Hamiltonian Diagonalizations & Variational Methods for Exotic and Conventional Hadrons

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Acknowledge

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Supported by DOE

INT-15-3
Seattle
September 20105

Motivation

- **detail how non-lattice techniques can provide hadron structure constraints and upper bounds**
- **obtain an insightful wavefunction picture of hadrons similar to positronium**
- **help discover exotic hadrons (non $q\bar{q}$ or qqq)**
- **possible framework for NSM structure studies?**

Philosophy/Approach

- **since a quintessential, nonperturbative many-body system, use a nuclear structure treatment**
- **three step process**
  - ∗ *obtain a calculable H_{eff} from H_{QCD}*
  - ∗ *introduce model space truncated in particle/hole (quasiparticle/anti-quasiparticle) excitations*

$$|J^{PC}>=|1p1h>+|2p2h>+... \rightarrow |q\bar{q}>+|q\bar{q}q\bar{q}>+|gg>+|q\bar{q}g>+\ ...$$

(quarks and gluons must be dressed to truncate)

  - ∗ *diagonalize H_{eff} in this truncated model space using a variety of techniques: BCS, TDA, RPA, coupled channels and variational*

Outline

- **Coulomb gauge QCD Hamiltonian**
- **Perturbative QCD kernel study**
- **Coulomb gauge Hamiltonian model**
- **Applications to J^{PC} spectra:**

  * *the vacuum (ground state)*
  * $q\bar{q}$ *mesons*
  * gg *glueballs and* ggg *oddballs*
  * $q\bar{q}g$ *hybrid mesons*
  * $q\bar{q}q\bar{q}$ *tetraquarks*
  * *meson and tetraquark mixing*

Coulomb gauge QCD

- **Hamiltonian**

$$
\begin{aligned}
H_{\rm QCD} &= H_q + H_g + H_{qg} + H_C \\
H_q &= \int d\mathbf{x}\Psi^\dagger(\mathbf{x})(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m)\Psi(\mathbf{x}) \\
H_g &= \frac{1}{2}\int d\mathbf{x}\big[\mathcal{J}^{-1}\mathbf{E}^a_{tr}(\mathbf{x})\cdot\mathcal{J}\mathbf{E}^a_{tr}(\mathbf{x}) + \mathbf{B}^a(\mathbf{x})\cdot\mathbf{B}^a(\mathbf{x})\big] \\
H_{qg} &= g\int d\mathbf{x}\,\mathbf{J}^a(\mathbf{x})\cdot\mathbf{A}^a(\mathbf{x}) \\
H_C &= -\frac{g^2}{2}\int d\mathbf{x}d\mathbf{y}\mathcal{J}^{-1}\rho^a(\mathbf{x})K^{ab}(\mathbf{x},\mathbf{y})\mathcal{J}\rho^b(\mathbf{y})
\end{aligned}
$$

- **Faddeev-Popov determinant and kernel**

$$
\mathcal{J} = \det(\mathcal{M}),\ \mathcal{M}^{ab} = \delta^{ab}\boldsymbol{\nabla}^{\mathbf{2}} - gf^{abc}\mathbf{A}^c\cdot\nabla
$$
$$
K^{ab}(\mathbf{x},\mathbf{y}) = \langle\mathbf{x},a|\mathcal{M}^{-1}\nabla^2\mathcal{M}^{-1}|\mathbf{y},b\rangle
$$

- **color charge density and current**

$$
\rho^a(\mathbf{x}) = \Psi^\dagger(\mathbf{x})T^a\Psi(\mathbf{x}) - f^{abc}\mathbf{A}^b(\mathbf{x})\cdot\mathbf{E}_{tr}{}^c(\mathbf{x})
$$
$$
\mathbf{J}^a = \Psi^\dagger(\mathbf{x})\boldsymbol{\alpha}T^a\Psi(\mathbf{x}),\quad T^a = \frac{\lambda^a}{2}
$$

- **non-abelian chromodynamic fields**

$$
\mathbf{E}^a_{tr} = -\dot{\mathbf{A}}^a + g(1-\nabla^{-2}\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot)f^{abc}A^b_0\mathbf{A}^c
$$
$$
\mathbf{B}^a = \nabla\times\mathbf{A}^a + \frac{1}{2}gf^{abc}\mathbf{A}^b\times\mathbf{A}^c
$$

- **bare parton normal mode expansions**

$$\Psi(\mathbf{x}) = \sum_{\lambda\mathcal{C}} \int \frac{d\mathbf{k}}{(2\pi)^3}[u_\lambda(\mathbf{k})b_{\lambda\mathcal{C}}(\mathbf{k}) + v_\lambda(-\mathbf{k})d^\dagger_{\lambda\mathcal{C}}(\mathbf{k})]e^{i\mathbf{k}\cdot\mathbf{x}}\hat{\epsilon}_{\mathcal{C}}$$

$$\mathbf{A}^a(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}[\mathbf{a}^a(\mathbf{k}) + \mathbf{a}^{a\dagger}(-\mathbf{k})]e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\mathbf{E}^a_{tr}(\mathbf{x}) = i\int \frac{d\mathbf{k}}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}[\mathbf{a}^a(\mathbf{k}) - \mathbf{a}^{a\dagger}(-\mathbf{k})]e^{i\mathbf{k}\cdot\mathbf{x}}$$

- **bare Fock operators**

$\underline{gluons}$: $a^a_\mu(\mathbf{k})$ *momentum* $\mathbf{k}$ *spin* $\mu = 0, \pm 1$

$\underline{quarks}$: $b_{\lambda\mathcal{C}}(\mathbf{k})\ d^\dagger_{\lambda\mathcal{C}}(\mathbf{k})$ *color* $\mathcal{C} = 1, 2, 3$ *spin* $\lambda = \pm 1$

- **gauge transverse condition and commutators**

$$\boldsymbol{\nabla} \cdot \mathbf{A}^a = 0 \Longrightarrow \mathbf{k} \cdot \mathbf{a}^a(\mathbf{k}) = (-1)^\mu k_\mu a^a_{-\mu}(\mathbf{k}) = 0$$

$$[a^a_\mu(\mathbf{k}), a^{b\dagger}_\nu(\mathbf{k}')] = (2\pi)^3\delta_{ab}\delta^3(\mathbf{k} - \mathbf{k}')D_{\mu\nu}(\mathbf{k})$$

$$D_{\mu\nu}(\mathbf{k}) = \delta_{\mu\nu} - (-1)^\mu\frac{k_\mu k_{-\nu}}{k^2}$$

Perturbative QCD study

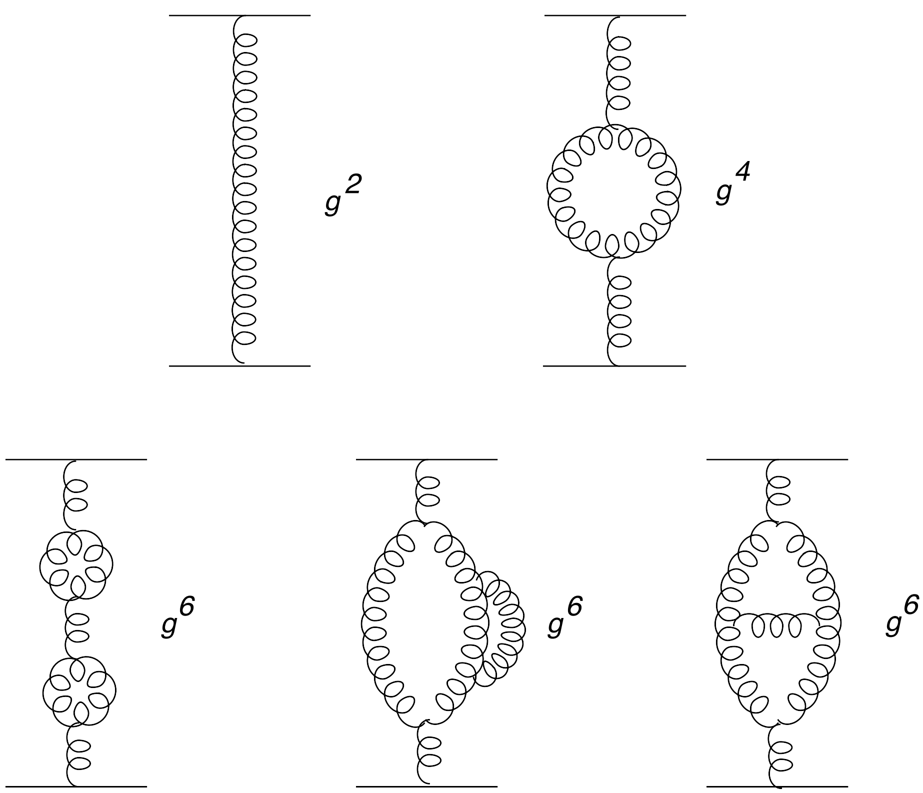
$$K^{ab}(\mathbf{x},\mathbf{y}) = g^2\langle \mathbf{x}, a|\mathcal{M}^{-1}\nabla^2\mathcal{M}^{-1}|\mathbf{y}, b\rangle$$
$$\mathcal{M}^{ab} = \delta^{ab}\boldsymbol{\nabla^2} - g\mathcal{A}^{ab}$$
$$\mathcal{A}^{ab} = f^{abc}\mathbf{A}^c \cdot \boldsymbol{\nabla}$$

- **expand in powers of $g\mathcal{A}$**

$$g^2\mathcal{M}^{-1}\nabla^2\mathcal{M}^{-1} = g^2\nabla^{-2} + 2g^3\nabla^{-2}\mathcal{A}\nabla^{-2} + 3g^4\nabla^{-2}\mathcal{A}\nabla^{-2}\mathcal{A}\nabla^{-2} + ...$$

- **calculate meson expectation value in MeV**

$$\langle q\bar{q}|H_C|q\bar{q}\rangle = 25.6g^2 + 13.2g^4 + (7.15 + 7.86 + .48)g^6 + ... \ .$$



Coulomb gauge model

- **simplify Faddeev-Popov determinant and kernel**

$$\mathcal{J} = \det(\mathcal{M}) \Longrightarrow 1,\ K^{ab}(\mathbf{x},\mathbf{y}) \Longrightarrow V(\mathbf{x},\mathbf{y})\delta^{ab}$$

- **simplify hyperfine interaction**

use Maxwell's equation for $A^a_i(\mathbf{x}) = \int G_{ij}(\mathbf{x},\mathbf{y})J^a_j(\mathbf{y})d\mathbf{y}$

$$H_{qg} = g\int d\mathbf{x}\mathbf{J}^a(\mathbf{x}) \ \cdot \mathbf{A}^a(\mathbf{x}) \Longrightarrow H^{CG}_{qg}$$

$$H^{CG}_{qg} = \frac{1}{2}\int d\mathbf{x}d\mathbf{y}\, J^a_i(\mathbf{x})G_{ij}(\mathbf{x},\mathbf{y})J^a_j(\mathbf{y})$$

kernel reflects the transverse gauge

$$G_{ij}(\mathbf{x},\mathbf{y}) = \left(\delta_{ij} - \frac{\nabla_i\nabla_j}{\nabla^2}\right)_{\boldsymbol{x}} U(|\mathbf{x}-\mathbf{y}|)$$

- **effective model Hamiltonian** $H_{QCD} \Longrightarrow H_{CG}$

$$H_{\rm CG} = H_q + H^{CG}_g + H^{CG}_{qg} + H^{CG}_C$$

$$H_q = \int d\mathbf{x}\Psi^\dagger(\mathbf{x})(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m)\Psi(\mathbf{x})$$

$$H^{CG}_g = \frac{1}{2}\int d\mathbf{x}\left[\mathbf{E}^a_{tr}(\mathbf{x})\cdot\mathbf{E}^a_{tr}(\mathbf{x}) + \mathbf{B}^a(\mathbf{x})\cdot\mathbf{B}^a(\mathbf{x})\right]$$

$$H^{CG}_{qg} = \frac{1}{2}\int d\mathbf{x}d\mathbf{y}\, J^a_i(\mathbf{x})G_{ij}(\mathbf{x},\mathbf{y})J^a_j(\mathbf{y})$$

$$H^{CG}_C = -\frac{1}{2}\int d\mathbf{x}d\mathbf{y}\,\rho^a(\mathbf{x})V(\mathbf{x},\mathbf{y}|)\rho^a(\mathbf{y})$$

model parameters and interactions

- **current quark masses**

$$m_u = m_d = 5 \ MeV$$

$$m_s = \textit{80 } MeV$$

- **Cornell-type confining interaction**

$$V(r = |\mathbf{x} - \mathbf{y}|) = -\alpha_s/r + \sigma r$$

$$= -4\pi\alpha_s/p^2 - 8\pi\sigma/p^4$$

$\alpha_s = 0.4$ **,** $\sigma = 0.135 \ GeV^{\ 2}$ *independently determined*

- **modified Yukawa hyperfine kernel**
  effective gluon exchange with $m_g = 600 \ MeV$

$$U(p) = \begin{cases} -\frac{8.07}{p^2}\frac{\ln^{-0.62}(\frac{p^2}{m_g^2}+0.82)}{\ln^{0.8}(\frac{p^2}{m_g^2}+1.41)} & p > m_g \\ -\frac{5.509}{p^2+m_g^2} & p < m_g \end{cases}$$

- **cut-off, regulating parameter**

$$\Lambda \approx 4 \ GeV$$

Application to $q\bar{q}$ mesons
(pure quark sector)

- **Bogoliubov-Valatin canonical transformation**
  BCS rotation to dressed quark operators

$$B_{\lambda\mathcal{C}}(\mathbf{k}) = \cos\frac{\theta_k}{2} b_{\lambda\mathcal{C}}(\mathbf{k}) - \lambda \sin\frac{\theta_k}{2} d^{\dagger}_{\lambda\mathcal{C}}(\mathbf{k})$$
$$D_{\lambda\mathcal{C}}(\mathbf{k}) = \cos\frac{\theta_k}{2} d_{\lambda\mathcal{C}}(\mathbf{k}) + \lambda \sin\frac{\theta_k}{2} b^{\dagger}_{\lambda\mathcal{C}}(\mathbf{k})$$

- **PQCD vacuum** $|0>$ $\Longrightarrow$ **BCS vacuum** $|\Omega>$

$$|\Omega_{quark}\rangle = \exp\left(- \int \frac{d\mathbf{k} \lambda \tan\frac{\theta_k}{2}}{(2\pi)^3} b^{\dagger}_{\lambda\mathcal{C}}(\mathbf{k}) d^{\dagger}_{\lambda\mathcal{C}}(-\mathbf{k}) \right) |0\rangle$$

- **minimize ground state energy**

$$\frac{\delta}{\delta\theta_k}\left(\frac{\langle\Omega|H_{CG} - E|\Omega\rangle}{\langle\Omega|\Omega\rangle}\right) = 0$$

- **generates quark gap equation for** $\phi_k = \phi(k)$
  and BCS angle $\theta_k = \theta(k)$ **with** $tan(\phi_k - \theta_k) = m/k$

$$k s_k - m c_k = \frac{2}{3}\int \frac{d\mathbf{q}}{(2\pi)^3} \left[(s_k c_q x - s_q c_k) V(|\mathbf{k}-\mathbf{q}|) - 2 c_k s_q U(|\mathbf{k}-\mathbf{q}|) + 2 c_q s_k W(|\mathbf{k}-\mathbf{q}|) \right]$$

$s_k = sin\, \phi_k$, $c_k = cos\, \phi_k$

- **solve gap eq. for quark constituent mass**

$$E_k = \sqrt{M_u(k)^2 + k^2} = M_u(k)/sin\phi_k$$
$$k \to \mathit{0} \;\; gives \;\;\; M_u(0) \cong 125\,\mathrm{MeV}$$

- **predict quark condensate (cooper pairs)**

$$\langle q\bar{q}\rangle = \langle\Omega|\bar{\Psi}(0)\Psi(0)|\Omega\rangle = -(177\,\mathrm{MeV})^3$$
$$comparable\; to\; QCD sum\; rule\; value\; -(236\,\mathrm{MeV})^3$$

- **TDA $q\bar{q}$ meson wavefunction**

$$|\Psi^{JPC}_{LS}\rangle = \sum_{\mathcal{C}\lambda\bar{\lambda}} \int \frac{d\mathbf{k}}{(2\pi)^3} \Phi^{JPC}_{\lambda\bar{\lambda}}(\mathbf{k}) B^{\dagger}_{\lambda\mathcal{C}}(\mathbf{k}) D^{\dagger}_{\bar{\lambda}\mathcal{C}}(-\mathbf{k})|\Omega\rangle$$

- **solve for meson masses M_{JPC}**

$$H_{CG}|\Psi^{JPC}_{LS}\rangle = M_{JPC}|\Psi^{JPC}_{LS}\rangle \;\; agrees\; with\; data$$

- **predict a Regge trajectory consistent with data**

$$J = \alpha(t) = bt + \alpha(0)$$
$$\alpha_M \approx .9t + .5 \qquad (t = M_J^2)$$

- **chiral charge operator** $Q_5 = \int d\mathbf{x}\Psi^\dagger(\mathbf{x})\gamma_5\Psi(\mathbf{x})$
  commutes with H_{CG}

- **TDA** $q\bar{q}$ **operator**

$$A^\dagger_{TDA} = \sum_{\mathcal{C}\lambda\bar{\lambda}} \int \frac{d\mathbf{k}}{(2\pi)^3} \Phi^{JPC}_{\lambda\bar{\lambda}}(\mathbf{k}) B^\dagger_{\lambda\mathcal{C}}(\mathbf{k}) D^\dagger_{\bar{\lambda}\mathcal{C}}(-\mathbf{k})$$

does not commute with Q_5 *yielding* $M_\pi \approx$ *500 MeV*

- **RPA** $q\bar{q}$ **operator**

$$A^\dagger_{RPA} = \sum_{\mathcal{C}\lambda\bar{\lambda}} \int \frac{d\mathbf{k}}{(2\pi)^3} [\mathcal{X}^{JPC}_{\lambda\bar{\lambda}}(\mathbf{k}) B^\dagger_{\lambda\mathcal{C}}(\mathbf{k}) D^\dagger_{\bar{\lambda}\mathcal{C}}(-\mathbf{k}) - \mathcal{Y}^{JPC}_{\lambda\bar{\lambda}}(\mathbf{k}) B_{\lambda\mathcal{C}}(\mathbf{k}) D_{\bar{\lambda}\mathcal{C}}(-\mathbf{k})]$$

does commute with Q_5 *yielding* $M_\pi \approx$ *150 MeV*
chiral symmetry is responsible for 2/3 of the π/ρ *splitting*

- **reasonable heavy meson hyperfine splittings**

in MeV	CG	lattice	NRQCD	data
$\eta_c - J/\Psi$	125	90	104	117.7
$\eta_b - \Upsilon$	$70^\dagger$	$61^\dagger$	$39^\dagger$	71.4
m_{η_b}	$9395^\dagger$	$9409^\dagger$	$9421 \pm 11^\dagger$	9389

† *predicted before bottomium discovery*

Application to gg glueballs (pure glue sector $\rightarrow$ quenched approximation)

- **Bogoliubov-Valatin canonical transformation**
  BCS rotation to dressed gluon operators

$$\alpha^a_\mu(\mathbf{k}) = \cosh\Theta(k)\, a^a_\mu(\mathbf{k}) + \sinh\Theta(k)\, a^{a\dagger}_\mu(-\mathbf{k})$$

- **PQCD vacuum** $|0>\Longrightarrow$ **BCS vacuum** $|\Omega_{gluon}>$

$$|\Omega_{gluon}\rangle = \exp\left(-\int\frac{d\mathbf{k}\tanh\Theta(k)}{2(2\pi)^3}D_{\mu\nu}(\mathbf{k})a^{a\dagger}_\mu(\mathbf{k})a^{a\dagger}_\nu(-\mathbf{k})\right)|0\rangle$$

- **minimize ground state energy**

$$\frac{\delta}{\delta\Theta(k)}\left(\frac{\langle\Omega|H_{CG}-E|\Omega\rangle}{\langle\Omega|\Omega\rangle}\right) = 0$$

- **generates a gap equation for** $\omega(k) = ke^{-2\Theta(k)}$

$$\omega(k)^2 = k^2 - \frac{3}{4}\int\frac{d\mathbf{q}}{(2\pi)^3}V(|\mathbf{k}-\mathbf{q}|)(1+(\hat{\mathbf{k}}\cdot\hat{\mathbf{q}})^2)\left(\frac{\omega(q)^2-\omega(k)^2}{\omega(q)}\right)$$

- **solve gap eq. for gluon constituent mass**

$$m_g \equiv \omega(0) \cong 800\, MeV$$

- **predict gluon condensate (cooper pairs)**

$$\langle \alpha G^a_{\mu\nu} G^{\mu\nu}_a \rangle = (433\ MeV)^4$$
$$[agrees\ with\ lattice\ (441\ MeV)^4]$$

- **TDA gg glueball wavefunction**

$$|\Psi^{JPC}_{LS}\rangle = \sum_{am_1m_2} \int \frac{d\mathbf{k}}{(2\pi)^3} \Phi^{JPC}_{LSm_1m_2}(\mathbf{k}) \alpha^{a\dagger}_{m_1}(\mathbf{k}) \alpha^{a\dagger}_{m_2}(-\mathbf{k}) |\Omega\rangle$$

- **solve for glueball mass M_{JPC}**

$$H_{CG}|\Psi^{JPC}_{LS}\rangle = M_{JPC}|\Psi^{JPC}_{LS}\rangle \ \ [agrees\ with\ lattice]$$

- **predict Regge trajectory close to the pomeron**

$$\alpha_P \approx .25t + 1$$

Application to $|ggg>$ glueballs (oddballs)

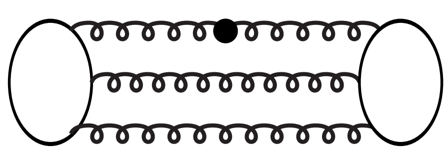
- **variational three gluon glueball wavefunction**

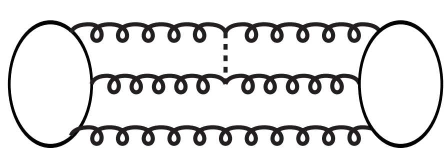
$$|\Psi^{JPC}\rangle = \int d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3 \delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)$$
$$F^{JPC}_{\mu_1\mu_2\mu_3}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) C^{abc} \alpha^{a\dagger}_{\mu_1}(\mathbf{q}_1)\alpha^{b\dagger}_{\mu_2}(\mathbf{q}_2)\alpha^{c\dagger}_{\mu_3}(\mathbf{q}_3)|\Omega\rangle$$

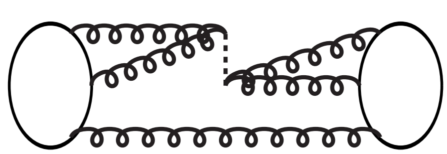
- **Bose statistics** $C^{abc} = f^{abc}$ **(**$C=1$**) or** d^{abc} **(**$C=-1$**)**
- **variational equation for the** J^{PC} **glueball**

$$\frac{\langle\Psi^{JPC}|H_{CG}|\Psi^{JPC}\rangle}{\langle\Psi^{JPC}|\Psi^{JPC}\rangle} = M_{JPC} \ \ [agrees\ with\ lattice]$$

- **self-energy, scattering & annihilation diagrams**







Predicted glueball masses (GeV)

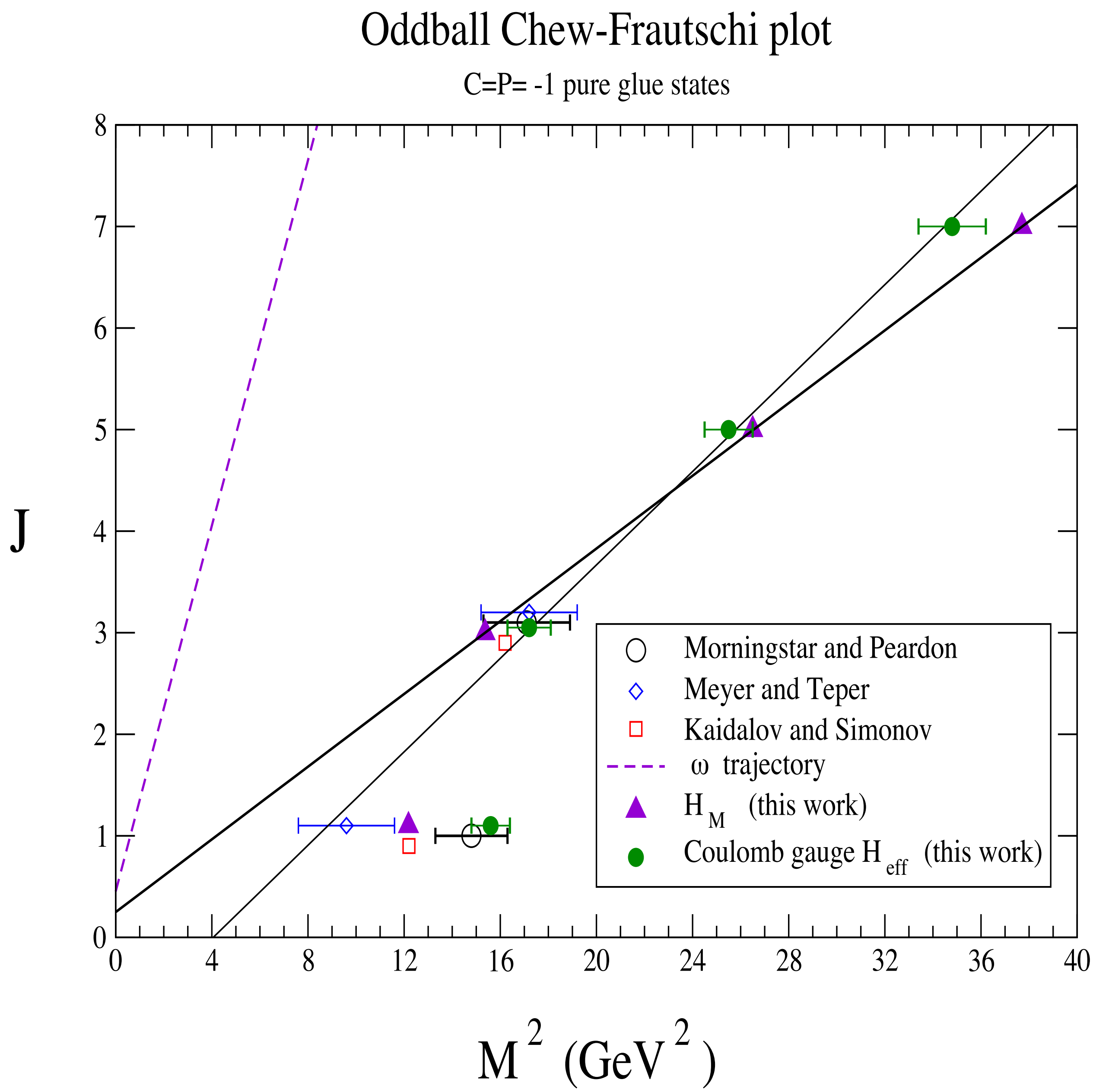
Model	J^{PC}	0^{-+}	1^{--}	2^{--}	3^{--}	5^{--}	7^{--}
our work	color	f	d	d	d	d	d
H_{CG}		3.90	3.95	4.15	**4.15**	5.05	5.90
H_M		3.40	3.49	3.66	**3.92**	5.15	6.14
lattice [1]		3.64	3.85	3.93	**4.13**		
lattice [2]		3.25	3.10	3.55	**4.15**		
Wilson loops[3]		3.77	3.49	3.71	**4.03**		

[1] C. J. Morningstar and M. Peardon, Phys. Rev. D **60**, *034509 (1999)*

[2] H. B. Meyer and M. Teper, Phys. Lett. B **605**, *344 (2005)*

[3] A. B. Kaidalov and Y. A. Simonov, Phys. Lett. B **477**, *163 (2000)*

lattice, H_{CG} and H_M odderons vs. ω trajectory



- **predict 1^{--}, 3^{--}, 5^{--}, 7^{--} states, yields leading Regge trajectory (odderon)**

$$\alpha_O = .23t - 0.88 \; [note\ negative\ intercept]$$

Summary of ggg calculations

- lattice, H_{CG} and H_M models predict an odderon
- starts with 3^{--} (not 1^{--}, need lattice 5^{--})
- odderon slope is similar to pomeron
- odderon intercept is low, below 0.5
- search for odderon where ω trajectory is not dominant (examine $\frac{d\sigma}{dt}$, not $\sigma_{total} \propto s^{\alpha_\omega(0)-1}$)

Application to $q\bar{q}g$ hybrid mesons (combined quark & gluon sector)

- **hybrid meson wavefunction**

  color structure $[(3 \otimes \bar{3})_8 \otimes 8]_1$

$$|\Psi^{JPC}\rangle = \int d\mathbf{q} d\bar{\mathbf{q}} d\boldsymbol{g} \delta(\mathbf{q} + \bar{\mathbf{q}} + \boldsymbol{g})$$

$$\Phi^{JPC}_{\lambda\bar{\lambda}\mu}(\mathbf{q}, \bar{\mathbf{q}}, \boldsymbol{g}) T^a_{\mathcal{C}\bar{\mathcal{C}}} B^\dagger_{\lambda\mathcal{C}}(\mathbf{q}) D^\dagger_{\bar{\lambda}\bar{\mathcal{C}}}(\bar{\mathbf{q}}) \alpha^{a\dagger}_\mu(\boldsymbol{g}) |\Omega\rangle$$

- **coordinates** $\mathbf{q}_- \equiv \mathbf{q} - \bar{\mathbf{q}}$, $\quad \mathbf{q}_+ \equiv (\mathbf{q} + \bar{\mathbf{q}})/2 = -\boldsymbol{g}/2$

  two orbital $L_\pm$

- **possible exotic quantum numbers**

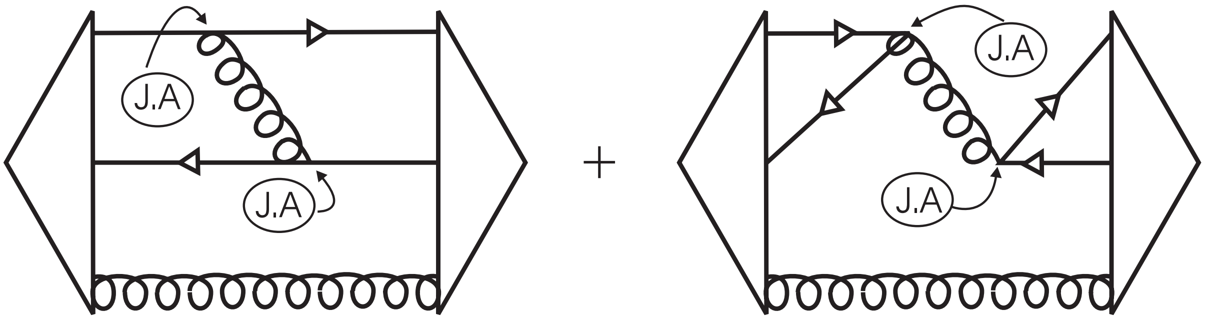
  $P = (-1)^{L_+ + L_-} \qquad C = (-1)^{L_- + S_{q\bar{q}} + 1}$

  $\mathbf{J} = \mathbf{L}_+ + \mathbf{L}_- + \mathbf{S}_{\mathbf{q}\bar{\mathbf{q}}} + \mathbf{S}_{\boldsymbol{g}} \quad \Rightarrow J^{PC} = 0^{--}, 1^{-+}, 3^{-+}$

- **variational equation for J^{PC} hybrids**

$$\frac{\langle \Psi^{JPC} | H_{CG} | \Psi^{JPC} \rangle}{\langle \Psi^{JPC} | \Psi^{JPC} \rangle} = M_{J^{PC}}$$

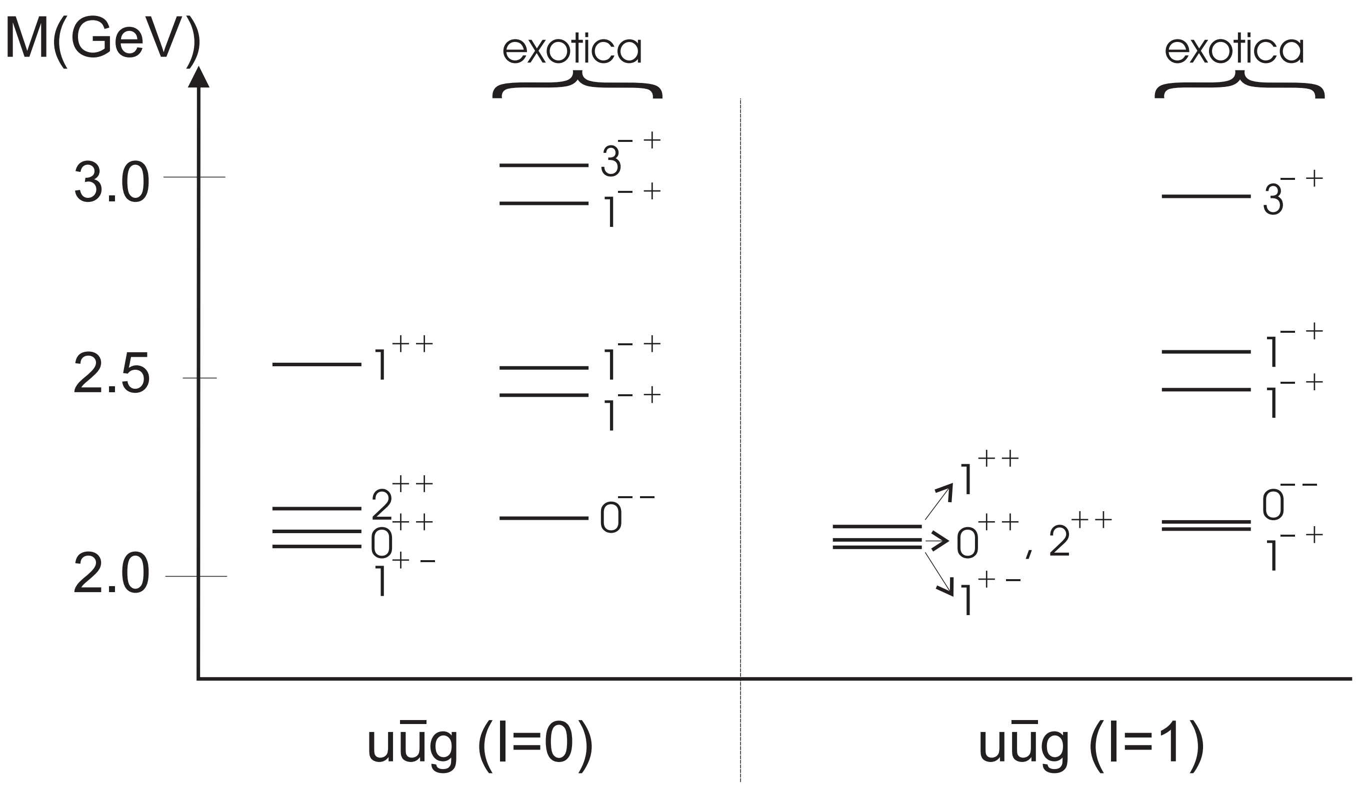
- **interaction & annihilation diagrams**



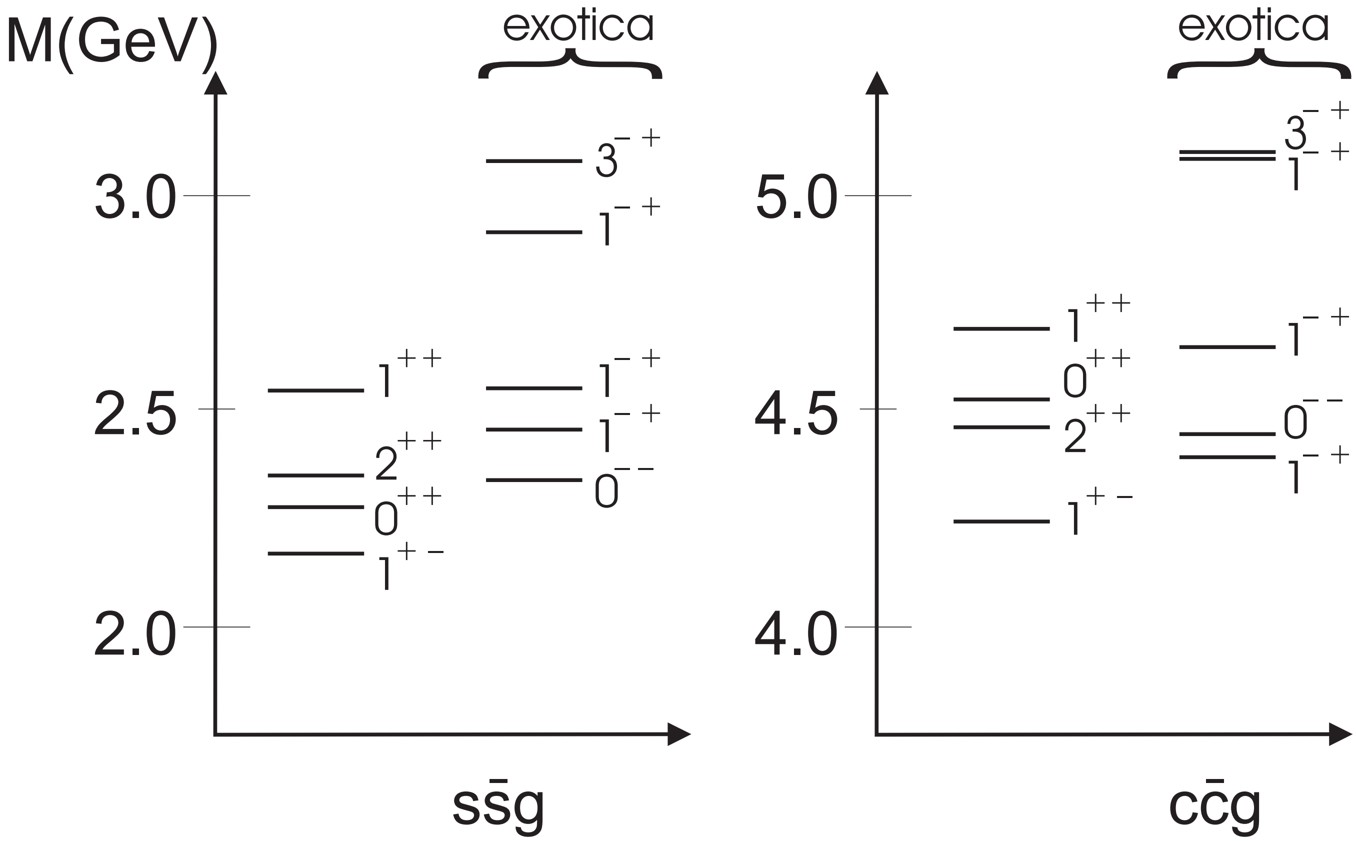
- **repulsive annihilation produces isospin splitting**

  $q\bar{q} \to g \to q\bar{q}$ *for* $I = 0, S_{q\bar{q}} = 1$ *states (repulsive)*
  $\Rightarrow I = 0$ *hybrids heavier (≤300 MeV) than* $I = 1$

- **predicted** $I = 0, 1$ $u\bar{u}g$ J^{PC} **spectrum**



• **predicted $s\bar{s}g$ and $c\bar{c}g$ J^{PC} spectrum**

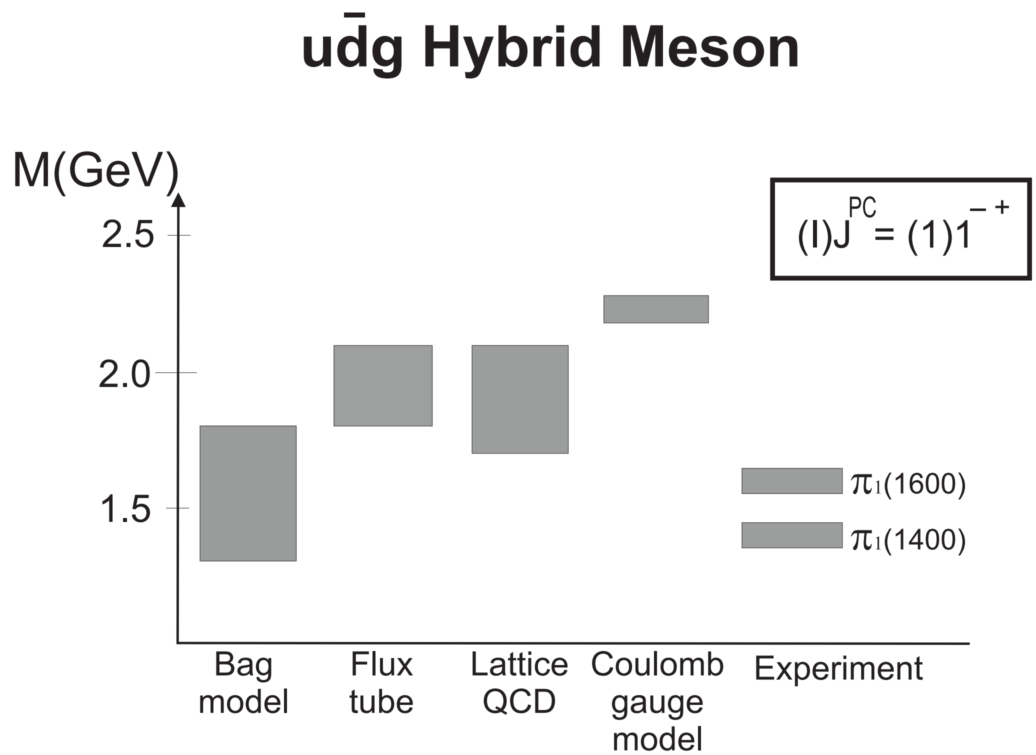


• **model comparison for $J^{PC} = 1^{-+}$ hybrids (GeV)**

Model	u/d hybrid	s hybrid	c hybrid
Coulomb Gauge	2.2	2.3	4.4
Lattice QCD	1.7 - 2.1	1.9	4.2 - 4.4
Flux Tube	1.8 - 2.1	2.1 - 2.3	4.1 - 4.5
Bag	1.3 - 1.8		3.9

Summary of $q\bar{q}g$ calculations

- model agreement for light and heavy hybrids (lattice, flux tube and Coulomb gauge)
- models predict 1^{-+} hybrid mass near 2 GeV
- $\Rightarrow \pi_1(1400)$ & maybe $\pi_1(1600)$ are $q\bar{q}q\bar{q}$ states



Application to $q\bar{q}q\bar{q}$ mesons (tetraquark results)

- $q\bar{q}q\bar{q}$ **wavefunction**

$$|\Psi^{JPC}\rangle = \int d\mathbf{q} d\bar{\mathbf{q}} d\mathbf{q}' d\bar{\mathbf{q}}' \delta(\mathbf{q}+\bar{\mathbf{q}}+\mathbf{q}'+\bar{\mathbf{q}}')$$

$$\Phi^{JPC}_{\lambda\bar{\lambda}\lambda'\bar{\lambda}'}(\mathbf{q},\bar{\mathbf{q}},\mathbf{q}',\bar{\mathbf{q}}') T^a_{\mathcal{C}\bar{\mathcal{C}}} T^a_{\mathcal{C}'\bar{\mathcal{C}}'} B^\dagger_{\lambda\mathcal{C}}(\mathbf{q}) D^\dagger_{\bar{\lambda}\bar{\mathcal{C}}}(\bar{\mathbf{q}}) B^\dagger_{\lambda'\mathcal{C}'}(\mathbf{q}') D^\dagger_{\bar{\lambda}'\bar{\mathcal{C}}'}(\bar{\mathbf{q}}')|\Omega\rangle$$

- **coordinates** $\mathbf{q}_- \equiv \mathbf{q}-\bar{\mathbf{q}}$, $\mathbf{q}'_- \equiv \mathbf{q}'-\bar{\mathbf{q}}'$, $\mathbf{q_r}$

  three orbital L_-, L'_-, L_r

- **possible exotic quantum numbers**

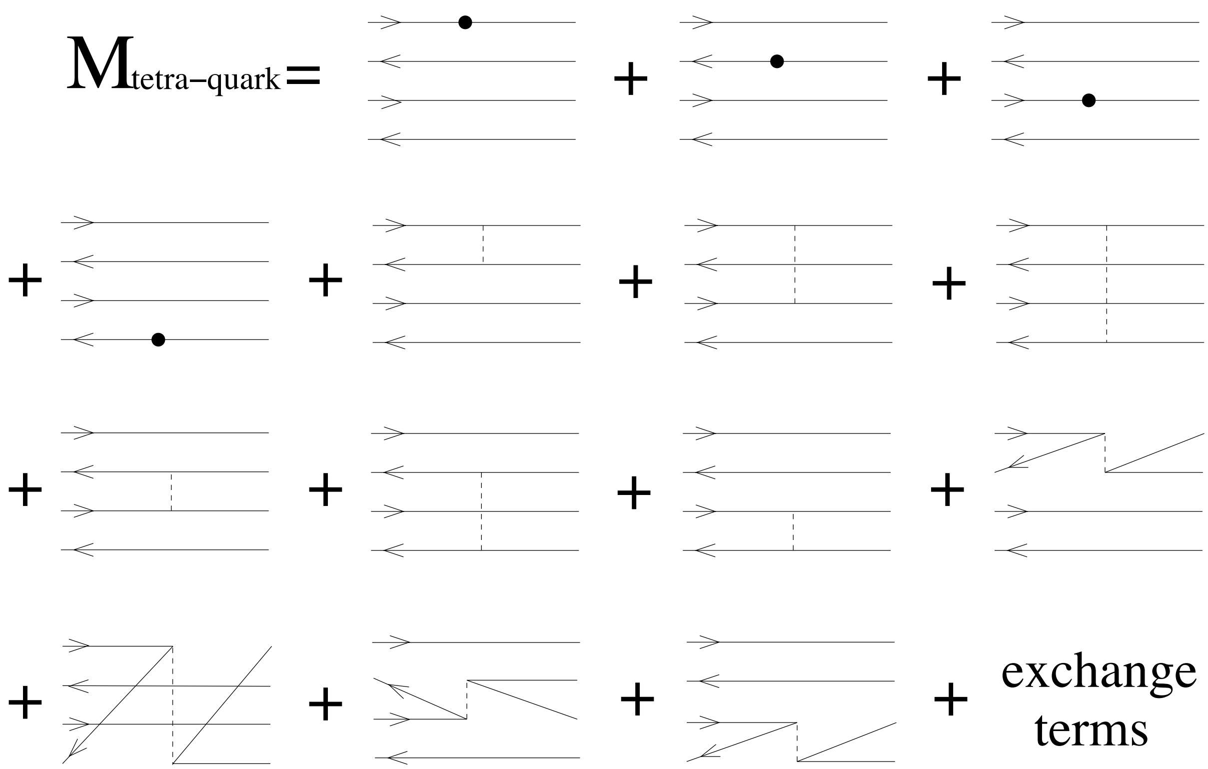
  $P = (-1)^{L_- + L'_- + L_r}$ $C = (-1)^{L_- + S_{q\bar{q}} + L'_- + S_{q'\bar{q}'}}$

  $\mathbf{J} = \mathbf{L}_- + \mathbf{L}'_- + \mathbf{L_r} + \mathbf{S_{q\bar{q}}} + \mathbf{S_{q'\bar{q}'}} \Rightarrow J^{PC} = 0^{--}, 1^{-+}, 3^{-+}$

- **variational equation for** J^{PC} **tetraquarks**

$$\frac{\langle\Psi^{JPC}|H_{CG}|\Psi^{JPC}\rangle}{\langle\Psi^{JPC}|\Psi^{JPC}\rangle} = M_{JPC}$$

- **self-energy, scattering & annihilation diagrams**



- **isospin $\boldsymbol{I = I_{q\bar{q}} + I_{q'\bar{q}'}}$**, $I_{q\bar{q}} = 1, 0$

  yields six isospins states
  I=2 (1⊗1), 1(1⊗1, 1⊗0, 0⊗1), 0(1⊗1, 0⊗0)

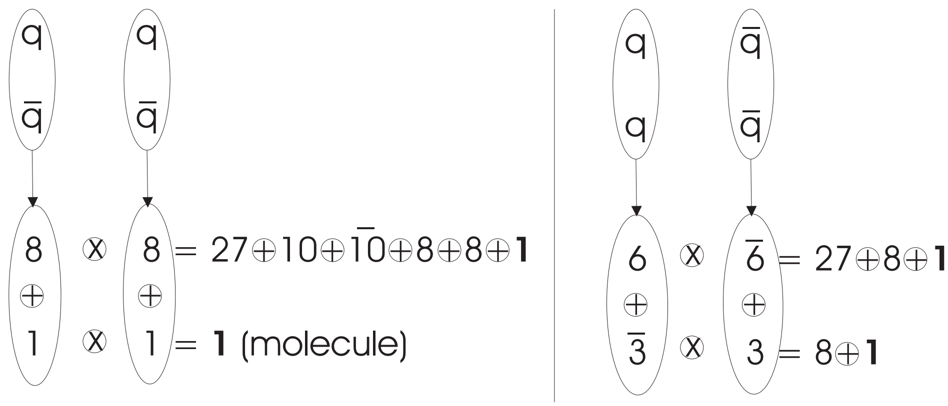
- **repulsive annihilation produces isospin splitting**

  I = 2 lightest, then I = 1(1⊗1),
  then 2 degenerate I =1 (1⊗0), I = 0 heaviest

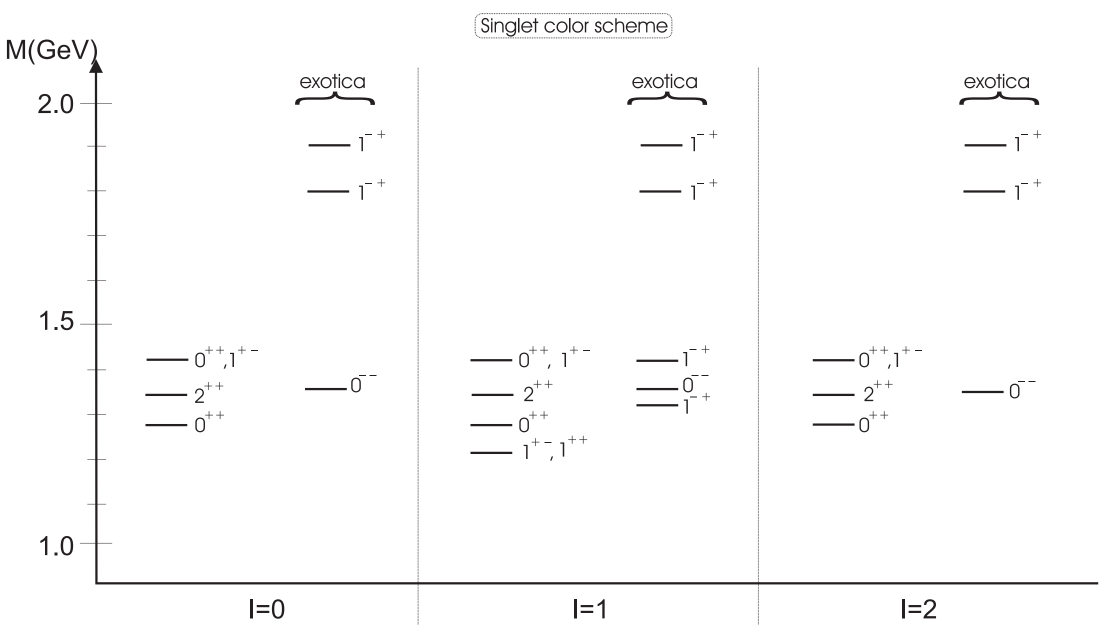
- **four possible color singlet configurations**

  $[(3\otimes\bar{3})_8\otimes(3'\otimes\bar{3}')_8]_1$ **exotic atom**
  $[(3\otimes\bar{3})_1\otimes(3'\otimes\bar{3}')_1]_1$ **meson-meson molecule**
  $[(3\otimes 3')_6\otimes(\bar{3}\otimes\bar{3}')_{\bar{6}}]_1$ **exotic diquark type atom**
  $[(3\otimes 3')_{\bar{3}}\otimes(\bar{3}\otimes\bar{3}')_3]_1$ **exotic diquark type atom**

(C parity forbids diquark type states for 1^{-+})



- **lightest is meson-meson molecule $J^{PC} = 1^{-+}$**



Summary of $q\bar{q}q\bar{q}$ calculations

- H_{CG} predicts exotic atoms ($q\bar{q}$ color octets) heavier than meson molecules ($q\bar{q}$ color singlets) (due to repulsion in color octet state)

- H_{CG} predicts isospin splitting for exotic atoms, I = 2 lightest, then I = 1, then I =0

- no isospin splitting for meson molecules due to color suppressed quark interactions and annihilation between mesons

- predicted 1^{-+} exotic atom mass 2.3 GeV comparable to exotic 1^{-+} hybrid mass 2.2 GeV

- predicted 1^{-+} meson molecule masses near 1.4 and 1.8 GeV

- implies $\pi_1(1400)$, $\pi_1(1600)$ are meson molecules

Application to $q\bar{q}c\bar{c}$ mesons

(X, Y and Z charmed tetraquarks)

EXPERIMENT

- **no states seen blelow strong decay thresholds $\omega J/\psi$, $\rho J/\psi$, $D\bar{D}$**

- **X(3872) detected by Belle Collaboration in 2003 confirmed by BaBar Collaboration in 2005**

- **several other heavier X and Y states have also been discovered and confirmed**

- **$Z_c^+(3900)$ detected by BESIII Collaboration and Belle in 2013, first charged tetraquark**

- **$Z_c^0(3900)$ detected by BESIII Collaboration in 2015, neutral $I = 1$ partner to $Z_c^+(3900)$**

- **observed X, Y, Z dominant decay channels are $\omega J/\psi$, $\rho J/\psi$, $D^*\bar{D}$**

THEORY

- **lattice (LQCD) gets no states below strong decay thresholds $\omega J/\psi$, $\rho J/\psi$, $D\bar{D}$**

- **above threshold more challenging but LQCD confirms X(3872), belief is a $D^*\bar{D}$ molecule**

- **LQCD (Prelovsek et al) does not get any I = 1 Z_c states below 4.2 GeV, a puzzle**

- **H_{CG} predicts X(3872) is a mixture of two molecular states with flavor/spin $\omega J/\psi$, $D^*\bar{D}$**

- **H_{CG} predicts $Z_c(3900)$ is a mixture of two molecular states with flavor/spin $\rho J/\psi$, $D^*\bar{D}$**

SUMMARY

- **need improved LQCD studies above threshold**

- **need H_{CG} mixing analysis**

isoscalar spectra

$\mathbf{J^{PC} = 0^{++}}$	$\mathbf{0^{-+}}$	$\mathbf{1^{--}}$
$f_0(600) \to \pi\pi$	$\eta \to \gamma\gamma, 3\pi$	$\omega(782) \to 3\pi$
$f_0(980) \to \pi\pi$	$\eta'(958) \to \eta\pi\pi$	$\phi(1020) \to K\bar{K}$
$f_0(1370) \to \rho\rho$	$\eta(1295)$	$\omega(1420) \to \rho\pi$
$f_0(1500) \to 4\pi, \pi\pi$	$\eta(1405)$	$\omega(1650)$
$f_0(1710)$	$\eta(1475) \to K\bar{K}\pi$	$\phi(1680) \to KK^*$
$f_0(2020)^\dagger$	$\eta(1760)^\dagger$	$\phi(2170)^\dagger$
$f_0(2100)^\dagger$	$\eta(2225)^\dagger$	
$f_0(2200)^\dagger$		
$f_0(2330)^\dagger$		

isovector spectra

(*no gluonic or $s\bar{s}$ components*)

$\mathbf{0^{++}}$	$\mathbf{0^{-+}}$	$\mathbf{1^{--}}$
$a_0(980) \to \eta\pi$	$\pi^0 \to \gamma\gamma$	$\rho(770) \to \pi\pi$
$a_0(1450)$	$\pi(1300)$	$\rho(1450)$
	$\pi(1800)$	$\rho(1570)^\dagger$
		$\rho(1700) \to \rho\pi\pi$
		$\rho(1900)^\dagger$
		$\rho(2150)^\dagger$

† **not considered established by PDG (2010)**

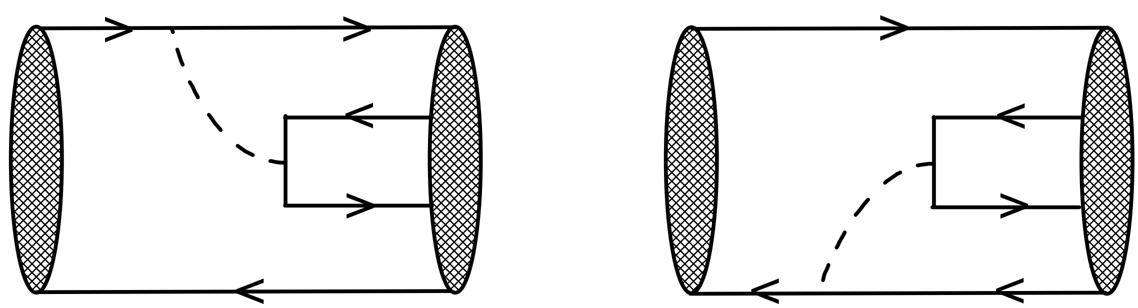
Application to $q\bar{q}$ and $q\bar{q}q\bar{q}$ mixing

- **expand using unmixed model states**

$$|J^{PC}\rangle = a|n\bar{n}\rangle + b|s\bar{s}\rangle + c_i|n\bar{n}n\bar{n}\rangle_{i=1,2} + d_i|n\bar{n}s\bar{s}\rangle_{i=1,2}$$
$$where \;\; n\bar{n} = (uu + dd)/\sqrt{2}$$

- **compute off-diagonal mixing elements**

$$M = \langle q\bar{q}|H_C^{\mathrm{CG}}|q\bar{q}q\bar{q}\rangle$$



$$M_1 = \frac{1}{2}\int d\boldsymbol{q}_1 d\boldsymbol{q}_2 d\boldsymbol{q}_3 V(k)\mathcal{U}^{\dagger}_{\lambda_1}(\boldsymbol{q}_1)\mathcal{U}_{\lambda'_1}(-\boldsymbol{q}_4)$$
$$\mathcal{U}^{\dagger}_{\lambda_3}(\boldsymbol{q}_3)\mathcal{V}_{\lambda_2}(\boldsymbol{q}_2)\Phi^{JPC\dagger}_{\lambda_1\lambda_2\lambda_3\lambda_4}(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)\Phi^{JPC}_{\lambda'_1\lambda_4}(-2\boldsymbol{q}_4)$$

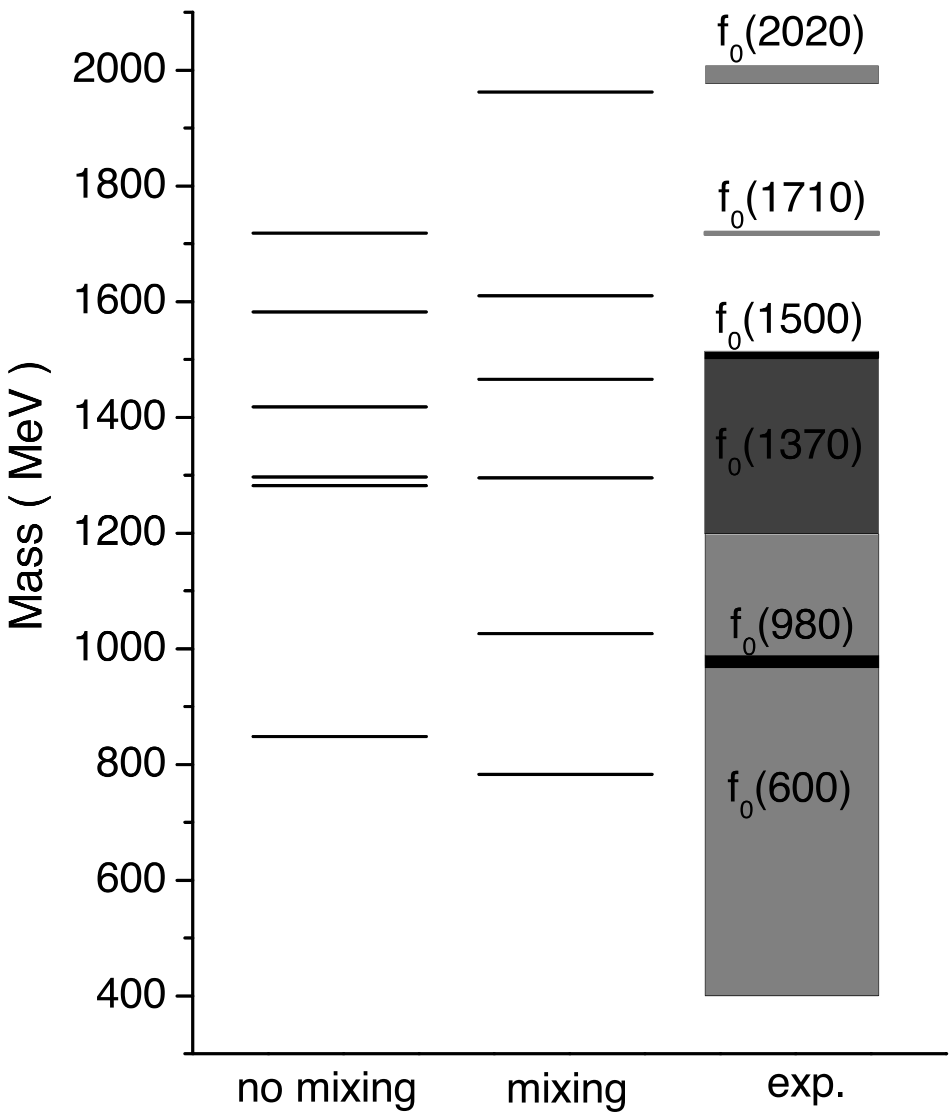
- **for** 0^{++} **states**

$$\langle s\bar{s}|H_C^{\mathrm{CG}}|n\bar{n}s\bar{s}\rangle = 365MeV$$
$$\langle n\bar{n}|H_C^{\mathrm{CG}}|n\bar{n}n\bar{n}\rangle = 166MeV$$
$$\langle n\bar{n}|H_C^{\mathrm{CG}}|n\bar{n}s\bar{s}\rangle = 45MeV$$

- **diagonalize**

$$< q\bar{q}\ and\ q\bar{q}q\bar{q}|H_{CG} - M_{J^{PC}}|J^{PC}\rangle = 0$$

- **obtain improved 0^{++} scalar meson spectrum**



- **for pseudoscalar states**

$$|0^{-+}\rangle = a|n\bar{n}\rangle + b|s\bar{s}\rangle + c|n\bar{n}n\bar{n}\rangle + d|n\bar{n}s\bar{s}\rangle$$

- **numerical results**

	$n\bar{n}$	$s\bar{s}$	$\|n\bar{n}n\bar{n}>$	$\|n\bar{n}s\bar{s}>$
no mixing	610	1002	1252	1552
mixing	531	970	1316	1598
exp.	η	η'	$\eta(1295)$	$\eta(1405)$
	548	958	1294	1410
coeff.	a	b	c	d
η	0.951	-0.046	0.279	0.126
η'	0.032	0.973	-0.046	0.223
$\eta(1295)$	-0.289	0.036	0.953	0.080
$\eta(1405)$	-0.108	-0.222	-0.105	0.963

- **for vector states calculated mixing is weak !**

CG model predicts ideal mixing $\omega \approx |n\bar{n}>$, $\phi \approx |s\bar{s}>$

General summary

- diagonalization/variational approach provides insightful framework for comprehensively studying systems with quarks and gluons
- is amendable to improvements in H_{eff} , quaisi-particle dressing and systematic Fock space expansion

Future work

- improved H_{eff} and Fock space expansion
- complete mixing $q\bar{q}$, gg, ggg, $q\bar{q}g$ and $q\bar{q}q\bar{q}$ and obtain improved vacuum
- apply to $q\bar{q}b\bar{b}$, pentaquarks $qqq\bar{q}q$, nucleon strangeness $|N>= \alpha|uud> +\beta|uuds\bar{s}>$ and hybrid baryons $qqqg$
- study dibaryons $uuddss$ and baryonium $qqq\bar{q}\bar{q}\bar{q}$
- predict decay signatures for identifying exotica
- applications to NSM physics (e.g. composite super symmetry particles)?

Exotica decay signatures

predict scalar (0^{++}) G_0 and tensor (2^{++}) G_2 glueball decay widths

use vector meson dominance, flavor independence and the pomeron/glubball connection to extract a phenomenological coupling constant

hadronic decay widths to 2 vector mesons

$(in\ MeV)\ VV' \rightarrow$	$\rho^0\rho^0$	$\omega\omega$	$\phi\phi$	$\omega\phi$
$\Gamma_{G(1700)\rightarrow VV'}$	44.3	34.2	$forbidden$	?
$\Gamma_{G_2(2010)\rightarrow VV'}$	26.2	25.8	10.3	33.0
$\Gamma_{G_2(2300)\rightarrow VV'}$	37.2	36.8	20.3	44.7

radiative decay widths

$(in\ keV)\ V \rightarrow$	ρ^0	ω	ϕ	γ
$\Gamma_{G(1700)\rightarrow V\gamma}$	1950	844	453	15.1
$\Gamma_{G_2(2010)\rightarrow V\gamma}$	298	129	91.6	1.72
$\Gamma_{G_2(2300)\rightarrow V\gamma}$	377	164	128	1.96

look for comparable $\rho\rho(4\pi)$ and $\omega\omega(6\pi)$ decays correlated with suppressed $\omega\gamma$ and $\phi\gamma$ relative to $\rho\gamma$

novel decay is to the $\omega\phi \rightarrow 3\pi K\bar{K}$ channel

Simple glueball constituent model
(see also Cornwall, Soni, Hou, PR D 29, 101 (1984))

$$H_M = \sum_i \frac{\mathbf{q}_i^2}{2m_g} + V_0 + \sum_{i<j} [\sigma r_{ij} - \frac{\alpha}{r_{ij}} + V_{ss} \mathbf{S}_i \cdot \mathbf{S}_j]$$

- **use** $q\bar{q}$ **funnel potential parameters**

  $V_0 =$ **-.9 GeV,** $\alpha =$ **.27,** $\sigma =$ **.25 GeV**^{2} $m_g =$ **.8 GeV**

  adjust $V_{ss} \rightarrow 0.085$ **GeV to produce** 2^{++} **&** 4^{++}
  $|gg>$ **glueballs yielding the pomeron**

  $$\alpha_P^M = .23t + 1.0$$

- **exactly diagonalize (Jacobi coordinates, large oscillator basis) to predict** $|ggg>$ **oddball** $J^{PC} = 1^{--}$, $3^{--}, 5^{--}, 7^{--}$ **states**
- **yields the leading Regge trajectory (odderon)**

  $$\alpha_O^M = .18t + 0.25 \text{ [\textbf{note low intercept}]}$$

- **odderon has similar slope to pomeron**
- **starts with** 3^{--} **like pomeron beginning with** 2^{++}
- 1^{--} **on daughter odderon**
  (like 0^{++} **on daughter pomeron)**

Selected references

PRL **96**, *081601 (2006);* **84**, *1102 (2000)*
PLB **725**, *148(2013);* **653**, *216(2007);* **504**,*15(2001)*
PRC **70**, *035202 (2004)*
EPJ A **31**, *656 (2007)*
EPJ C **55**, *409 (2008); C* **51**, *347 (2007)*