Recent Developments in Numerical Relativity

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Contents

Outline

What is numerical relativity

Numerical Relativity:

solving numerically the full GR equations, typically for dynamical spacetimes in the strong field regime, where no approximations hold.

Goals:

understanding gravity in its full non-linear glory.

Challenges:

very difficult problem…

Why numerical relativity

Study of systems with strong and dynamical gravitational fields

- Gravitational radiation
  - Astrophysics, gravitational wave astronomy
- Mathematical and theoretical Physics
  - Cosmic censorship,
  - Instabilities (Black hole interior, Myers-Perry)
- High-energy particle systems
  - AdS/CFT correspondence;
  - Black hole production at the LHC;

Gravitational waves

- Accelerated bodies emit gravitational radiation
- Detected indirectly by measurements of the Hulse-Taylor binary system (1993 Nobel Prize)
- Interact weakly with matter $\Rightarrow$ carry unique information about astronomical phenomena
    - $\Rightarrow$ New window to the universe

Gravitational waves

- Difficult to detect
  - $\Rightarrow$ Need theoretical models for the structure of the waveform

Mathematical and theoretical Physics

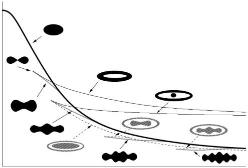
- Cosmic censorship hypothesis:
  - does it hold under extreme conditions?

    Sperhake, Cardoso, Pretorius, Berti, Gonzales, 2008
- No no-hair theorem for $D > 4 \Rightarrow$ black hole solutions with non-spherical topology.
  - (Non-)Linear stability of higher-dimensional black objects:
    - Black string

      Choptuik, Lehner, Olabarrieta, Petryk, Pretorius, Villegas, 2003

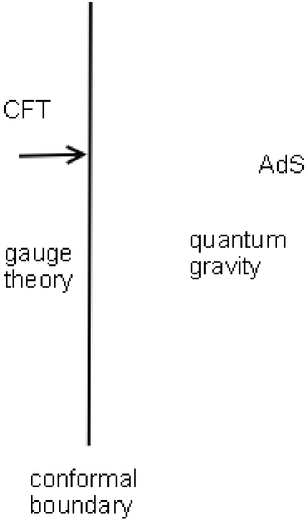
      Lehner, Pretorius 2010
    - Myers-Perry black hole

      Shibata & Yoshino, 2010
    - Black ring
    - ...



Emparan & Reall, 2008

AdS/CFT duality



- Properties of strongly coupled thermal gauge theories are related to the physics of higher-dimensional black holes
- Formation of quark-gluon plasma at the RHIC
  $\Leftrightarrow$ black hole collisions in AdS_5

Issues with numerical simulations in AdS:

- AdS is not globally hyperbolic
- The boundary plays an active role

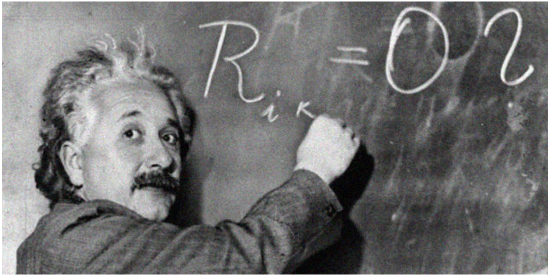
Outline

History and milestones

1915	Einstein's equations are published	Einstein
1964	First documented attempts at numerical simulations: evolving two wormholes	Hahn & Lindquist
1976	Head-on collision of two black holes (in axisymmetry)	Smarr & Eppley
1990's	"Binary Black Hole Grand Challenge Project"	Matzner et al
1993	Critical phenomena in gravitational collapse	Choptuik
1997	Release of Cactus 1.0	Seidel et al
1998	Generic (3D) single BH simulation (using a characteristic approach)	Gomez et al
1999	BSSN evolution system	Baumgarte & Shapiro; Shibata & Nakamura
2005	First simulations of BH binaries through inspiral, merger and ringdown (Two-body problem in GR) (GHG code)	Pretorius
2006	"Moving puncture" simulations (BSSN code)	UTB/RIT; NASA Goddard
2008	High-energy collision of two BHs	Berti, Cardoso, Gonzalez, Sperhake, Pretorius
2010	Collision of gravitational shock waves in AAdS5 spacetimes (2+1 code)	Chesler & Yaffe
2010	Black hole collisions in higher dimensions	Witek, M.Z. et al; Yoshino & Shibata
2012	Simulations of AAdS5 spacetimes (GHG code)	Bantilan, Pretorius, Gubser
2015	Off-center collisions of shock waves in AAdS5 spacetimes (4+1 code)	Chesler &Yaffe

Outline

Einstein's equations



$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi\, T_{\mu\nu}$$

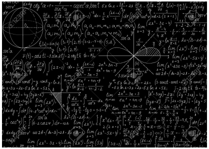
Einstein's equations

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} \sum_{\delta=t,x^1,\ldots,x^{D-1}} g^{\alpha\delta} \left(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma}\right)$$

$$8\pi T_{\alpha\beta} = \sum_{\delta} \left[\partial_\delta \Gamma^{\delta}_{\alpha\beta} - \partial_\alpha \Gamma^{\delta}_{\delta\beta} + \sum_{\gamma} (\Gamma^{\delta}_{\alpha\beta}\Gamma^{\gamma}_{\delta\gamma} - \Gamma^{\delta}_{\gamma\beta}\Gamma^{\gamma}_{\delta\alpha})\right]$$
$$- \frac{1}{2} g_{\alpha\beta} \sum_{\delta,\gamma} \left\{ g^{\delta\gamma} \sum_{\mu} \left[\partial_\mu \Gamma^{\mu}_{\delta\gamma} - \partial_\delta \Gamma^{\mu}_{\mu\delta} + \sum_{\nu} (\Gamma^{\mu}_{\delta\gamma}\Gamma^{\nu}_{\mu\nu} - \Gamma^{\mu}_{\nu\gamma}\Gamma^{\nu}_{\mu\delta})\right]\right\}$$

Before numerical evolution. . .

- Write Einstein's equations as a well-posed Initial Boundary Value Problem (IBVP):
  - solution's behaviour depends continuously with the initial data;
  - numerically suitable gauge conditions.
- Discretize resulting PDEs
- Specify constraint preserving, and physically correct, boundary conditions
- Find a way to deal with singularities

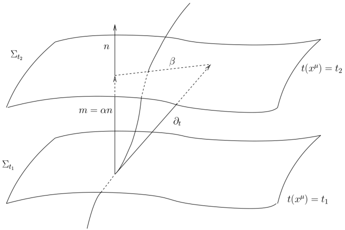


During numerical evolution. . .

- Compute constraint-satisfying initial data representing snapshot of physical system
- Implement mesh refinement, or similar, to efficiently handle different length scales (and parallelize resulting algorithms)
- Extract physical results in gauge-invariant fashion from numerical data

Outline

3+1 decomposition



We write the metric as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} \left(dx^i + \beta^i dt \right) \left(dx^j + \beta^j dt \right) ,$$

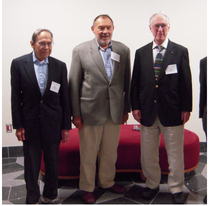
- γ_{ij} is the metric on surfaces of $t = \mathrm{const}$
- K_{ij} is the extrinsic curvature

ADM-York evolution equations

Evolution equations

$$(\partial_t - \mathcal{L}_\beta)\,\gamma_{ij} = -2\alpha K_{ij},$$

$$(\partial_t - \mathcal{L}_\beta)\,K_{ij} = -\nabla_i\nabla_j\alpha + \alpha\Big[R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j + \frac{8\pi}{D-2}\left((S-E)\gamma_{ij} - 2S_{ij}\right)\Big],$$



Constraints

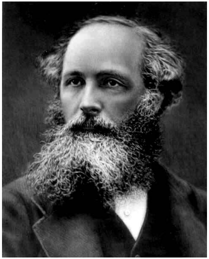
$$R + K^2 - K_{ij}K^{ij} = 16\pi E,$$

$$\nabla_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi p^i.$$

Electromagnetic analogy

Evolution equations

$$-\partial_t \vec{E} + \nabla \times \vec{H} = 4\pi \vec{j}$$
$$-\partial_t \vec{H} + \nabla \times \vec{E} = 0$$



Constraints

$$\nabla \cdot \vec{E} = 4\pi \rho$$
$$\nabla \cdot \vec{H} = 0$$

Well-posedness

Consider evolution equations of the form

$$\partial_t \phi + M^i \partial_i \phi = S(\phi)$$

The system is well-posed iff

$$\|\phi(t, x^i)\| \leq k e^{\alpha t} \|\phi(0, x^i)\|$$

ie, the norm of the solution can be bounded by the same exponential for all initial data.

Hyperbolicity

Construct matrix $P(n_i) = n_i M^i$, for arbitrary unit vector n^i (principal symbol).

Strongly hyperbolic system:

if P has real eigenvalues and complete set of eigenvectors for all n^i.

Weakly hyperbolic system:

if P has real eigenvalues but does not have a complete set of eigenvectors.

If a system is strongly hyperbolic, it is well-posed.

BSSN equations

...the ADM equations are only weakly hyperbolic...

- Alternative formulations to ADM system started being explored in the late 1980s, before full impact of the hyperbolicity properties of the different formulations had been realized.
- Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation is derived from the ADM equations but works with conformally rescaled variables, a trace split of the extrinsic curvature and promotes the contracted Christoffel symbols to the status of independent variables.

BSSN equations

$$\partial_t \tilde{\gamma}_{ij} = \beta^k \partial_k \tilde{\gamma}_{ij} + 2\tilde{\gamma}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k - 2\alpha \tilde{A}_{ij},$$

$$\partial_t \chi = \beta^k \partial_k \chi + \frac{2}{3}\chi(\alpha K - \partial_k \beta^k),$$

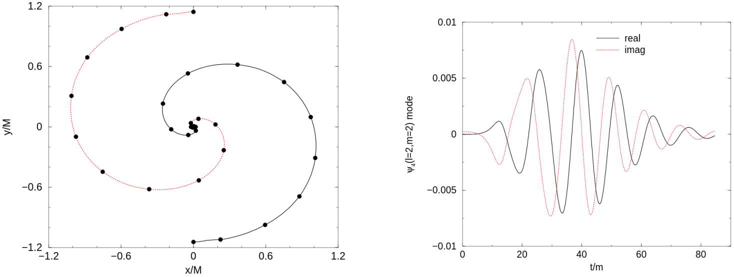
$$\partial_t \tilde{A}_{ij} = \beta^k \partial_k \tilde{A}_{ij} + 2\tilde{A}_{k(i}\partial_{j)}\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k \beta^k + \chi\left(\alpha R_{ij} - \nabla_i \partial_j \alpha\right)^{\mathrm{TF}}$$
$$+ \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_i{}^k \tilde{A}_{kj}\right) - 8\pi\alpha\left(\chi S_{ij} - \frac{S}{3}\tilde{\gamma}_{ij}\right),$$

$$\partial_t K = \beta^k \partial_k K - \nabla^k \partial_k \alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha(E + S),$$

$$\partial_t \tilde{\Gamma}^i = \beta^k \partial_k \tilde{\Gamma}^i - \tilde{\Gamma}^k \partial_k \beta^i + \frac{2}{3}\tilde{\Gamma}^i \partial_k \beta^k + 2\alpha \tilde{\Gamma}^i_{jk}\tilde{A}^{jk} + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j \partial_k \beta^k$$
$$+ \tilde{\gamma}^{jk}\partial_j \partial_k \beta^i - \frac{4}{3}\alpha \tilde{\gamma}^{ij}\partial_j K - \tilde{A}^{ij}\left(3\alpha\chi^{-1}\partial_j \chi + 2\partial_j \alpha\right) - 16\pi\alpha\chi^{-1} j^i$$

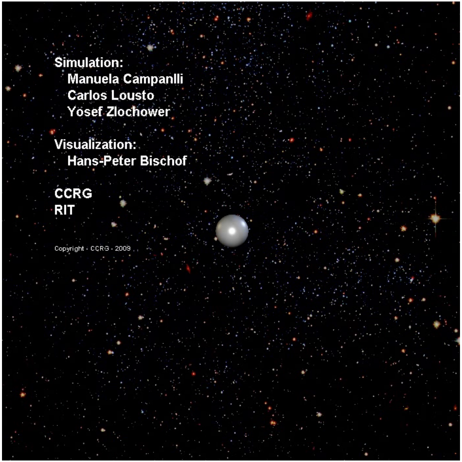
Moving Punctures

This system, together with the moving puncture gauge conditions, allowed for the 2005-06 breakthrough simulations of the Brownsville/RIT and NASA-Goddard groups.



Brownsville/RIT 2006

Kick configuration



RIT

Generalized Harmonic Gauge (GHG)

Imposing coordinates satisfying condition (harmonic coordinates)

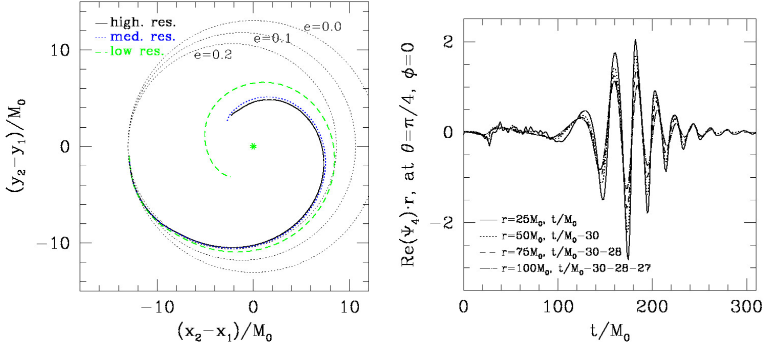
$$H^\alpha \equiv \Box x^\alpha = -g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$$

and promoting these to independently evolved variables, the (generalized) Einstein equations take the form:

$$\begin{aligned} g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} &= -2\partial_\nu g_{\mu(\alpha}\partial_{\beta)}g^{\mu\nu} - 2\partial_{(\alpha}H_{\beta)} + 2H_\mu\Gamma^\mu_{\alpha\beta} - 2\Gamma^\mu_{\nu\alpha}\Gamma^\nu_{\mu\beta} \\ &\quad -8\pi T_{\alpha\beta} + 4\pi T g_{\alpha\beta} - 2\kappa\left[2n_{(\alpha}\mathcal{C}_{\beta)} - \lambda g_{\alpha\beta}n^\mu\mathcal{C}_\mu\right] \end{aligned}$$

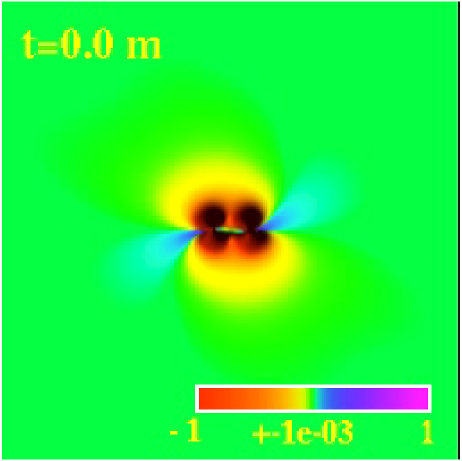
Generalized Harmonic Gauge (GHG)

- Set of 2nd order wave equations for each component of spacetime metric
- Equations are symmetric hyperbolic
- Used in Pretorius' 2005 breakthrough simulations



Pretorius 2005

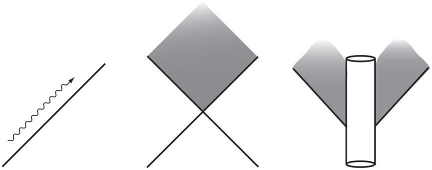
Black Hole Inspiral



F. Pretorius

Outline

Characteristic Initial Value Problem



$$ds^2 = -\left(e^{2\beta}\frac{V}{r} - r^2 h_{AB}U^A U^B\right) du^2 - e^{2\beta} du dr - 2r^2 h_{AB} U^B du dx^A + r^2 h_{AB} dx^A dx^B$$

Schematic evolution equations:

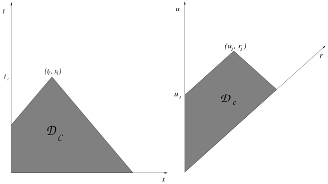
$$\partial_r F = H_F(F, G)$$
$$\partial_u \partial_r G = H_G(F, G, \partial_u G)$$

Advantages of characteristic evolution

- Initial data is free (no elliptic constraints on the data);
- No second time derivatives (therefore smaller number of basic variables);
- Rigorous description of gravitational waves on null hypersurfaces (ideal for wave-extraction);
- Equations have convenient hierarchical structure in which variables are integrated in turn in terms of characteristic data from prior members of the hierarchy.

First stable 3D evolutions of black holes (including moving and rotating configurations) were achieved with characteristic methods in 1997, which proved remarkably stable.

Toy problem (wave equation)



Cauchy:

$$\partial^2_{tt}\phi - \partial^2_{xx}\phi = 0$$

Characteristic ($u = t - x$, $x = r$):

$$2\partial_u\partial_r\phi - \partial^2_{rr}\phi = 0$$

- solution: $\phi(u, r) = F(u) + G(u + 2r)$.
  - F – outgoing wave; G – ingoing wave
- Solved provided $\phi(u = u_0, r)$ and boundary data at $r = 0$ are given.

Outline

Outline

Why Cactus/ET

Typical problem in Numerical Relativity...

- mesh refinement
- efficiently parallelize
- large input/output
- somewhat complex tools for analysis

Typical workflow:

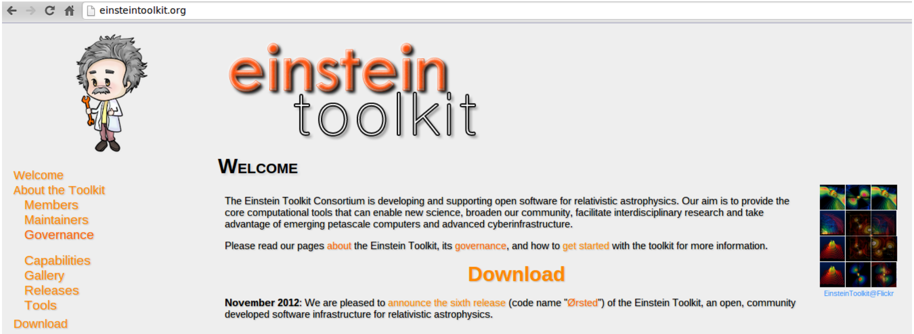
1. Compute initial data
2. Evolve equations
3. Analysis

What is Cactus

Cactus:

- general framework for the development of portable, modular applications
- programs are split into independent components (thorns)
- thorns are developed independently and should be interchangeable with others with same functionality
- thorns don't directly interact with each other
- Cactus framework (flesh) provides the "glue"
- supports C, C++, Fortran

Obtaining ET



Obtaining ET

einstein toolkit

Download

The Einstein Toolkit is hosted on many different machines around the world. We provide a script called GetComponents to simplify downloading the toolkit. This page just describes how to download the toolkit - you may also be interested in the Tutorial for New Users which leads you through these steps and more on the Queen Bee supercomputer.

Enter the directory on your machine in which you would like to download the ET (for example, your home directory), and type the commands listed below. This will create a directory called Cactus in which the components of the Einstein Toolkit are downloaded.

Current release: Ørsted (released on November 8th, 2012)

This is the recommended version of the toolkit for most users. See the release notes for more information.

```
curl -O https://raw.github.com/gridaphobe/CRL/master/GetComponents

chmod a+x GetComponents

./GetComponents --parallel https://svn.einsteintoolkit.org/manifest/branches/ET_2012_11/einsteintoolkit.th
```

ET contents: arrangements

Provided arrangements ($\approx$ collection of thorns):

- Several Cactus thorns (I/O, Method of Lines, . . .)
- Carpet (Adaptive Mesh Refinement driver)
- EinsteinBase
- EinsteinInitial
- EinsteinEvolve
- EinsteinAnalysis
- McLachlan (BSSN implementation)
- . . .

Cactus arrangements

Main core Cactus arrangements:

CactusBase
: Infrastructure thorns for boundary conditions, coordinates, IO, symmetries and time

CactusNumerical
: Numerical infrastructure thorns: time integration, dissipation, symmetry boundary conditions, spherical surfaces, local interpolation, Method of Lines (MoL), . . .

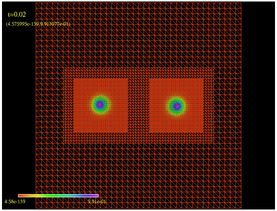
CactusUtils
: Utility thorns: formaline, nan-checking, termination triggering and timer reports

ExternalLibraries
: Provides external libraries: Lapack, GSL, HDF5, FFTW, Lorene (for initial data) and others

Carpet

Berger-Oliger Adaptive Mesh Refinement (AMR) driver:

- Splits grid functions and arrays among the MPI processes
- Setups mesh refinement grid hierarchy
- Communicates ghost cell information between MPI processes
- Communicates between refinement levels by prolongation and restriction
- Modifies grid hierarchy (regridding) when requested
- Performs parallel IO



ET Arrangements

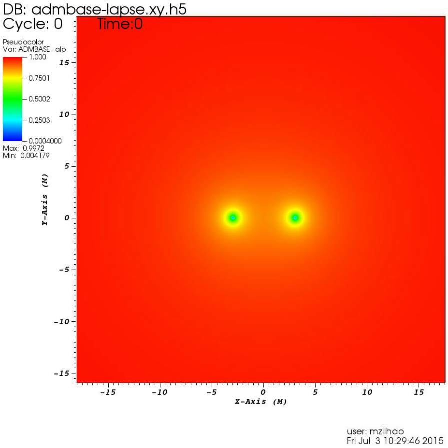
EinsteinBase
: defines basic spacetime variables

EinsteinInitial
: computing initial data

EinsteinEvolve
: evolves variables in time (typically with Method Of Lines)

EinsteinAnalysis
: diagnostic tools

Demo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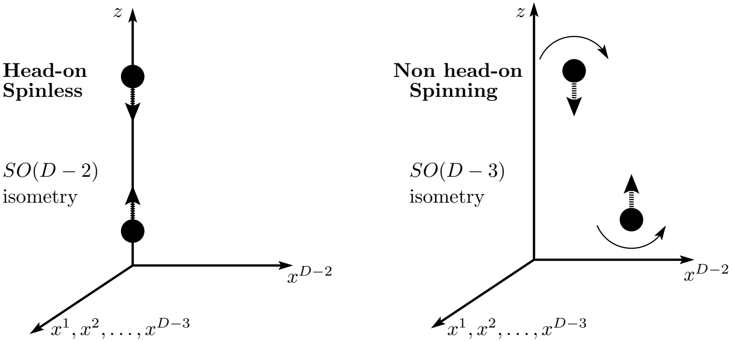
Outline

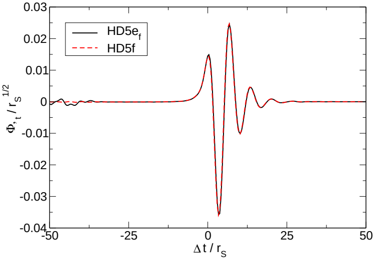
Higher-dimensional BH collisions

Axial symmetry $SO(D-2)$ and $SO(D-3)$



- Highly symmetric systems;
- Can be reduced to effective $3+1$ systems;
  $\Rightarrow$ We can use existing numerical codes (with adaptations);

$D = 5$ head-on collision from rest Witek, MZ et al 2010



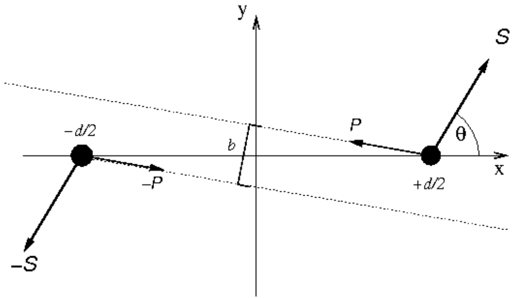
- $l = 2$ mode of $\Phi_{,t}$
  $d = 6.37\, r_S$ (black)
  $d = 10.37\, r_S$ (red)

characteristic ringdown frequency:

$$r_S\omega = 0.955 \pm 0.005 - i(0.255 \pm 0.005)$$
$$(r_S\omega = 0.9477 - i0.2561, \text{ e.g., Berti et al. 2009})$$

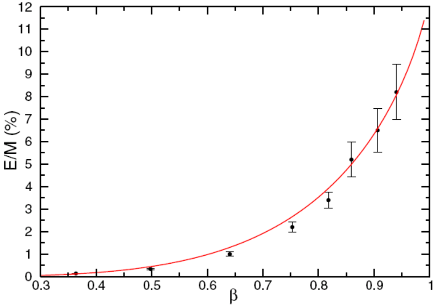
Black hole collisions in $D = 4$

- two black holes
  Total rest mass: $M_0 = M_\mathrm{A} + M_\mathrm{B}$
  Initial position: $\pm x_0$
  Linear momentum: $\mp P[\cos\alpha,\ \sin\alpha,\ 0]$
- Impact parameter: $b \equiv \frac{L}{P}$



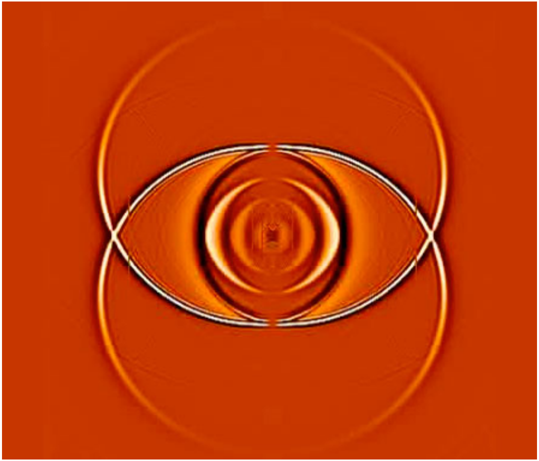
Head-on collisions: $b = 0, \quad \vec{S} = 0$

- Total radiated energy: 14 ± 3 % for $v \to 1$ Sperhake et al. 2008



- Agreement with approximative methods Berti et al. 2010
- Recently revisited with improved initial data by RIT group Ruchlin et al 2015

$D = 4$ Boosted collisions Sperhake et al 2008

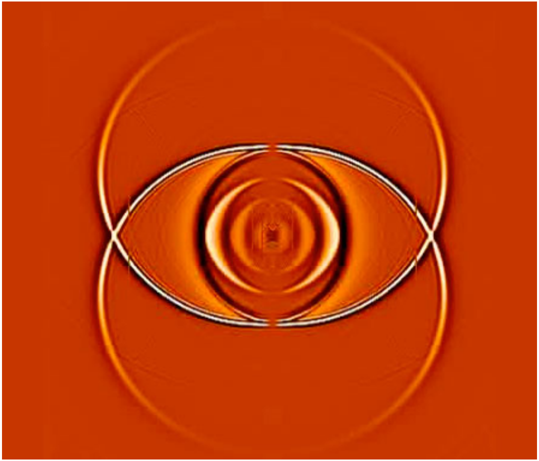


U. Sperhake

Grazing collisions

- Two distinct end-states to BH scattering problem: one BH or two BHs
- Near the critical impact parameter:
  - sensitivity to initial conditions
  - enhanced gravitational wave emission (even in scattering cases)

Whirl, merger



U. Sperhake

Black String

Class of higher-dimensional black hole solutions

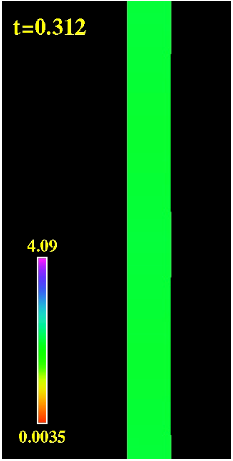
$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2 + dw^2$$

- Shown to be unstable to long-wavelength perturbations

  Gregory & Laflamme 1993

- A lot of debate about the end-state

Black String Lehner & Pretorius 2010



F. Pretorius

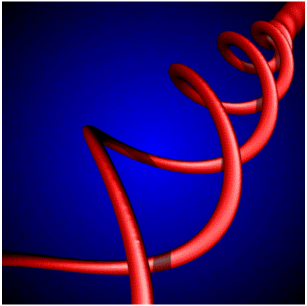
Further developments

- BH Collisions in AAdS spacetimes Bantilan & Romatschke 2015
- Off-center collisions of shock waves in AAdS5 spacetimes Chesler & Yaffe
- `GRChombo`: NR with Adpative Mesh Refinement Clough, Figueras et al 2015
- Accretion disks around binary BHs of unequal mass: General relativistic MHD simulations Gold et al 2014
- . . .

Outline

Final Remarks

- Numerical modelling of gravitational systems is as old as the advent of computing itself.
- 2005 breakthroughs in NR marked a phase transition in the field.
- This allowed for the discovery of unexpected results (superkicks of thousands of km/s, zoom-whirl behaviour, etc.)
- Motivation for long-term NR efforts came originally mostly from the modeling of gravitation wave sources.
- Nowadays, NR is finding applications to other fields, such as high-energy physics, higher-dimensional gravity, and AdS/CFT.
- What surprises will next 10 years of NR reveal?



M. Thierfelder