gamma\gamma}}{2} = \frac{e^2}{8\pi^2 F_\pi}$

- Similar procedure for $\omega, \phi \to 3\pi$ and $\omega, \phi \to \pi^0\gamma^*$ Schneider, Kubis, Niecknig 2012
- Additional complications due to **decay kinematics**





- Puzzle of steep rise in $F_{\omega\pi^0}$
- Measurement of $F_{\phi\pi^0}$ would be extremely valuable

Phenomenological analysis of the singly-virtual form factor

- General virtualities: how to fix the **normalization**?

  $\hookrightarrow$ $F_{3\pi}$ for $\gamma\pi \to \pi\pi$, widths for $\omega, \phi \to 3\pi$

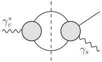
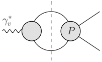
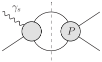
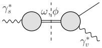
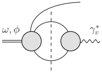
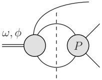
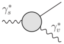
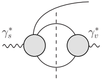
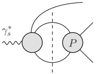
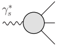
- Fit to $e^+e^- \to 3\pi$

$$a(q^2) = \alpha + \beta q^2 + \frac{q^4}{\pi}\int_{s_{\text{thr}}}^{\infty} \text{d}s' \frac{\text{Im}\,\mathcal{A}(s')}{s'^2(s'-q^2)}$$

$$\mathcal{A}(q^2) = \frac{c_\omega}{M_\omega^2 - q^2 - i\sqrt{q^2}\Gamma_\omega(q^2)} + \frac{c_\phi}{M_\phi^2 - q^2 - i\sqrt{q^2}\Gamma_\phi(q^2)}$$

- α fixed by $F_{3\pi}$, $\Gamma_{\omega/\phi}(q^2)$ include 3π, $K\bar{K}$, $\pi^0\gamma$ channels
- Good analytic properties, free parameters: β, c_ω, c_ϕ
- Valid up to 1.1 GeV, also fit including ω', ω'' to estimate uncertainties

Pion transition form factor: unitarity relations

process	unitarity relations	SC 1	SC 2
			$F_{\pi^0\gamma\gamma}$
		$F_{3\pi}$	$\sigma(\gamma\pi\to\pi\pi)$
			$\Gamma_{\pi^0\gamma}$
		$\Gamma_{3\pi}$	$\frac{d^2\Gamma}{ds dt}(\omega,\phi\to 3\pi)$
			$\sigma(e^+e^-\to\pi^0\gamma)$
		$\sigma(e^+e^-\to 3\pi)$	$\sigma(\gamma\pi\to\pi\pi)$ $\frac{d^2\Gamma}{ds dt}(\omega,\phi\to 3\pi)$
		$F_{3\pi}$	$\sigma(e^+e^-\to 3\pi)$

$\gamma\pi\to\pi\pi$

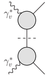
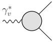
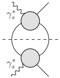
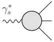
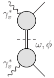
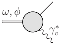
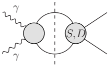
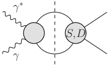
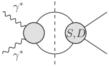
$\omega\to 3\pi$, $\phi\to 3\pi$

$\gamma^*\to 3\pi$

resummation of

$\pi\pi$ rescattering

$\gamma^*\gamma^* \to \pi\pi$ partial waves: unitarity relations

process	building blocks and SC
	
	
	
	$\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2$
	$\alpha_1(q^2) \pm \beta_1(q^2)$, ChPT $e^+e^- \to \pi\pi\gamma$ $e^+e^- \to e^+e^-\pi\pi$
	ChPT $(e^+e^- \to \pi\pi\gamma)$ $e^+e^- \to e^+e^-\pi\pi$

left-hand cut

π

2π

$3\pi\ (\sim \omega, \phi)$

unitarity relations

on-shell

singly-virtual

doubly-virtual

$\pi\pi$ intermediate states: non-diagonal terms

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_i \left(A_{i,s}^{\mu\nu\lambda\sigma} \Pi_i(s) + A_{i,t}^{\mu\nu\lambda\sigma} \Pi_i(t) + A_{i,u}^{\mu\nu\lambda\sigma} \Pi_i(u) \right)$$

- Need to choose $A_i^{\mu\nu\lambda\sigma}$ so that Π_i are **free of kinematic singularities**
- General procedure for finding such a basis Bardeen, Tung 1968, Tarrach 1975
- Results in **non-diagonal terms**

$$\Pi_1(s) = \frac{s - q_3^2}{\pi} \int\limits_{4M_\pi^2}^{\infty} \frac{ds'}{s' - q_3^2} \left(K_1(s', s) \text{Im}\, \bar{h}^0_{++,++}(s') + \frac{2\xi_1\xi_2}{\lambda(s', q_1^2, q_2^2)} \text{Im}\, \bar{h}^0_{00,++}(s') \right)$$

Example: $\gamma^*\gamma^* \to \pi\pi$

- Similar analysis for $\gamma^*\gamma^* \to \pi\pi$: Bardeen–Tung–Tarrach basis

  $\hookrightarrow$ partial-wave dispersion relations (**Roy–Steiner equations**)
- Find similar non-diagonal kernels

Example: $\gamma^*\gamma^* \to \pi\pi$

- Similar analysis for $\gamma^*\gamma^* \to \pi\pi$: Bardeen–Tung–Tarrach basis

  $\hookrightarrow$ partial-wave dispersion relations (**Roy–Steiner equations**)

- Find similar non-diagonal kernels
- Check within 1-loop ChPT

$$\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \left\{ \left(\frac{1}{t'-t} - \frac{t'-q_1^2-q_2^2}{\lambda(t',q_1^2,q_2^2)} \right) \mathrm{Im}\, h_1(t';q_1^2,q_2^2) + \frac{2q_1^2q_2^2}{\lambda(t',q_1^2,q_2^2)} \mathrm{Im}\, h_2(t';q_1^2,q_2^2) \right\}$$

$$= 1 + 2\left(M_\pi^2 + \frac{tq_1^2q_2^2}{\lambda(t,q_1^2,q_2^2)}\right) C_0(t,q_1^2,q_2^2) + \frac{t(q_1^2+q_2^2)-(q_1^2-q_2^2)^2}{\lambda(t,q_1^2,q_2^2)} \bar{J}(t)$$

$$- \frac{q_1^2(t+q_2^2-q_1^2)}{\lambda(t,q_1^2,q_2^2)} \bar{J}(q_1^2) - \frac{q_2^2(t+q_1^2-q_2^2)}{\lambda(t,q_1^2,q_2^2)} \bar{J}(q_2^2)$$

$$\mathrm{Im}\, h_1(t;q_1^2,q_2^2) = 2\left(M_\pi^2 + \frac{tq_1^2q_2^2}{\lambda(t,q_1^2,q_2^2)}\right) \mathrm{Im}\, C_0(t,q_1^2,q_2^2) + \frac{t(q_1^2+q_2^2)-(q_1^2-q_2^2)^2}{\lambda(t,q_1^2,q_2^2)} \mathrm{Im}\, \bar{J}(t)$$

$$\mathrm{Im}\, h_2(t;q_1^2,q_2^2) = -\frac{1}{\lambda(t,q_1^2,q_2^2)} \left[\left(t^2 - (q_1^2-q_2^2)^2\right) \mathrm{Im}\, C_0(t,q_1^2,q_2^2) + 4t\,\mathrm{Im}\, \bar{J}(t) \right]$$

$\hookrightarrow$ non-diagonal kernels crucial for doubly-virtual case

- Another doubly-virtual complication: **anomalous thresholds** in time-like region

Colangelo, MH, Procura, Stoffer arXiv:1309.6877

Subtraction functions

Omnès representation for *S*-wave

$$h_{0,++}(s) = \Delta_{0,++}(s) + \Omega_0(s)\Bigg[\frac{1}{2}(s-s_+)a_+(q_1^2,q_2^2) + \frac{1}{2}(s-s_-)a_-(q_1^2,q_2^2) + q_1^2q_2^2 b(q_1^2,q_2^2)$$

$$+ \frac{s(s-s_+)}{2\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_+)(s'-s)|\Omega_0(s')|} + \frac{s(s-s_-)}{2\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_-)(s'-s)|\Omega_0(s')|}$$

$$+ \frac{2q_1^2q_2^2 s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,00}(s')}{s'(s'-s_+)(s'-s_-)|\Omega_0(s')|}\Bigg] \qquad s_\pm = q_1^2 + q_2^2 \pm 2\sqrt{q_1^2q_2^2}$$

- Inhomogeneities $\Delta_{0,++}(s)$, $\Delta_{0,00}(s)$ include left-hand cut
- **Subtraction functions**
  - $b(q_1^2,q_2^2)$ and $a_+(q_1^2,q_2^2) - a_-(q_1^2,q_2^2)$ multiply $q_1^2q_2^2$ and $\sqrt{q_1^2q_2^2}$
    $\hookrightarrow$ inherently doubly-virtual observables $\Rightarrow$ need ChPT (or lattice)
  - However: $a(q_1^2,q_2^2) = (a_+(q_1^2,q_2^2) + a_-(q_1^2,q_2^2))/2$ fixed by singly-virtual measurements
    $\hookrightarrow$ compare with chiral prediction, uncertainty estimates for the other functions

Subtraction functions: chiral constraints

- 1-loop result for arbitrary q_i^2, e.g.

$$a^{\pi^0}(q_1^2,q_2^2) = -\frac{M_\pi^2}{8\pi^2 F_\pi^2 (q_1^2-q_2^2)^2}\bigg\{q_1^2+q_2^2+2\Big(M_\pi^2(q_1^2+q_2^2)+q_1^2q_2^2\Big)C_0(q_1^2,q_2^2) + q_1^2\bigg(1+\frac{6q_2^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_1^2)+q_2^2\bigg(1-\frac{6q_1^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_2^2)\bigg\}$$

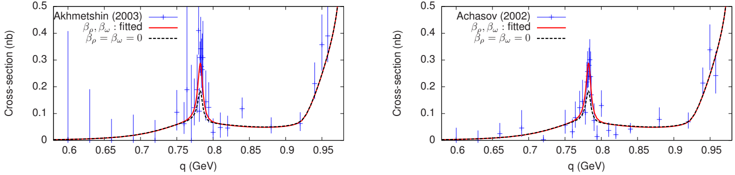
- Special case: $q_1^2=q_2^2=0$

$$a^{\pi^\pm}(0,0)=\frac{\bar{l}_6-\bar{l}_5}{48\pi^2F_\pi^2}+\cdots=\frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^\pm} \qquad b^{\pi^\pm}(0,0)=0$$

$$a^{\pi^0}(0,0)=-\frac{1}{96\pi^2F_\pi^2}+\cdots=\frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^0} \qquad b^{\pi^0}(0,0)=-\frac{1}{1440\pi^2F_\pi^2M_\pi^2}+\cdots$$

$\hookrightarrow$ resume higher chiral orders into **pion polarizabilities**

Subtraction functions: dispersive representation



Moussallam 2013

- Singly-virtual case: phenomenological representation with chiral constraints

  $\hookrightarrow$ parameters fixed from $e^+e^- \to \pi^0\pi^0\gamma$ (CMD2 and SND) Moussallam 2013

- **Dispersive representation**: imaginary part from $2\pi, 3\pi, \ldots$

  $\hookrightarrow$ analytic continuation from time-like to space-like kinematics

- Example: $I = 2 \Rightarrow$ isovector photons $\Rightarrow 2\pi \sim \rho$

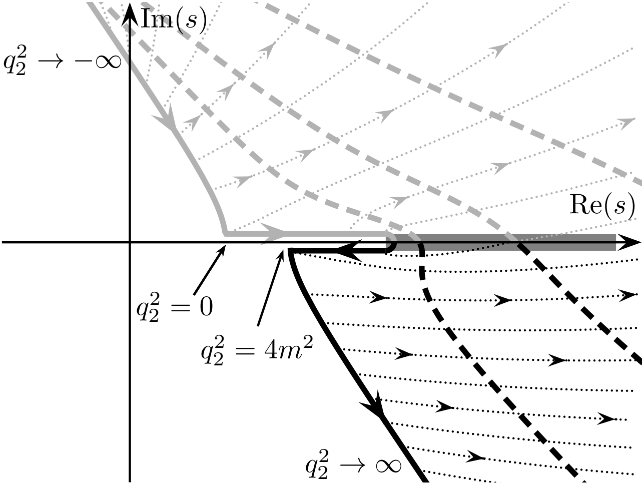
$$a^2(q_1^2, q_2^2) = \alpha_0 \Big[\alpha^2 + \alpha\Big(q_1^2 \mathcal{F}^\rho(q_1^2) + q_2^2 \mathcal{F}^\rho(q_2^2)\Big) + q_1^2 q_2^2 \mathcal{F}^\rho(q_1^2)\mathcal{F}^\rho(q_2^2)\Big]$$

$$\mathcal{F}^\rho(q^2) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds \frac{q_{\pi\pi}^3(s)\big(F_\pi^V(s)\big)^*\Omega_1(s)}{s^{3/2}(s-q^2)} \qquad q_{\pi\pi}(s) = \sqrt{\frac{s}{4} - M_\pi^2}$$

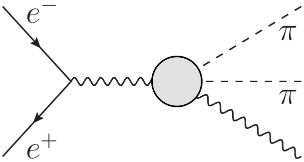
$\hookrightarrow$ α_0 and α can be determined from $a^2(q^2, 0)$ alone!

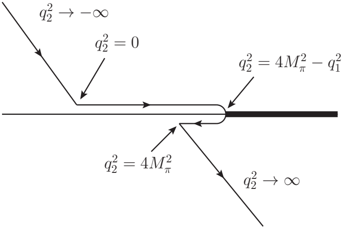
Wick rotation: anomalous thresholds

Trajectory of the triangle anomalous thresholds for $0 < q_1^2 < 4m^2$



Anomalous thresholds





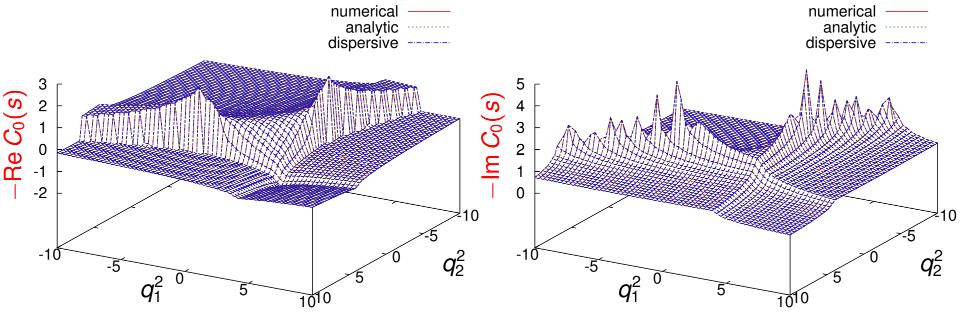
- **Analytic continuation** in q_i^2 in time-like region non-trivial in doubly-virtual case
- Singularities from second sheet move onto first one

  $\hookrightarrow$ need to **deform** the **integration contour**
- Problem already occurs for a simple triangle loop function $C_0(s)$

  $\hookrightarrow$ extra factor $t_\ell(s)/\Omega_\ell(s)$ is well defined in the whole complex plane

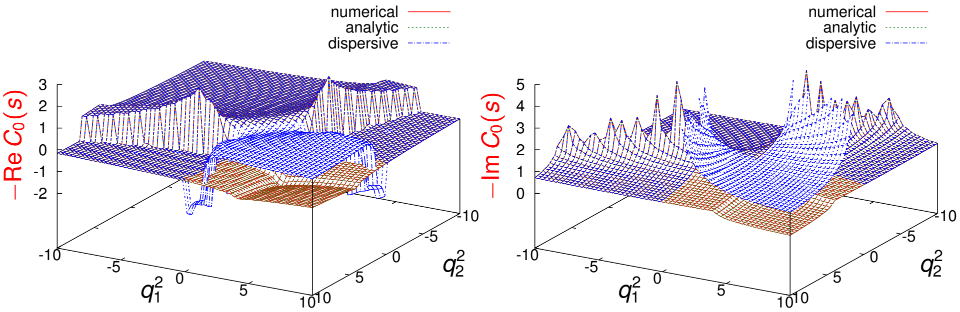
  $\hookrightarrow$ remedy in case of $C_0(s)$ can be taken over to full Omnès solution
- Becomes relevant for $e^+e^+ \to e^+e^-\pi\pi$ in time-like kinematics

Numerical check of anomalous thresholds



- Comparison for $s = 5$, $M_\pi = 1$

$\hookrightarrow$ **dispersive reconstruction** of $C_0(s)$ works!



- Ignore anomalous piece

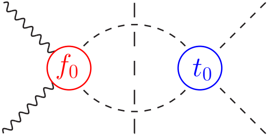
  $\hookrightarrow$ substantial deviations for **large virtualities**!

$\gamma^*\gamma^* \to \pi\pi$

- **Left-hand cut** approximated by **pion pole** + **resonances**
- **Unitarity** for $\gamma^*\gamma^* \to \pi\pi$ system: Watson's theorem

$$\text{disc}\, f_0\big(s;q_1^2,q_2^2\big) = 2i\sigma_s f_0\big(s;q_1^2,q_2^2\big)\, t_0^*(s)$$

$$t_0(s) = \frac{1}{\sigma_s} e^{i\delta_0(s)} \sin\delta_0(s) \qquad \sigma_s = \sqrt{1-\frac{4M_\pi^2}{s}}$$



$\hookrightarrow$ solution in terms of **Omnès function**, e.g. for pion pole only

$$f_0\big(s;q_1^2,q_2^2\big) = \frac{\Omega_0(s)}{\pi}\int\limits_{4M_\pi^2}^{\infty} ds' \frac{N_0\big(s';q_1^2,q_2^2\big)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|} \qquad N_0\big(s;q_1^2,q_2^2\big) = \frac{2L}{\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}$$

$$L = \log\frac{s-q_1^2-q_2^2+\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}{s-q_1^2-q_2^2-\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}} \qquad \lambda(x,y,z) = x^2+y^2+z^2-2(xy+xz$$

- **Analytic continuation** in q_i^2?

$\gamma^*\gamma^* \to \pi\pi$: analytic continuation

$$L = \log \frac{s-q_1^2-q_2^2+\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}{s-q_1^2-q_2^2-\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}} \overset{q_2^2\to 0}{\longrightarrow} \pm\log\frac{1+\sigma_s}{1-\sigma_s}$$

- Singularities of the log: **anomalous thresholds**

$$s_\pm = q_1^2 + q_2^2 - \frac{q_1^2 q_2^2}{2M_\pi^2} \pm \frac{1}{2M_\pi^2}\sqrt{q_1^2\big(q_1^2-4M_\pi^2\big)q_2^2\big(q_2^2-4M_\pi^2\big)}$$

$\hookrightarrow$ usual Omnès derivation breaks down

- Idea: consider first the **scalar loop function**

$$C_0(s) \equiv C_0\big((q_1+q_2)^2; q_1^2, q_2^2\big) = \frac{1}{i\pi^2}\int \frac{d^4k}{\big(k^2-M_\pi^2\big)\big((k+q_1)^2-M_\pi^2\big)\big((k-q_2)^2-M_\pi^2\big)}$$

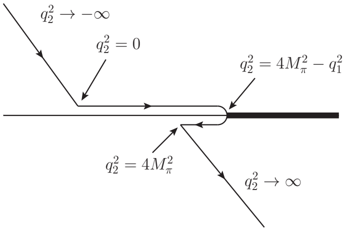
$$\text{disc}\, C_0(s) = -\frac{2\pi i}{\sqrt{\lambda(s,q_1^2,q_2^2)}} L = -\pi i \sigma_s N_0\big(s;q_1^2,q_2^2\big)$$

$\gamma^*\gamma^* \to \pi\pi$: anomalous thresholds

$$s_+ = q_1^2 + q_2^2 - \frac{q_1^2 q_2^2}{2M_\pi^2} + \frac{1}{2M_\pi^2}\sqrt{q_1^2(q_1^2 - 4M_\pi^2)q_2^2(q_2^2 - 4M_\pi^2)}$$

- **Anomalous threshold** usually on the **second sheet**
- Trajectory of $s_+(q_2^2)$ for $0 \leq q_1^2 \leq 4M_\pi^2$

  $\hookrightarrow$ moves through unitarity cut onto first sheet



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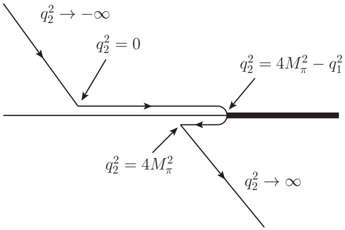
- Need to deform the contour



$$C_0(s) = \frac{1}{2\pi i}\int_{4M_\pi^2}^{\infty} ds' \frac{\text{disc}\, C_0(s')}{s'-s}$$

$$+\,\theta(q_1^2+q_2^2-4M_\pi^2)\frac{1}{2\pi i}\int_0^1 dx \frac{\partial s_x}{\partial x}\frac{\text{disc}_{\text{an}}\, C_0(s_x)}{s_x - s}$$

$$s_x = x4M_\pi^2 + (1-x)s_+ \qquad \text{disc}_{\text{an}}\, C_0(s) = \frac{4\pi^2}{\sqrt{\lambda(s,q_1^2,q_2^2)}}$$

$\gamma^*\gamma^* \to \pi\pi$: back to the Omnès representation

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi}\int\limits_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

- Integrand similar to the scalar-loop example

$$\frac{N_0(s;q_1^2,q_2^2)\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\, C_0(s)}{\pi i \sigma_s}\frac{\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\, C_0(s)}{\pi i}\frac{t_0(s)}{\Omega_0(s)}$$

$\hookrightarrow$ additional factor **independent of q_i^2** and **well-defined in the whole s-plane**

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Omnès representation for $\gamma^*\gamma^* \to \pi\pi$

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

$$+\,\theta\big(q_1^2+q_2^2-4M_\pi^2\big)\frac{\Omega_0(s)}{2\pi i}\int_0^1 dx\,\frac{\partial s_x}{\partial x}\frac{\text{disc}_{\text{an}}\, f_0(s_x;q_1^2,q_2^2)}{s_x-s}$$

$$s_x = x4M_\pi^2+(1-x)s_+ \qquad \text{disc}_{\text{an}}\, f_0(s;q_1^2,q_2^2) = -\frac{8\pi}{\sqrt{\lambda(s,q_1^2,q_2^2)}}\frac{t_0(s)}{\Omega_0(s)}$$