# anomalies AND NON-equilibrium



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## Motivation

- Hydrodynamics is a very universal effective field theory used to describe heavy-ion collisions and condensed matter systems
- Inclusion of anomalies of discrete symmetries
- Applications to quark-gluon plasma, non-equilibrium modelling of chiral phenomena
- Classical manifestations anomalies
- Anomalies in turbulence
- Relation between partition functions and entropy current constraints

## COINCIDENCES



#### VORTEX-STRECHING described in the double limit of small Knudsen number *Kn* and and Yau [1] have rigorously derived these equations in such  $\overline{R}$ ORIEX-SIRECHING

In the long-wavelength, non-relativistic limit the fluid equations are: that the coarse-grained velocity fields in the model must satisfy some Leray solution [2] of (1), *even if* the latter develops verengely non relatividue initi-

$$
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \triangle \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0.
$$

small Mach number *Ma* by the incompressible Navier–Stokes

Experiments suggest that energy dissipation Experime<sup>®</sup> There is no apparent limitation on the Reynolds number *Re* in

 $\varepsilon = v|\nabla u|^2$ nish in the limit of vanishing viscosity, for a variety, of turbulent flows. This fact was a basic assumption in the 1941 theory of aue to A. Nollhogorov. does not v  $turbulent$ turbulenc  $2$  appears not to vanish in the limit in the limit is  $2$ does not vanish in the limit of vanishing viscosity, for a variety, of turbulence due to A. Kolmogorov.

Onsager noticed that if we divide the above **Election State British fluid-dynamics** equation by viscosity and take the zero-viscosity **the semi-phenomenology in a semi-phenomenology** There is no approxime to a paper is no come of the Reynolds number  $\mathcal{L}$  and  $\mathcal{L}$  $\frac{1}{\sqrt{2\pi}}$  Onsager r limit the velocity becomes non-differentiable.

We can also rewrite the RHS of the energy **EXAMPLE 2008** as proportional to enstrophy. Thus *And for a set of the set of a meta-box se* France is a mechanism of enstrophy. Thus<br>turbulence is a mechanism of enstrophy which is know as vortex-streching. Institut for Physics  $\frac{1}{2}$  is the short  $\frac{1}{2}$  in a surface short  $\frac{1}{2}$  is the surface stress in a surface stre  $\overline{\phantom{a}}$  -  $\overline{\phantom{a}}$ dissipation as proportional to enstrophy. Thus generation, which is know as vortex-streching. Source University of Münster,



source University of Münster, Institut for Physics

#### Effective equations that any of the hydrodynamic energy must be converted to heat  $\epsilon$  +  $\epsilon$ numerical simulation of homogeneous, isotropic turbulence concise assertions above. PER HOUATIONS is in terms of effective "coarse-grained" equations obtained  $CFTCCTIVICCT$ is in terms of the coarse- $\epsilon$  from the incompart of  $\epsilon$ CEFECLIVE EQUATIONS in a periodic domain, as summarized by Sreenivasan [12]. The recent numerical study of  $\mathbb{R}^n$ et al. [13] on a 4096<sup>3</sup> spatial grid has confirmed that from the incompressible Navier–Stokes equation, as in [17,18]. LIVE FOUALIONS

from the incompressible Navier–Stokes equation, as in [17,18].

Consider a locally space-averaged velocity et al. [13] on a 4096<sup>3</sup> spatial grid has confirmed that Consider a locally space-averaged (low-pass filtered) velocity  $x^2$  nsider a locally space-averaged velocity #  $\overline{C}$ Consider a locally <u>(</u><br>P-averas <sup>d</sup>*drG*\$*(*r*)*u*(*<sup>x</sup> <sup>+</sup> <sup>r</sup>*).* (4)

in a periodic domain, as summarized by Sreenivasan [12].

et al. [13] on a 4096<sup>3</sup> spatial grid has confirmed that

$$
\overline{\mathbf{u}}_{\ell}(\mathbf{x}) = \int d^d r G_{\ell}(\mathbf{r}) \mathbf{u}(\mathbf{x} + \mathbf{r})
$$

where  $G_{\ell}(\mathbf{r}) = \ell^{-d} G(\mathbf{r}/\ell)$  is an averaging kernel that is nonnegative, smooth and rapidly decaying. Furthermore, the mean kinetic energy remains bounded in the mean kinetic energy remains bounded in the mean  $\sim$ where  $G_{\ell}(\mathbf{r}) = \ell^{-\alpha} G(\mathbf{r}/\ell)$  is an averaging kerner mat is non-<br>negative smooth and rapidly decaying where  $G_{\ell}(\mathbf{r}) = \ell^{-d}G(\mathbf{r}/\ell)$  is an averaging kernel that is nonnegative, smooth and rapidly decaying. the hydrodynamic modes but, instead, is being transferred to where  $G_{\ell}(\mathbf{r}) =$ negative, smooth and rapidly decaying.  $\frac{1}{2}$  $\sigma(\Gamma/\ell)$  is all averaging kenier that is non-

#

We can averege out the Navier-Stokes equations  $\frac{1}{\sqrt{3}}$  is the Navier–Stockes equation  $\frac{1}{\sqrt{3}}$  is the Navier–Stockes equation yields equation  $\frac{1}{\sqrt{3}}$ we can avelege out the We can averege out the Navier-Stokes equations

$$
\partial_t \overline{\mathbf{u}}_\ell + \nabla \cdot [\overline{\mathbf{u}}_\ell \overline{\mathbf{u}}_\ell + \tau_\ell] = -\nabla \overline{p}_\ell + \nu \Delta \overline{\mathbf{u}}_\ell, \quad \nabla \cdot \overline{\mathbf{u}}_\ell = 0
$$
  
where we introduced subscale stress-tensor

where we introd where we introduced subscale stress-tensor<br>——————————————————— where we introduced subscale stress-tensor  $\alpha$  introduced subscale stress-tensor

$$
\boldsymbol{\tau}_{\ell} = \overline{(\mathbf{u} \otimes \mathbf{u})}_{\ell} - \overline{\mathbf{u}}_{\ell} \otimes \overline{\mathbf{u}}_{\ell}
$$

This is analogous to the Wilson-Kadanoff RG approach. The viscous term can be show the atom bulk of the interwas foundations for the walls of the walls of<br>The walls of the wa be irrelevant in terms of KG analysis. This results the inertial range of scales This is analogous to the Wilson-Kadanoff RG approach. The viscous term can be show to be irrelevant in term in a simplification in the inertial range of m a sampamenten in the merican image of m can be show to be irrelevant in terms of RG analysis. This result a simplification in the inertial range of scales term can be show in a simplification This is analogous to the Wilson-Kadanoff RG approach. The viscous term can be show to be irrelevant in terms of RG analysis. This results in a simplification in the inertial range of scales  $n$  alogous to the Wilson-Kadanoff  $RG$  annroach. The viscous  $\frac{1}{2}$  be show to be irrelevant in terms of RG analysis. This results *ination* in the inertial range of s

$$
\partial_t \overline{\mathbf{u}}_\ell + \nabla \cdot [\overline{\mathbf{u}}_\ell \overline{\mathbf{u}}_\ell + \boldsymbol{\tau}_\ell] = -\nabla \overline{p}_\ell, \quad \nabla \cdot \overline{\mathbf{u}}_\ell = 0
$$

#### Dissipative anomaly  $\left($ ip $\lambda$  Tive  $\lambda$  Nom $\lambda$  IV is comparable with unity of is comparable with unity." In RG language, one may regard the Euler equations as "bare"  $\overline{A}$  space-integrated by Onsager to C. VIAL)  $\sum_{i=1}^{n} \frac{1}{i} \int_{0}^{1} f(t) \, dt = \sum_{i=1}^{n} \frac{1}{i} \int_{0}^{1} f(t) \, dt$  $PISIFAU$ considerable time elapsed before his ideas were rediscovered.  $\mathbf{I}$  showed that spectral energy flux Π *(k)* → 0 as *k* → ∞ for an Euler solution with energy ∂*<sup>t</sup> e*" + ∇ · J" = −Π"  $MAL$ <sup>2</sup> is large-scale energy density per mass,

C. Lin in a private letter in 1945. See [16] for a reprinting

The effective equations are equivalent to the ones obtained by coarsegraining procedure of incompressible Euler equations. However, the equations are not well-defined for the singular velocity fields and only meaningful in a sense of distributions.<br> $\frac{1}{2}$ The effective equations are equivalent to the ones obtained by coarseedure of incompressible Euler equations. However, the sequence of the only resolved that requestions. The weiver, the *f* hot wen-defined for the singular se que de la composition having *first* taken the limit ν → 0*.* equations are not wen-defined for the singular velocity fields and only<br>meaningful in a sense of distributions of this letter. Onsager himself never published his proof and tained by coarseis. However, the spectral exponent *n >* 8*/*3. Eyink [28] showed that spectral flux The effective equations are equivalent to the ones obtained by coarsethe Euler solution use of the energy of the space of the space of the space of large-scale energy space of the<br>
in space of incompressible Euler equations. However, the  $\frac{1}{100}$  derived and twell defined for the meanir ngful in a sense of distribu<sup>.</sup> wer the same  $\alpha$  $\int$  and only stronger than Holder continuity with exponent ¨ α *>* 1*/*3. He also showed that  $\mathcal{O}_\mathcal{A}$  result is optimal by constructing is optimal by constructing  $\mathcal{O}_\mathcal{A}$ elocity fields and only

As realized by Onsager, the Euler equations in this generalized sense<br>do not guarantee the concernation of energy It can be shown that the do not guarantee the conservation of energy. It can be shown that the generalized energy balance equation has the form  $\frac{1}{2}$  be dependence of the form the form do not guarantee the conservation of energy. It can be shown that the  $\frac{1}{2}$ ealized by (  $\mathbf{R}$ re of unstributions.<br>er, the Fuller equations in this generalized sense generalized energy balance equation has the form d sense<br>that the that the s generanzed sense cascade is the dynamical transfer of kinetic energy from large-

$$
\partial_t \left( \frac{1}{2} |\mathbf{u}|^2 \right) + \nabla \cdot \left[ \left( \frac{1}{2} |\mathbf{u}|^2 + p \right) \mathbf{u} \right] = -D(\mathbf{u}),
$$

where  $D(u)$  is a non-vanishing distribution (Duchon, Robert)  $\int d\mu \, (\nabla \Omega)$  (**u**)  $\int \rho_{\text{max}}(\mathbf{r}) | \rho_{\text{max}}(\mathbf{r}) |^2$  $\int d^{a} r \, (\mathbf{V} \, G)_{\ell}(\mathbf{r}) \cdot [\delta \mathbf{u}(\mathbf{r}) | \delta \mathbf{u}(\mathbf{r})]^{2}] \cdot \delta \mathbf{u}(\mathbf{r}; \mathbf{x}) \equiv \mathbf{u}(\mathbf{x} + \mathbf{r})$ first derived by Duchon and Robert  $\overline{\mathcal{L}}$  and  $\overline{\mathcal{L}}$  $).$  $D(u) = \lim_{\epsilon \to 0} \frac{1}{u} \int d^d r \left( \nabla G \right) e(\mathbf{r}) \cdot \left[ \delta \mathbf{u}(\mathbf{r}) |\delta \mathbf{u}(\mathbf{r})|^2 \right]$  $L(u) = \lim_{\ell \to 0} \overline{4\ell} \int u \int (v \omega) \ell(u) \cdot \left[ \frac{\partial u(u)}{\partial u(u)} \right]$ .  $D(\mathbf{u}) = \lim_{\ell \to 0} \frac{1}{4\ell} \int d^d r \, (\nabla G)_{\ell}(\mathbf{r}) \cdot \left[ \delta \mathbf{u}(\mathbf{r}) |\delta \mathbf{u}(\mathbf{r})|^2 \right] \cdot \delta \mathbf{u}(\mathbf{r}; \mathbf{x}) \equiv \mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x}).$  $\sum_{k=1}^{n}$  $\ell \rightarrow 0$ 1  $4\ell$ %  $d^d r \left( \nabla G \right)_{\ell} (\mathbf{r})$  ·  $\overline{\Gamma}$  $\delta$ **u**(**r**)| $\delta$ **u**(**r**)|<sup>2</sup>  $\Big]$ .  $\delta$ **u**(**r**; **x**)  $\equiv$ **u**(**x** + **r**)  $-$ **u**(**x**). **''**  $r(f)$ and  $y$  as an obtained the Sulem–Frisch result  $\mathcal{Z}$ further important results related to the zero-viscosity limit. It A key realization of Onsager was that this energy flux depends only upon *velocity-increments*

a dissipative anoma  $\Box$ *B* non-cons  $\frac{1}{1}$  $\sigma$ *f* symmetr  $\overline{e}$ he anomalous non-conservation of symmetries in QFT. This is called a dissipative anomaly. Polyakov pointed out that the non-conservation of the symmetries of Euler equation is analogous to UISCI VALIUII UI  $\overline{\phantom{a}}$  $AT<sub>2</sub>$  as another corollary. This is called a dissipative anomaly. Polyakov pointed out that the non-conservation of the symmetries of Euler equation is analogous to the anomalous non-conservation of symmetries in QFT. conservation-law and the conservation of the conservation and the theory, such anomalies in the such as the such as at the setthat, if a sequence of  $\mathcal{L}$ ogous to inted out that the  $1 \overline{\text{OF}}$ <sup>d</sup>*drG*"*(*r*)*δu*(*r*)* <sup>⊗</sup> <sup>δ</sup>u*(*r*)*

#### Khokhlov Saw-tooth dissipation for *Re* → ∞ but for which K41 theory fails. It is a useful counterexample! The KHOKHLOV SAW a function on the interval [−*L, L*] with *u*(*±L, t*) = 0 a sharp discontinuity of size "*u* = (2*L*)*/t* at *x* is called a station is called a shock of the shock of it is case, and it is in

*u* Let us consider the so-called Burgers equation. It is a 1d model, in which we can se the dissipative anomaly. model is the 1-dimensional Burgers equation: *it* is a rainbard, in which we can se the dissipative anomaly Let us consider the so-

$$
\partial_t u + u \partial_x u = \nu \partial_x^2 u.
$$

This equation has a simple solution known as Khokhlov saw-tooth. This equation has a This equation has a simple solution known as Khokhlov sav

$$
u(x,t) = \frac{1}{t} [x - L \tanh(\frac{Lx}{2\nu t})].
$$

∂*t*( <sup>1</sup>



#### NUCLEUS-NUCLEUS COLLISION



## PARITY-ODD HYDRO

Relativistic fluid with one conserved charge described by conservation laws

$$
\partial_{\mu}T^{\mu\nu} = 0
$$

$$
\partial_{\mu}J^{\mu} = 0
$$

plus equations that express  $T^{\mu\nu}$  and  $J^{\mu}$  in terms of local temperature  $T$ , chemical potential  $\mu$ , and fluid velocity  $u^{\mu}$ :

$$
T^{\mu\nu} = (\epsilon + P)u^{\mu}u^{\nu} + Pg^{\mu\nu} + \tau^{\mu\nu}
$$

$$
J^\mu = n u^\mu + \nu^\mu
$$

The definition of velocity is ambiguous beyond leading order. We fix it by imposing (Landau frame)

$$
u_\mu \tau^{\mu\nu} = 0
$$

### Vorticity

$$
\partial_{\mu}\left(s u^{\mu} - \frac{\mu}{T} \nu^{\mu}\right) = -\partial_{\mu}\left(\frac{\mu}{T}\right) \nu^{\mu} - \frac{1}{T} \partial_{\mu} u_{\nu} \tau^{\mu \nu}
$$

The left hand side is then interpreted as the divergence of the entropy current  $\partial_{\mu}J_{s}^{\mu}$ . When the current is chiral, or when the fundamental theory does not preserve parity, it is possible to construct one additional Lorentz structure that may appear in the current  $J_\mu$ 

$$
\omega^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta
$$

The new term is consistent with Lorentz symmetry, but its divergence is now:

$$
\partial_{\mu}J^{\mu}_{s} = \ldots - \xi(T,\mu)\partial_{\mu}\left(\frac{\mu}{T}\right)\omega^{\mu}
$$

One has to revisit the entropy current argument

$$
J^\mu_s = \ldots + D(T,\mu)\omega^\mu
$$

#### Hydrodynamics with anomalies

$$
\partial_\mu T^{\mu\nu} = F^{\nu\lambda}j_\lambda
$$

$$
\partial_\mu J^\mu = C_{anom} E^\mu B_\mu
$$

There are only two new terms consistent with symmetry that can be added to the entropy current

$$
J_s^{\mu} = su^{\mu} - \frac{\mu}{T} \nu^{\mu} + D\omega^{\mu} + D_B B^{\mu}
$$

Requiring that contributions with undetermined signs cancel on both side we find  $\bar{\mu} = \mu/T$ 

$$
D(\bar{\mu}) = T^2 \frac{1}{3} C_{anom} \bar{\mu}^3; \qquad D_B(\bar{\mu}) = T^2 \frac{1}{2} C_{anom} \bar{\mu}^2
$$
  
We have new transport coefficients (vortical and magnetic conductivities) up to an integration constant

$$
\xi = C_{anom} \left( \mu^2 - \frac{2}{3} \frac{n \mu^3}{\epsilon + P} \right); \qquad \xi_B = C_{anom} \left( \mu - \frac{1}{2} \frac{n \mu^2}{\epsilon + P} \right)
$$

#### Gravitational anomalies

In the previous calculation we had two integration constants which we cannot constrain by hydrodynamic reasoning. However, we can calculate them using linear response theory in weakly coupled field theory. It turns out these constants emerge as a consequence of gravitational anomalies

$$
\xi_M = \lim_{k_n \to 0} \sum_{ij} \epsilon_{ijk} \frac{-i}{2k_n} \left\langle J_M^i T^{0j} \right\rangle|_{\omega=0}
$$

$$
\xi_{MN}^B = \lim_{k_n \to 0} \sum_{ij} \epsilon_{ijk} \frac{-i}{2k_n} \left\langle J_M^i J_N^j \right\rangle|_{\omega=0}
$$

In the case of vortical conductivity we get  $T^2$  correction

$$
\xi_M = \frac{1}{8\pi^2} \sum_{f=1}^N T_M^f f\left( (\mu^f)^2 + \frac{\pi^2}{3} T^2 \right)
$$

## Kinetic theory

Kinetic theory treats the evolution of the one-particle distribution function, which can be associated with the number of on-shell particles per unit phase space

$$
f(\vec{p},\vec{x};t)=\frac{dN}{d^3pd^3x}
$$

If collisions between particles can be neglected and there is no Berry phase effects, the evolution of  $f(\vec{p}, \vec{x}; t)$  follows from Liouville's theorem

Given this interpretation the particle number density should be proportional to !

$$
\int d^3p f(\vec{p},\vec{x};t)
$$

Summing instead with a weight of particle energy, one expects a result proportional to the product of number density and energy, or energy density, which is a part of the energy-momentum tensor.

### $HYDRO \Leftrightarrow KNNETIC THEORY$

We can derive hydrodynamic quantities from kinetic theory e.g.

$$
T^{\mu\nu}\equiv \int \frac{d^4p}{(2\pi)^3} p^\mu p^\nu \delta(p^\mu p_\mu - m^2) 2\theta(p^0) f(p, x)
$$

If we take the distribution function in equilibrium we recover energymomentum tensor of a perfect fluid. One can derive the correspondence between kinetic theory out of equilibrium and viscous hydrodynamics by considering small departures from equilibrium where

$$
f(p^{\mu}, x^{\mu}) = f_{\text{eq}}\left(\frac{p^{\mu}u_{\mu}}{T}\right)[1 + \delta f(p^{\mu}, x^{\mu})]
$$

This procedure allows one to study dissipative effects (first order in the derivatives of fields). Performing the integral one gets perfect fluid contribution plus shear tensor

$$
T^{\mu\nu} = T^{\mu\nu}_{(0)} + \int \frac{d^4p}{(2\pi)^3} p^{\mu} p^{\nu} f_{\text{eq}} \delta f = T^{\mu\nu}_{(0)} + \pi^{\mu\nu}
$$

### Anomalous part

Solving the Weyl equation we obtain

$$
\psi=\int_0^\infty\frac{dE_p}{2\pi}\frac{1}{\sqrt{2E_p}}\left[a_pe^{ip.x}+b_p^\dagger e^{-ip.x}\right]_{p^\mu=E_p\left[u^\mu+\chi_{d=2}\epsilon^{\mu\nu}u_\nu\right]}
$$

Populating these states leads to anomalous correction to hydrodynamics

$$
T^{\mu\nu} = \sum_{species} \int_0^\infty \frac{dE_p}{2\pi} (f_q + f_{-q}) E_p \left[ u^\mu + \chi_{d=2} \epsilon^{\mu\alpha} u_\alpha \right] \left[ u^\nu + \chi_{d=2} \epsilon^{\nu\lambda} u_\lambda \right]
$$
  
=  $\epsilon u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu) + q_{anom}^\mu u^\nu + q_{anom}^\nu u^\mu$ 

$$
J^{\mu} = \sum_{species} \int_0^{\infty} \frac{dE_p}{2\pi} (q f_q - q f_{-q}) [u^{\mu} + \chi_{d=2} \epsilon^{\mu \alpha} u_{\alpha}]
$$
  
=  $nu^{\mu} + J_{anom}^{\mu}$  helicity current

$$
J_S^{\mu} = -\sum_{species} \int_0^{\infty} \frac{dE_p}{2\pi} (\mathcal{H}_q + \mathcal{H}_{-q}) \left[ u^{\mu} + \chi_{d=2} \epsilon^{\mu \alpha} u_{\alpha} \right]
$$
  
=  $s u^{\mu} + J_{S,anom}^{\mu}$ 

### GIBBS CURRENT

The above anomalous quantities can be generated from

$$
\bar{\mathcal{G}}_{anom} = \sum_{F} \int_0^\infty \frac{dE_p}{2\pi} g_q \,\chi_{d=2} u
$$
\n
$$
\bar{J}_{anom} = -\frac{\partial \bar{\mathcal{G}}_{anom}}{\partial \mu}, \quad \bar{J}_{S,anom} = -\frac{\partial \bar{\mathcal{G}}_{anom}}{\partial T}
$$
\n
$$
\bar{q}_{anom} = \bar{\mathcal{G}}_{anom} + T\bar{J}_{S,anom} + \mu \bar{J}_{anom}
$$

where  $g_q \equiv -\frac{1}{\beta} \ln \left[1 + e^{-\beta(E_p - q\mu)}\right]$  and we used Hodge duals for simplicity.  $\frac{1}{\beta}\ln\Big[$  $1 + e^{-\beta(E_p - q\mu)}$  $\mathbf{I}$ 

We have to evaluate one thermal integral to get

*species*

$$
\bar{g}_{anom} = -2\pi \left[ \frac{\mu^2}{2!(2\pi)^2} \left( \sum_{species} \chi_{d=2} q^2 \right) + \frac{T^2}{4!} \left( \sum_{species} \chi_{d=2} \right) \right] u
$$

Crucial observation : the anomalous contribution is completely proportional to the U(1) anomaly coefficient  $\sum_{\chi_{d=2}q^2}$  and the Lorentz anomaly coefficient *species*  $\sqrt{ }$  $\chi_{d=2}$ 

#### Anomaly polynomials Y POLYNO

The anomaly coefficients of a system are summarised by a polynomial in gauge field strength and space-time curvature:  $\mathbb{R}^d$  $\text{ce}-1$  $\lim_{t \to 0}$  cui vai .<br>.<br>.

1d strength and space-time curvature:

\n
$$
\mathcal{P}_{anom}(F, \mathfrak{R}) \equiv -2\pi \left[ \frac{F^2}{2!(2\pi)^2} \left( \sum_{species} \chi_{d=2} q^2 \right) - \frac{p_1(\mathfrak{R})}{4!} \left( \sum_{species} \chi_{d=2} \right) \right]_{2d}
$$

Using this we can write a rule to get from the anomaly polynomial to the anomaly induced Gibbs current where we have in addition used the fact that any 2n form made of purely spatial forms B  $\frac{1}{1}$  is a set of the other hand dependent

$$
\bar{\mathcal{G}}_{anom} = u \ \mathcal{P}_{anom} \left( F \mapsto \mu \ , \ p_1(\mathfrak{R}) \mapsto -T^2 \right)
$$

Motivated by this result and Berry phase calculation we can generalise the Gibbs current to higher dimensions introducing concept of chiral spectral current, repeat the analysis and match to hydrodynamics  $\overline{\phantom{a}}$ this r  $\frac{1}{2}$ B<sub>e</sub>  $\mu_n \left( F \mapsto \mu \,\, , \,\, p_{\scriptscriptstyle 1}({\frak R}) \mapsto -T^2 \right) \ \hbox{and} \ {\rm try \,\, phase \,\, calculation \,\, we \,\, can \,\, generalise}$ 

$$
\mathcal{G}^\mu_{anom} = \sum_F \int_0^\infty dE_p \mathcal{J}^\mu_q \; g_q
$$

Using adiabaticity

$$
\bar{\mathcal{J}}_q = \frac{\chi_{d=2n}}{2\pi} \left( \frac{qB + 2\omega E_p}{2\pi} \right)^{n-1} \wedge \frac{u}{(n-1)!}
$$

#### EXAMPLES Ep % = 1 τ  $\lambda$   $\Gamma$  $\overline{\phantom{a}}$  $\mathcal{A}$  $\overline{L}$  $FX^{\Delta}MP16$  $V$  $\overline{\phantom{0}}$ τ  $\overline{10}$  $\prod$  $LE$ <u>−</u> Γ

Cardy entropy

The structure of Gibbs functionals in higher dimensions The structure of Gibbs functionals in bigher

$$
(\bar{g}_{anom})_{d=2} = -2\pi \sum_{species} \chi_{d=2} \left[ \frac{1}{2!} \left( \frac{q\mu}{2\pi} \right)^2 + \frac{T^2}{4!} \right] u \qquad \text{formula + first}
$$
\n
$$
(\bar{g}_{anom})_{d=4} = -2\pi \sum_{species} \chi_{d=4} \left[ \frac{1}{3!} \left( \frac{q\mu}{2\pi} \right)^3 + \left( \frac{q\mu}{2\pi} \right) \frac{T^2}{4!} \right] (2\omega) \wedge u \qquad \text{thermodynamics}
$$
\n
$$
-2\pi \sum_{species} \chi_{d=4} \left[ \frac{1}{2!} \left( \frac{q\mu}{2\pi} \right)^2 + \frac{T^2}{4!} \right] (qB) \wedge u
$$
\n
$$
(\bar{g}_{anom})_{d=6} = -2\pi \sum_{species} \chi_{d=6} \left[ \frac{1}{4!} \left( \frac{q\mu}{2\pi} \right)^4 + \frac{1}{2!} \left( \frac{q\mu}{2\pi} \right)^2 \frac{T^2}{4!} + \frac{7}{8} \frac{T^4}{6!} \right] (2\omega)^2 \wedge u
$$
\n
$$
-2\pi \sum_{species} \chi_{d=6} \left[ \frac{1}{3!} \left( \frac{q\mu}{2\pi} \right)^3 + \left( \frac{q\mu}{2\pi} \right) \frac{T^2}{4!} \right] (2\omega) \wedge (qB) \wedge u
$$
\n
$$
-2\pi \sum_{species} \chi_{d=6} \left[ \frac{1}{2!} \left( \frac{q\mu}{2\pi} \right)^2 + \frac{T^2}{4!} \right] \frac{(qB)^2}{2!} \wedge u
$$

## QCD PHASE STRUCTURE



source Shanghai Jiao Tong University, Institut of Nuclear and Particle Physics, Astronomy and Cosmology (INPAC)

Usually, "the phase diagram of QCD" is drawn in the plane spanned by the temperature T and the baryon chemical potential μ. But various additional directions, i.e., higher-dimensional versions of the phase diagram, are of interest as well, for example chiral chemical potential.

#### Information geometry  $\mathbf{S}$ ical nature of anomalies makes transport due to anomalies makes transport due to anomalies makes transport due<br>Disponsion due to anomalies makes transport due to anomalies makes transport due to anomalies makes transport  $\sum_{i=1}^{n} \sum_{i=1}^{n} \sum_{i=1}^{n}$ UNIMALIUN GEUMEL  $MITQD$ INFUNIALION GEUN lies very special. In particular it is non-dissipative and insensitive to the changes of the coupling, which makes the analysis analytically tractable. In fact we can not ric which is precisely the Fisher-Rao information metric. FORMALION GEOME central role and contains the information about phase of the information about phase of the information

The field of information geometry was developed in order to study the phase space of statistical systems using geometry. A given statistical ensemble is represented as a point on a Riemannian manifold. This manifold is endowed with a metric which is precisely the Fisher-Rao information metric. The system is characterised by a set of thermodynamic parameters  $β$  which include inverse temperature and generalized chemical potentials for the conserved quantities. One can write down a Gibbs measure for this system  $i$ the analysis analytically tractable. In fact we can not  $\frac{1}{3}$ ensemble is represented as a point on a Kiemannian manifold. Th write do  $\sigma$ f statistical systems their assembly  $\Lambda$  aircre statistic of statistical systems asing geometry. The iven statistic  $t$ mic parameters  $\beta$  which include inverse temperature chemical potentials for the conserved quantities. One  $s$  Cibbs measure for transitions. In order to see the see the see how it would be a started to see the set of the started to see the<br>In order to see the started to see the started to start with the started to see the started to see the started  $\sigma$  $\frac{1}{2}$  . The system is characterised by a settlem in the settlem is contained by a settlem in the **of the thermodynamic product** include include include include international include include in the international include include include include include include include international include include include include includ nformation metric. The system is characterised by a set o hermodynamic parameters ß which include inverse tem ;eneralized chemical potentials for the conserved quan  $p$ (*x*<sup>|</sup>|β)  $p(x)$  $T_{\rm{loc}}$  $\frac{111}{1}$ Partition functions are primary objects in studying field the statistical system nto ther gen writ  $t$  of information cooperatures developed in order to of a or information geometry was developed in order to st bace of statistical systems using geometry. A given statis le is represented as a point on a Kiemannian manifold. d is endowed with a metric which is precisely the Fishe tion metric. The system is characterised by a set of dynamic parameters potentials<br>Fibbs measure fo *r* this system

$$
p(x|\beta) = \exp\left(-\sum_{i} \beta^{i} H_{i}(x) - \ln \mathcal{Z}(\beta)\right),
$$

 $where$ We defin *y* includes hammonian and conservery currents. reeft Fisher information matrix ability measure  $\left(\frac{\partial^2 \ln p(x|\beta)}{\partial x}\right)$ where  $H_i(x)$  includes hamiltonian and conserverd currents. *Z R R R R R R Fisher information matrix* ve achility measure information matrix<br> $\sqrt{2^2 \ln n(x|S)}$ We define the Fisher information matrix  $\mathbf{w}$ he  $\overline{M_0}$  $\overline{\phantom{a}}$ ability measure we can define Fisher information matrix  $\rho$  is the fisher information matrix  $\sigma$  $\left\langle \frac{\partial^2 \ln p(x|\beta)}{\beta} \right\rangle$ *.* (2)

$$
G_{ij}(\beta) = -\left\langle \frac{\partial^2 \ln p(x|\beta)}{\partial \beta_i \partial \beta_j} \right\rangle.
$$

 $\overline{\text{H}}$  is a m  $\sim$  10  $\mu$  m. *f* form to be ∂β*i*∂β*<sup>j</sup>*  $G \circ B$ ∂β*i*∂β*<sup>j</sup>* lt is etric and can be proven to be unique. It is a metric and can be proven to be unique.

#### RICCI SCALAR *D.C. Brody, A. Ritz / Journal of Geometry and Physics 47 (2003) 207–220* 215

This manifold is endowed with a metric which is precisely the Fisher-Rao information metric. In such a geometrization a scalar curvature plays a central role and contains the information about phase transitions. where the spin variable spin variables of the spin variable primerical primeric and *primerical* spin variables, and the manufacture of the spin variables, and the manufacture of the spin variables of the spin variables o metric are obtained by differentiating  $\frac{1}{2}$  ln  $\frac{1}{2}$  ln  $\frac{1}{2}$  ln  $\frac{1}{2}$  ln  $\frac{1}{2}$ *D.C. Brody, A. Ritz / Journal of Geometry and Physics 47 (2003) 207–220* 215 lays a central role and contains the information about phase<br>The contral fole and contains the information about phase e<br>pric which is precisely the Fisher– etric which is precisely the Fisher-<br>…… diali  $\mathbf{1}$  $\alpha$ i $\beta$ lia $\alpha$ 

Example: 1d Ising model in magnetic field  $\frac{1}{\sqrt{2}}$ 

where

 $G_{ij} =$ 1 N  $\partial_i \partial_j \{N\beta + \ln\left[ (\cosh h + \eta)^N + (\cosh h - \eta)^N \right] \},\,$  $\overline{G}$  $i =$  $\lim_{n \to \infty} \frac{1}{\alpha^2} \cos^2\left(\frac{1}{n}\right) \cos^2\left(\frac{1}{n}\right) \sin^2\left(\frac{1}{n}\right) \sin^2\$ where  $\eta = \sqrt{\sinh^2h + e^{-4\beta}}$ .  $\sin n + \eta$  + (cosn  $n - \eta$ ) 1},

e  $\eta = \sqrt{\sinh^2h + e^{-4\beta}}$ .<br>If we consider the metric in the theorem in the theorem of the metric of the theorem is not the theorem in the theorem is not the theorem in the theorem is not the theorem is no see that the theore her modynamic curvature reads:  $\kappa = 1 + \eta$  cosh h The presence of this divergence may be understood by taking a r ansionit of the risher-Kao metric, which is given by the<br>natrix of the entropy. One observes that the nondegeneracy non ior this metric is precisely the concavity condition for the  $\frac{1}{2}$ uropy, and thus its breakdown, where the curvature diverges, does indeed signal a phase transition point. be thermodynamic curvature reads:  $\mathcal{R} = 1 + n^{-1} \cosh h$ ne uren:<br>' dre transform of the Fisher–Rao metric, which is given by the websian maans of are strate py. The exect restation increases respectively  $\frac{1}{2}$ The thermodynamic curvature reads:  $\mathcal{R} = 1 + \eta^{-1} \cosh h$ concavity condition for the the curvature diverges does  $\alpha$  and  $\alpha$  is a function of  $\alpha$  (set  $\alpha$ ) are shown in Fig. 3.3. are shown in Fig. 3.3. and 3.3. an Legendre transform of the Fisher–Rao metric, which is given by the Hessian matrix of the entropy. One observes that the nondegeneracy condition for this metric is precisely the concavity condition for the entropy, and thus its breakdown, where the curvature diverges, does

#### CHIRAL VORTICAL EFFECT CHIKAL VOKUCAL CFFEC The anomaly polynomial in two dimensions allows one to  $\overline{\phantom{a}}$ the expansion after the collision the plasma reaches ther- $\blacksquare$  $V$   $V$   $V$   $I$   $I$   $C$   $A$   $L$   $C$ Manifold *M* Ricci scalar *R*anom Critical point  $CHIR A$ <sup>[</sup>  $A$ <sup>2</sup> *T<sup>c</sup>* =  $^{\prime}$ 3 "<sup>1</sup>*/*<sup>2</sup> !*<sup>c</sup><sup>A</sup>* "<sup>1</sup>*/*<sup>2</sup> *<sup>M</sup>µ,*<sup>Ω</sup> <sup>3</sup>β<sup>Ω</sup> <sup>2</sup>π(β4*c*<sup>2</sup> *Aµ*4+6β2*cAcgµ*2−3*c*<sup>2</sup> *g*) TABLE I: Scalar curvature invariants and corresponding critical points FINAL VUNUDI

"<sup>1</sup>*/*<sup>2</sup> !*<sup>c</sup><sup>A</sup>*

"<sup>1</sup>*/*<sup>2</sup>

*µ*

To make a direct connection between fluids with anomalies and io make a direct connection between fluids with anomalies and<br>information geometry in 1 + 1 and 3 + 1 dimension it is convenient to take fluid configurations on  $R \times S^1$  and  $R \times S^3$ . single generational. To make a direct connection functional and connectional connectional. To make a direct co It provides a natural cut-off and permits to calculate the partition effective action exactly  $\mathbb{R}$  dynamics. We therefore  $\mathbb{R}$  $\overline{10}$   $\overline{11}$ dk per want to all the phase structure. It prov.  $\mathfrak{a}$  $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$  (16) with divided configurations on  $R \times S^+$  and  $R \times S^0$ . now the more present interesting and the anomaly interesting case of  $\mathbf{r}$ <sup>2</sup> *T<sup>c</sup>* = ∂ ma  $ke$  *a* direct *cg µ* nformation <sub>{</sub> <sup>2</sup> *T<sup>c</sup>* = ake fl  $\mu$ *id* config *µ* It provides a natural cut-off and permits to calculate the partition  $\frac{1}{2}$  make a divect connection between € TO IIIAKE A UITECT CONTIECTION DELWEEN II information geometry in  $1 + 1$  and  $3 + 1$  dimension it is convenient to *Io* make a direct connection between fluids with anomalies and mondeum geometry in 1 + 1 di take fluid configurations on  $R\times$ effective action exactly and

!3+2√<sup>3</sup>

modynamic equilibrium and can be described by hydro-

<sup>2</sup> *<sup>T</sup><sup>c</sup>* <sup>=</sup> <sup>√</sup><sup>3</sup>

*<sup>M</sup>*<sup>β</sup>*,µ,*<sup>Ω</sup> <sup>β</sup>Ω(β<sup>6</sup>(−*c*<sup>3</sup>

π

*A*)*µ*6−25β4*c*<sup>2</sup>

*cg*

*µ*

*gµ*2+9*c*<sup>3</sup>

*g*)

<sup>2</sup> *T<sup>c</sup>* =

*Acgµ*4+33β2*cAc*<sup>2</sup>

$$
W = \beta \left[ p \frac{2\pi^2 R^3}{(1 - R^2 \Omega_1)(1 - R^2 \Omega_2)} \right]
$$

$$
- \beta \left[ \mathfrak{F}_{anom}^{\omega} \frac{2\pi R^2 \Omega_1}{1 - R^2 \Omega_1} \frac{2\pi R^2 \Omega_2}{1 - R^2 \Omega_2} \right].
$$

$$
\mathfrak{F}_{anom}^{\omega} = \tilde{c}_A \mu^3 + c_m \frac{\mu}{\beta^2}, \qquad \tilde{c}_A = -\frac{1}{3!(2\pi)^2} \sum \chi_i q_i^3, \qquad c_m = -\frac{1}{4!} \sum \chi_i q_i
$$

*F* We have two parts in the effective action. One is universal and fixed we have two parts in the effective action. One is universal and fixed *anom* sur<br> which is specific to microscopic details. It can be fixed in holograph we are interested in the universal part that dominates for large rad torms c  $\frac{1}{1}$  + 1  $\alpha$  parte in the offective action Ope is universe. the function of the control we are the state of the angular the an erms of data due to anomalies, the other is proportiona which is specific to microscopic details. It can be fixed in associated Ricci scalars, which with the contract of  $\frac{1}{\sqrt{2}}$ we are interested in the universal part that dominates it  $\overline{1}$  (2010) *x*<sup>*i*</sup> *idve iwo* parts in the enective act terms of data due to anomalies, the o  $M_0$  have ty  $\frac{1}{2}$ per we want to ask how it affects the phase structure.  $\mathbf{O}^{\dagger}$ terms of data due to anomalies the other is proportional to de to microscopic details. It can be fixed in no we are interested in the universal part that dominates for la We have two parts in the effectiv  $\mathbf{1} \cdot \mathbf{1} \cdot \mathbf{1}$ which is specific to interoscopic details. It can be fixed in holography but  $\alpha$  coefficients and mixed and fixed in ensinos. When we can do written the partition of the partition of the partition function  $f(x)$ terms of data due to anomalies, the other is proportional to pressure,  $2R^2$  $251011a1gC1a01u3.$ We have two parts in the effective action. One is universal and fixed in which is specific to microscopic details. It can be fixed in holography but we are interested in the universal part that dominates for large radius.

### CRITICAL POINTS

$$
\begin{array}{|c|c|}\hline \text{Manifold } \mathcal{M} & \text{Critical point} \\ \hline \mathcal{M}^{\beta,\mu} & T_c = \left(\frac{3(3-2\sqrt{2})\tilde{c}_A}{c_m}\right)^{1/2} \mu \text{ ; } & T_c = \left(\frac{3(3+2\sqrt{2})\tilde{c}_A}{c_m}\right)^{1/2} \mu \\ \mathcal{M}^{\beta,\Omega_i} & T_c = \left(\frac{(2\sqrt{3}-3)\tilde{c}_A}{3c_m}\right)^{1/2} \mu \\ \mathcal{M}^{\mu,\Omega_i} & \mu_c = \left(\frac{(2\sqrt{3}-3)c_m}{3\tilde{c}_A}\right)^{1/2} T \\ \mathcal{M}^{\Omega_1,\Omega_2} & \text{no critical point} \\ \mathcal{M}^{\beta,\Omega_1,\Omega_2} & T_c = \left(\frac{\tilde{c}_A}{c_m}\right)^{1/2} \mu \text{ ; } & T_c = \left(\frac{(3+2\sqrt{3})\tilde{c}_A}{3c_m}\right)^{1/2} \mu \\ \mathcal{M}^{\beta,\Omega_1,\Omega_2} & T_c = \left(\frac{(\sqrt{33}-5)\tilde{c}_A}{4c_m}\right)^{1/2} \mu \\ \mathcal{M}^{\mu,\Omega_1,\Omega_2} & \mu_c = \left(\frac{c_m}{3\tilde{c}_A}\right)^{1/2} T \\ \mathcal{M}^{\beta,\mu,\Omega_1,\Omega_2} & \text{no critical point} \\ \hline \end{array}
$$

We obtain a set of critical points fixed in terms of anomaly coefficients. Perhaps on could check on the lattice. TABLE III: Critical points of anomalous hydrodynamics in 3+ 1 dimenvve

### GLASMA



#### INFORMATION GEOMETRY OF GLASMA 1 ∂ log *q* ∂ log *q*  $\overline{I}$  **N** *p*  $\overline{I}$  **D**  $\overline{I}$  **D**  $\overline{I}$  **C**  $\overline{I}$  **b**  $\overline{I}$  **p**  $\overline{I}$

The next step is to construct the information geometry for Color Glass Condensate and Glasma. Proposal of Peschanski  $G$ <sup> $\Gamma$ </sup> rt the i *dµ*(*z*) *q*(*z*) *COMPERENE COLOR Glass* Peschanski gave a definition of relative entropy for Color Glass Condensate Peschanski gave a definition of relative entropy for Color Glass Condensate Per depends on definition of Deeds on det  $\overline{\mathcal{G}}$  info  $\inf$  $\lim_{\alpha \to 0} \alpha$ *ry* for Color Glass

*<sup>D</sup>KL*(*p||q*) " *<sup>G</sup>*Fisher

!

*MN* ≡

$$
\Sigma^{Y_1 \to Y_2} = \kappa \left\{ \left( R_1^2 / R_2^2 - 1 \right) - \log \left( R_1^2 / R_2^2 \right) \right\}
$$

**.** 

∂ξ*<sup>M</sup>*

∂ξ*<sup>N</sup> .* (1.3)

$$
\Sigma_{glassma}^{Y_1 \rightarrow Y_2} \sim \kappa_{gl} \ Q_2^2/Q_1^2
$$

 $W<sup>h</sup>$   $R$  = 1*/Q*  $(V)$  and  $Q$  denotes saturation momentum. where  $I_q = 1/\sqrt{8(1)}$  and  $\sqrt{8}$  denotes saturation momentum.  $Rela$ *Q<sup>S</sup>* = *QS*(*Y,* α¯*S*)*.* (1.6) where  $R_i = 1/Q_s(Y_i)$  and  $Q_s$  denotes saturation momentum. information geometry of these models. The saturation sacle is a function of rapidity and coupling where  $R_i = 1/Q_s(Y_i)$  and  $Q_s$  denotes saturation momentum. information geometry of these models. The saturation sacred is a function sacred is a function of  $\mathbf{r}$ <u>selative</u>  $A = \frac{1}{2} \sqrt{2} \sqrt{4} \sqrt{2} \sqrt{4} \sqrt{2}$ Relative entropy can be used to defin  $Y_i$ ) and  $Q_S$  de: *n d*(*z*) *dtes saturation m n* momentum. IN *particular compare closed to define I folice meeting* Relative entropy can be used to define Fisher metric

$$
D_{KL}(p||q) \equiv \Sigma = \int d\mu(z)p(z) \log \frac{p(z)}{q(z)}
$$

$$
D_{KL}(p||q) \simeq G_{MN}^{\text{Fisher}} \delta \xi^M \delta \xi^N
$$

$$
G_{MN}^{\text{Fisher}} \equiv \int d\mu(z) \ q(z) \frac{\partial \log q}{\partial \xi^M} \frac{\partial \log q}{\partial \xi^N}.
$$

#### summary and goals

- Partition functions are very useful in the analysis of anomalies. Many questions are left unanswered
- Close anologs between QFT and turbulence
- Role of parity anomalies
- Phase transitions from information geometry
- Information geometry of CGC and glasma
- Role of magnetic field