

# Coupling constant corrections to the universal identity in second order fluid dynamics

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Based on:

S.Grozdanov and A.Starinets:

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arXiv: 1412.5685 [hep-th]

arXiv: 15XX.XXXX [hep-th]

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Fluid dynamics is an effective theory valid in the long-wavelength, long-time limit

Fundamental degrees of freedom = densities of conserved charges

Equations of motion = conservation laws + constitutive relations<sup>\*</sup>

### Example I

$$\left. \begin{aligned} \partial_a J^a &= 0 \\ J^i &= -D \nabla^i J^0 + \dots \end{aligned} \right\} \partial_t J^0 = D \nabla^2 J^0 + \dots$$

### Example II

$$\left. \begin{aligned} \partial_a T^{ab} &= 0 \\ T^{ab} &= \varepsilon u^a u^b + P(\varepsilon) (g^{ab} + u^a u^b) + \Pi^{ab} + \dots \end{aligned} \right\} \begin{array}{l} \text{Navier-Stokes eqs} \\ \text{Burnett eqs} \\ \dots \end{array}$$

\* Modulo assumptions e.g. analyticity

\*\* E.o.m. universal, transport coefficients depend on underlying microscopic theory

Consider relativistic neutral conformal fluid in a d-dimensional (curved) space-time

$$T^{ab} = \varepsilon u^a u^b + P(\varepsilon) (g^{ab} + u^a u^b) + \Pi^{ab} + \dots$$

Including only terms with first and second derivatives of fluid velocity:

$$\begin{aligned} \Pi^{ab} = & -\eta \sigma^{ab} \\ & + \eta \tau_{\Pi} \left[ \langle D \sigma^{ab} \rangle + \frac{1}{d-1} \sigma^{ab} (\nabla \cdot u) \right] \\ & + \kappa \left[ R^{\langle ab \rangle} - (d-2) u_c R^{c \langle ab \rangle d} u_d \right] \\ & + \lambda_1 \sigma^{\langle a}_c \sigma^{b \rangle c} + \lambda_2 \sigma^{\langle a}_c \Omega^{b \rangle c} + \lambda_3 \Omega^{\langle a}_c \Omega^{b \rangle c} \end{aligned}$$

Transport coefficients (in conformal case):  $\eta, \tau_{\Pi}, \kappa, \lambda_1, \lambda_2, \lambda_3$

Non-conformal case: 2 first order coefficients, 15 (10) second order coefficients  
(see S.Bhattacharyya, 1201.4654 [hep-th])

## Beyond second order hydrodynamics

Tensors structures appearing in the derivative expansion have been analyzed using computer algebra in [1507.02461 \[hep-th\]](#) by Grozdanov & Kaplis.

At third order, there are 20 relevant structures in the conformal case and 68 in the non-conformal one.

This still needs an entropy current analysis similar to the one in S.Bhattacharyya, [1201.4654 \[hep-th\]](#)

Example: dispersion relations in conformal case

$$\omega = -i \frac{\eta}{\varepsilon + P} k^2 - i \left[ \frac{\eta^2 \tau_{\Pi}}{(\varepsilon + P)^2} - \frac{\theta_1}{2(\varepsilon + P)} \right] k^4 + \dots$$

$$\omega = \pm c_s k - i \Gamma k^2 \mp \frac{\Gamma}{2c_s} (\Gamma - 2c_s^2 \tau_{\Pi}) k^3 - i \left[ \frac{8\eta^2 \tau_{\Pi}}{9(\varepsilon + P)^2} - \frac{\theta_1 + \theta_2}{3(\varepsilon + P)} \right] k^4 + \dots$$

Here  $c_s = 1/\sqrt{3}$        $\Gamma = \eta/(\varepsilon + P)$



## Notations used in the derivative expansion

$$D \equiv u^a \nabla_a$$

$$\Delta^{ab} \equiv g^{ab} + u^a u^b$$

$$A^{\langle ab \rangle} \equiv \frac{1}{2} \Delta^{ac} \Delta^{bd} (A_{cd} + A_{dc}) - \frac{1}{d-1} \Delta^{ab} \Delta^{cd} A_{cd} \equiv \langle A^{ab} \rangle$$

$$\sigma^{ab} = 2 \langle \nabla^a u^b \rangle$$

$$\Omega^{ab} = \frac{1}{2} \Delta^{ac} \Delta^{bd} (\nabla_c u_d - \nabla_d u_c)$$

\*Hydro definitions differ in the literature – see footnote 91 on page 128 of M.Haehl, R.Loganayagam, M.Rangamani, 1502.00636 [hep-th]

Transport coefficients  $\eta, \tau_{\Pi}, \kappa, \lambda_1, \lambda_2, \lambda_3$  are non-trivial functions of the parameters of the underlying microscopic theory, even in the simplest case of conformal liquids

$$\frac{\eta}{s} = F \left( N_c, N_f, \frac{m}{T}, \frac{\Lambda}{T}, \dots \right)$$

Some information can be obtained from kinetic theory at **weak coupling** and from gauge-string duality at **strong coupling** (for theories with string or gravity duals).

In the latter case, it is natural to look for “**universal**” (independent of the specific string construction) results.

In the limit of infinite **N** (gauge group rank) and infinite **coupling** (i.e. in the supergravity approximation from dual string theory point of view)

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}$$

Policastro, Kovtun, Son, AOS, 2001-2008  
 Buchel, J.Liu, 2003; Buchel, 2004  
 Iqbal, H.Liu, 2008

$$\frac{\sigma}{\chi} = \frac{1}{4\pi T} \frac{d}{d-2}$$

[for conformal relativistic fluids]: Kovtun, Ritz, 2008

$$2\eta\tau_{\Pi} - 4\lambda_1 - \lambda_2 = 0$$

[for conformal relativistic fluids]: Haack, Yarom, 2008

## More speculative statements inspired by holography

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$$

Kovtun, Son, AOS, 2004 (**violated by some models with Higher derivative gravity**, Kats and Petrov, 2007; Brigante, Myers, H.Liu, Myers, Shenker, Yaida, 2008)

$$\frac{\zeta}{\eta} \geq 2 \left( \frac{1}{d-1} - c_s^2 \right)$$

Buchel, 2008 (**counterexample**: Buchel, 2012)  
See also Kanitscheider and Skenderis, 2009

$$c_s^2 \leq c_{s,conf}^2 = \frac{1}{d-1}$$

Hohler and Stephanov, 2009;  
Cherman, Cohen, Nellore, 2009

$$\frac{\sigma}{\chi} \geq \frac{\hbar v^2}{4\pi T} \frac{d}{d-2}$$

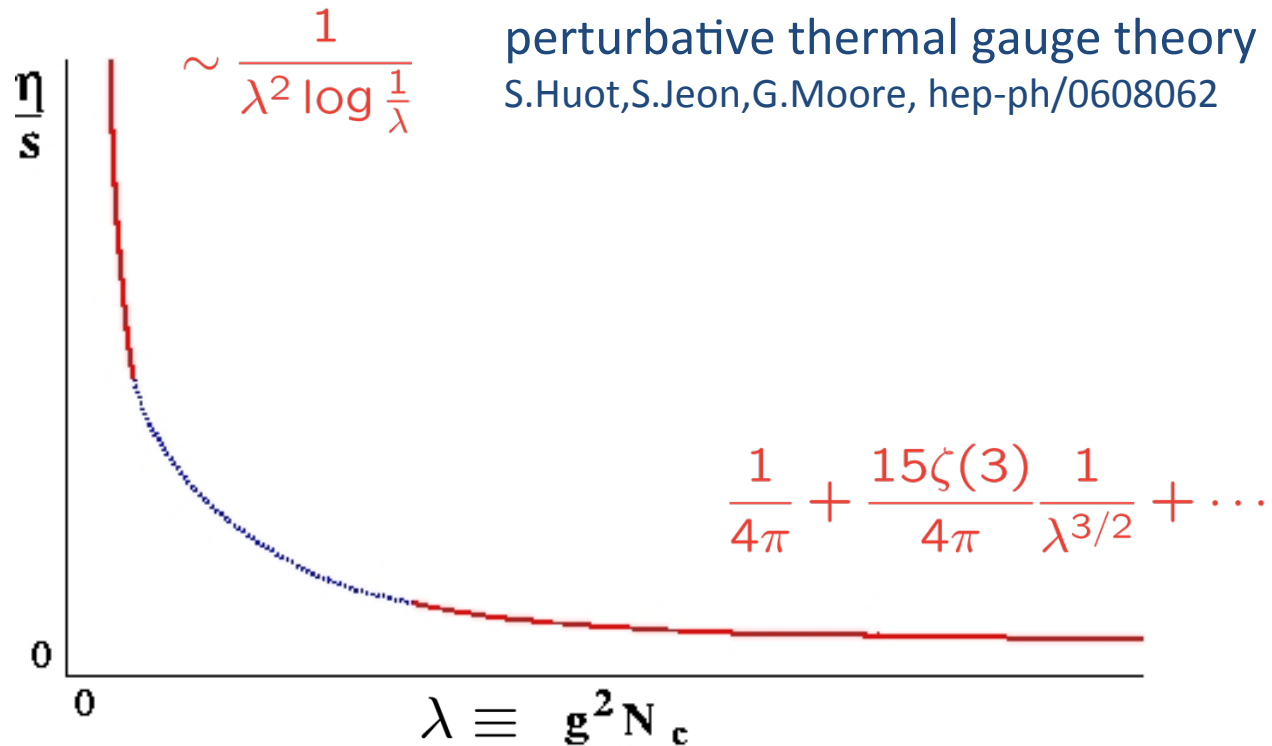
Kovtun and Ritz, 2008

$$D \gtrsim \frac{\hbar v_F}{k_B T}$$

Hartnoll, 2014

See also “Fluctuation bounds...” by Kovtun 1407.0690 [hep-th]

# Shear viscosity in $\mathcal{N} = 4$ SYM



Correction to  $1/4\pi$ : Buchel, Liu, A.S., hep-th/0406264

Buchel, 0805.2683 [hep-th]; Myers, Paulos, Sinha, 0806.2156 [hep-th]



# Electrical conductivity in $\mathcal{N} = 4$ SYM

Weak coupling:

$$\sigma \approx 1.28349 \frac{e^2 (N_c^2 - 1) T}{\lambda^2 [\ln \lambda^{-1/2} + O(1)]}$$

$\lambda \ll 1$

Strong coupling:

$$\sigma = \frac{e^2 N_c^2 T}{16\pi} + O\left(\frac{1}{\lambda^{3/2}}\right)$$

$\lambda \gg 1$

\* Charge susceptibility can be computed independently:

$$\Xi = \frac{N_c^2 T^2}{8}$$

D.T.Son, A.S., hep-th/0601157

Einstein relation holds:

$$\frac{\sigma}{e^2 \Xi} = D_{U(1)} = \frac{1}{2\pi T}$$

# First and second order transport coefficients of *conformal* holographic fluids *to leading order* in supergravity approximation

$$\eta = s/4\pi ,$$

$$\tau_{\Pi} = \frac{d}{4\pi T} \left( 1 + \frac{1}{d} \left[ \gamma_E + \psi \left( \frac{2}{d} \right) \right] \right) ,$$

$$\kappa = \frac{d}{d-2} \frac{\eta}{2\pi T} ,$$

$$\lambda_1 = \frac{d\eta}{8\pi T} ,$$

$$\lambda_2 = \left[ \gamma_E + \psi \left( \frac{2}{d} \right) \right] \frac{\eta}{2\pi T} ,$$

$$\lambda_3 = 0$$

Bhattacharyya et al, 2008

Note:  $2\eta\tau_{\Pi} - 4\lambda_1 - \lambda_2 = 0$

# How to compute second order transport coefficients?

Fluid-gravity correspondence [Bhattacharyya et al, 2007]

Quasinormal spectrum [Baier et al, 2007]

Kubo formulas & three-point functions

[Moore, Sohrabi, Saremi, 2010, 2011; Arnold, Vaman, Wu, Xiao, 2011]

## Coupling constant corrections to N=4 SYM transport coefficients

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left( R + \frac{12}{L^2} + \gamma \mathcal{W} \right) \quad \gamma = \lambda^{-3/2} \zeta(3)/8$$

$$\eta = \frac{\pi}{8} N_c^2 T^3 (1 + 135\gamma + \dots)$$

$$\tau_{\Pi} = \frac{(2 - \ln 2)}{2\pi T} + \frac{375\gamma}{4\pi T} + \dots$$

$$\kappa = \frac{N_c^2 T^2}{8} (1 - 10\gamma + \dots)$$

$$\lambda_1 = \frac{N_c^2 T^2}{16} (1 + 350\gamma + \dots)$$

$$\lambda_2 = -\frac{N_c^2 T^2}{16} (2 \ln 2 + 5(97 + 54 \ln 2)\gamma + \dots)$$

$$\lambda_3 = \frac{25N_c^2 T^2}{2} \gamma + \dots$$



Note:  $2\eta\tau_{\Pi} - 4\lambda_1 - \lambda_2 = 0$



## Kubo formulas for second order transport coefficients

First order transport coefficients can be computed from two-point functions of the corresponding operators using Kubo formulas

$$\eta = \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int dt d^3x e^{i\omega t} \langle [T_{xy}(t, x), T_{xy}(0, 0)] \rangle$$

Similarly, second transport coefficients can be computed from three-point functions

$$\lambda_2 = 2\eta\tau_{\Pi} - 4 \lim_{p, q \rightarrow 0} \frac{\partial^2}{\partial p^0 \partial q^z} G_{RAA}^{xy, ty, xz}(p, q)$$

Moore, Sohrabi, Saremi, 2010, 2011; Arnold, Vaman, Wu, Xiao, 2011

### Schwinger-Keldysh generating functional

$$G_{RA\dots}^{ab, cd, \dots}(0, x, \dots) = \frac{(-i)^{n-1} (-2i)^n \delta^n W}{\delta h_{A ab}(0) \delta h_{R cd}(x) \dots} \Big|_{h=0} = (-i)^{n-1} \langle T_R^{ab}(0) T_A^{cd}(x) \dots \rangle$$

$$W[h^+, h^-] = \ln \int \mathcal{D}\phi^+ \mathcal{D}\phi^- \mathcal{D}\varphi \exp \left\{ i \int d^4x^+ \sqrt{-g^+} \mathcal{L}[\phi^+(x^+), h^+] - \int_0^\beta d^4y \mathcal{L}_E[\varphi(y)] - i \int d^4x^- \sqrt{-g^-} \mathcal{L} \right.$$

## Curvature squared corrections to transport coefficients of a (hypothetical) strongly coupled liquid

$$S_{R^2} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[ R - 2\Lambda + L^2 \left( \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \right]$$

$$\eta = \frac{r_+^3}{2\kappa_5^2} (1 - 8(5\alpha_1 + \alpha_2)) + \dots$$

$$\eta\tau_{\text{II}} = \frac{r_+^2 (2 - \ln 2)}{4\kappa_5^2} \left( 1 - \frac{26}{3} (5\alpha_1 + \alpha_2) \right) - \frac{r_+^2 (23 + 5 \ln 2)}{12\kappa_5^2} \alpha_3 + \dots$$

$$\kappa = \frac{r_+^2}{2\kappa_5^2} \left( 1 - \frac{26}{3} (5\alpha_1 + \alpha_2) \right) - \frac{25r_+^2}{6\kappa_5^2} \alpha_3 + \dots$$

$$\lambda_1 = \frac{r_+^2}{4\kappa_5^2} \left( 1 - \frac{26}{3} (5\alpha_1 + \alpha_2) \right) - \frac{r_+^2}{12\kappa_5^2} \alpha_3 + \dots$$

$$\lambda_2 = -\frac{r_+^2 \ln 2}{2\kappa_5^2} \left( 1 - \frac{26}{3} (5\alpha_1 + \alpha_2) \right) - \frac{r_+^2 (21 + 5 \ln 2)}{6\kappa_5^2} \alpha_3 + \dots$$

$$\lambda_3 = -\frac{28r_+^2}{\kappa_5^2} \alpha_3 + \dots$$

Note:  $2\eta\tau_{\text{II}} - 4\lambda_1 - \lambda_2 = 0$

## Gauss-Bonnet corrections to transport coefficients of a (hypothetical) strongly coupled liquid

$$S_{GB} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} + \frac{\lambda_{GB}}{2} L^2 (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \right]$$

$$\eta = \frac{r_+^2}{2\kappa_5^2} (1 - 4\lambda_{GB})$$

$$\eta\tau_{\Pi} = \frac{r_+^2 (2 - \ln 2)}{4\kappa_5^2} - \frac{r_+^2 (25 - 7 \ln 2)}{8\kappa_5^2} \lambda_{GB} + \dots$$

$$\kappa = \frac{r_+^2}{2\kappa_5^2} - \frac{17r_+^2}{4\kappa_5^2} \lambda_{GB} + \dots$$

$$\lambda_1 = \frac{r_+^2}{4\kappa_5^2} - \frac{9r_+^2}{8\kappa_5^2} \lambda_{GB} + \dots$$

$$\lambda_2 = -\frac{r_+^2 \ln 2}{2\kappa_5^2} - \frac{7r_+^2 (1 - \ln 2)}{4\kappa_5^2} \lambda_{GB} + \dots$$

$$\lambda_3 = -\frac{14r_+^2}{\kappa_5^2} \lambda_{GB} + \dots$$

Brigante, Myers, H.Liu, Myers, Shenker, Yaida, 2008

Shaverin, Yarom, 2012

Note:  $2\eta\tau_{\Pi} - 4\lambda_1 - \lambda_2 = 0$



# Non-perturbative Gauss-Bonnet corrections to transport coefficients of a (hypothetical) strongly coupled liquid

$$S_{GB} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} + \frac{\lambda_{GB}}{2} L^2 (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \right]$$

$$\eta = \frac{s}{4\pi} \gamma^2 = \frac{s}{4\pi} (1 - 4\lambda_{GB})$$

Brigante et al, 2008

$$\tau_{\Pi} = \frac{1}{2\pi T} \left( \frac{1}{4} (1 + \gamma) \left( 5 + \gamma - \frac{2}{\gamma} \right) - \frac{1}{2} \log \left[ \frac{2(1 + \gamma)}{\gamma} \right] \right)$$

Banerjee and Dutta, 2011

$$\lambda_1 = \frac{\eta}{2\pi T} \left( \frac{(1 + \gamma) (3 - 4\gamma + 2\gamma^3)}{2\gamma^2} \right)$$

Grozdanov and AOS, 2014

$$\lambda_2 = -\frac{\eta}{\pi T} \left( -\frac{1}{4} (1 + \gamma) \left( 1 + \gamma - \frac{2}{\gamma} \right) + \frac{1}{2} \log \left[ \frac{2(1 + \gamma)}{\gamma} \right] \right)$$

Grozdanov and AOS, 2014

$$\lambda_3 = -\frac{\eta}{\pi T} \left( \frac{(1 + \gamma) (3 + \gamma - 4\gamma^2)}{\gamma^2} \right)$$

Grozdanov and AOS, 2014

$$\kappa = \frac{\eta}{\pi T} \left( \frac{(1 + \gamma) (2\gamma^2 - 1)}{2\gamma^2} \right)$$

Banerjee and Dutta, 2011

$$H(\lambda_{GB}) = 2\eta\tau_{\Pi} - 4\lambda_1 - \lambda_2 = -\frac{\eta}{\pi T} \frac{(1 - \gamma_{GB}) (1 - \gamma_{GB}^2) (3 + 2\gamma_{GB})}{\gamma_{GB}^2} = -\frac{40\lambda_{GB}^2 \eta}{\pi T} + \dots$$



## Gauss-Bonnet gravity and dissipationless fluids

$$S_{GB} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} + \frac{\lambda_{GB}}{2} L^2 (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \right]$$

$$\frac{\eta}{s} = \frac{1 - 4\lambda_{GB}}{4\pi}$$

Brigante, Myers, H.Liu, Myers, Shenker, Yaida, 2007

The existence of dissipationless fluids (fluids with zero entropy production and yet non-trivial second order transport coefficients) was conjectured by [Bhattacharyya et al, 1211.1020 \[hep-th\]](#).

Holographic Gauss-Bonnet fluid is a natural candidate (modulo problems with UV)

Conditions:  $\kappa = 2\lambda_1, H(\lambda_{GB}) = 0$

In the limit  $\lambda_{GB} \rightarrow 1/4$

$$\eta\tau_{\Pi} = 0, \quad \lambda_1 = \frac{3\pi^2 T^2}{2\sqrt{2}\kappa_5^2}, \quad \lambda_2 = 0, \quad \lambda_3 = -\frac{3\sqrt{2}\pi^2 T^2}{\kappa_5^2}, \quad \kappa = -\frac{\pi^2 T^2}{\sqrt{2}\kappa_5^2}$$

Thus, GB fluid is not dissipationless...

## From strong to weak coupling

Is  $H(\lambda)$  a non-trivial function of the coupling?

$$H(\lambda_{GB}) = 2\eta\tau_{\Pi} - 4\lambda_1 - \lambda_2 = -\frac{\eta}{\pi T} \frac{(1 - \gamma_{GB})(1 - \gamma_{GB}^2)(3 + 2\gamma_{GB})}{\gamma_{GB}^2} \leq 0$$

Conformal kinetic theory:  $2\eta\tau_{\Pi} + \lambda_2 = 0$  and  $\lambda_1/\eta\tau_{\Pi} \approx 1$

[Baier et al, 2007; York and Moore, 2008; Betz et al, 2008]

Also, in (perturbative) QED, QCD and in conformal kinetic theory:  $\kappa = 0$ ,  $\lambda_3 = 0$

[York and Moore, 2008]

Recall that at strong coupling  $\lambda_3 = 0$  but  $\kappa \neq 0$

# Breaking conformal invariance

Bigazzi, Cotrone, 1006.4634 [hep-th]

$\frac{\eta}{s}$	$\frac{1}{4\pi}$	$T\tau_\pi$	$\frac{2-\log 2}{2\pi} + \frac{3(16-\pi^2)}{64\pi}\delta$	$\frac{T\kappa}{s}$	$\frac{1}{4\pi^2}\left(1 - \frac{3}{4}\delta\right)$
$\frac{T\lambda_1}{s}$	$\frac{1}{8\pi^2}\left(1 + \frac{3}{4}\delta\right)$	$\frac{T\lambda_2}{s}$	$-\frac{1}{4\pi^2}\left(\log 2 + \frac{3\pi^2}{32}\delta\right)$	$\frac{T\lambda_3}{s}$	0
$\frac{T\kappa^*}{s}$	$-\frac{3}{8\pi^2}\delta$	$T\tau_\pi^*$	$-\frac{2-\log 2}{2\pi}\delta$	$\frac{T\lambda_4}{s}$	0
$\frac{\zeta}{\eta}$	$\frac{2}{3}\delta$	$T\tau_\Pi$	$\frac{2-\log 2}{2\pi}$	$\frac{T\xi_1}{s}$	$\frac{1}{24\pi^2}\delta$
$\frac{T\xi_2}{s}$	$\frac{2-\log 2}{36\pi^2}\delta$	$\frac{T\xi_3}{s}$	0	$\frac{T\xi_4}{s}$	0
$\frac{T\xi_5}{s}$	$\frac{1}{12\pi^2}\delta$	$\frac{T\xi_6}{s}$	$\frac{1}{4\pi^2}\delta$		

Table 1: The transport coefficients, in the notation of (1)-(3), for a marginally (ir)relevant deformation of a conformal theory, at leading order in the deformation parameter  $\delta \equiv (1 - 3c_s^2)$ . The holographic equation of state is  $\varepsilon = 3(1 + \delta)p$ .

**Note:**  $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$



## Near-equilibrium entropy production and black holes

$$\frac{\nabla_a s^a}{s} = \frac{\eta}{2sT} \sigma_{ab} \sigma^{ab} + \frac{\kappa - 2\lambda_1}{4sT} \sigma_{ab} \sigma_c^a \sigma^{bc} + \left( \frac{A_1}{2s} + \frac{\kappa - \eta\tau_\Pi}{2sT} \right) \sigma_{ab} \left[ \langle D\sigma^{ab} \rangle + \frac{1}{3} \sigma^{ab} (\nabla \cdot u) \right]$$

[Romatschke, 2009]

For dissipationless liquids:  $\kappa = 2\lambda_1$        $2\eta\tau_\Pi - 4\lambda_1 - \lambda_2 = 0$

Weakly coupled N=4 SYM:  $\eta/s \sim 1/\lambda^2 \ln \lambda^{-1}$ ,  $\lambda_1 \sim T^2/\lambda^4 \ln^2 \lambda^{-1}$ ,  $\kappa \sim T^2/\lambda^2$

[York and Moore, 2008]

Strongly coupled N=4 SYM:  $\eta/s = O(\lambda^{-3/2})$ ,  $\kappa - 2\lambda_1 = O(\lambda^{-3/2})$ ,  $H(\lambda) = 0$

*Conjecture: near-equilibrium hydrodynamic entropy production is generically suppressed at strong coupling*

[See also Haehl, Loganayagam, Rangamani, 1412.1090 [hep-th]]



## Open questions

Can we prove (holographically or otherwise – in conformal perturbation theory?) that  $H(\lambda) = 0$  at NLO at strong coupling?

What happens for non-conformal theories?

Are black holes minimizing entropy production?

Consequences for the entropy production in heavy ion collisions?

**THANK YOU!**