Nonlocal Wave Turbulence in QCD

(In preparation)

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Equilibration Mechanisms in Weakly and Strongly Coupled Quantum Field Theory
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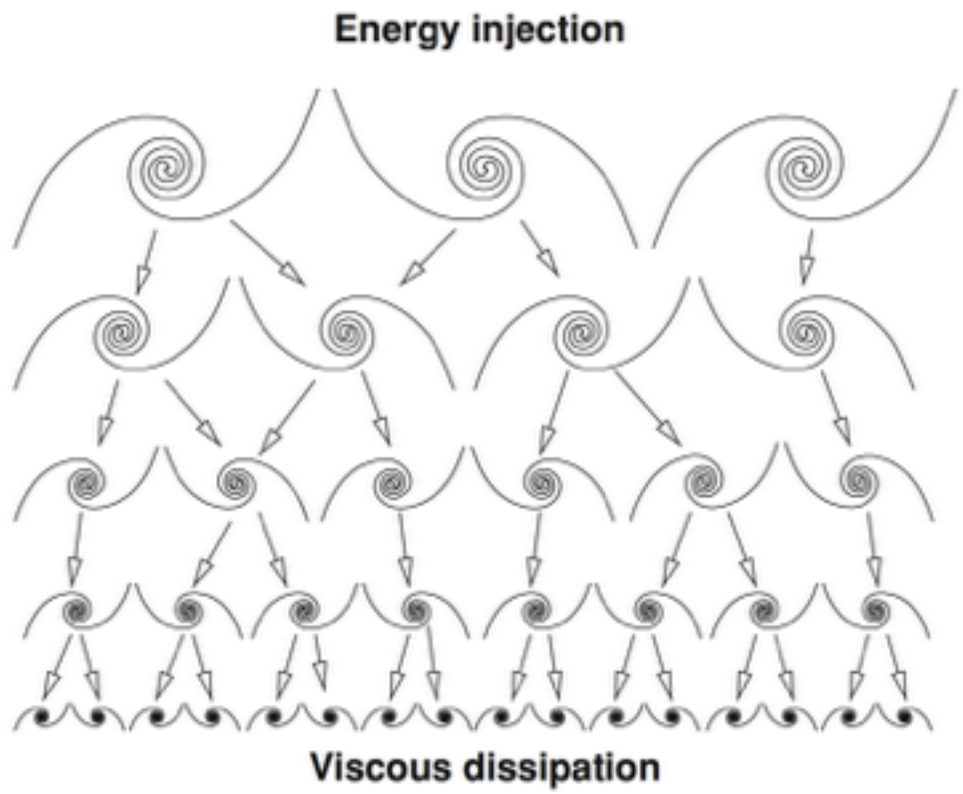
Wave Turbulence (I)

- Out-of-equilibrium statistics of random non-linear waves



- Similarity with fluid turbulence: inviscid transport of conserved quantities from large to small scales through the so-called transparency window or inertial range

- Examples:
  - Atmospheric Rossby waves
  - Water surface gravity and capillary waves
  - Waves in plasmas
  - Nonlinear Schrödinger equation (NL Optics, BEC)

Wave Turbulence (II)

- Waves are excited by external processes. Driven turbulence: Open system with source and sink $\rightarrow$ away from thermodynamical equilibrium

- Steady states characterized by constant fluxes P and Q rather than temperature and thermodynamical potentials

- Kolmogorov-Obukhov (KO41) theory relies on Locality of interactions: Only eddies (waves) with comparable sizes (wavelengths) interact. Steady state power spectra in momentum space depend on the fluxes and not on the pumping and dissipation scales

- Weak (Wave) Turbulence Theory: Kinetic description in the limit of weak nonlinearities (No theory for strong turbulence)

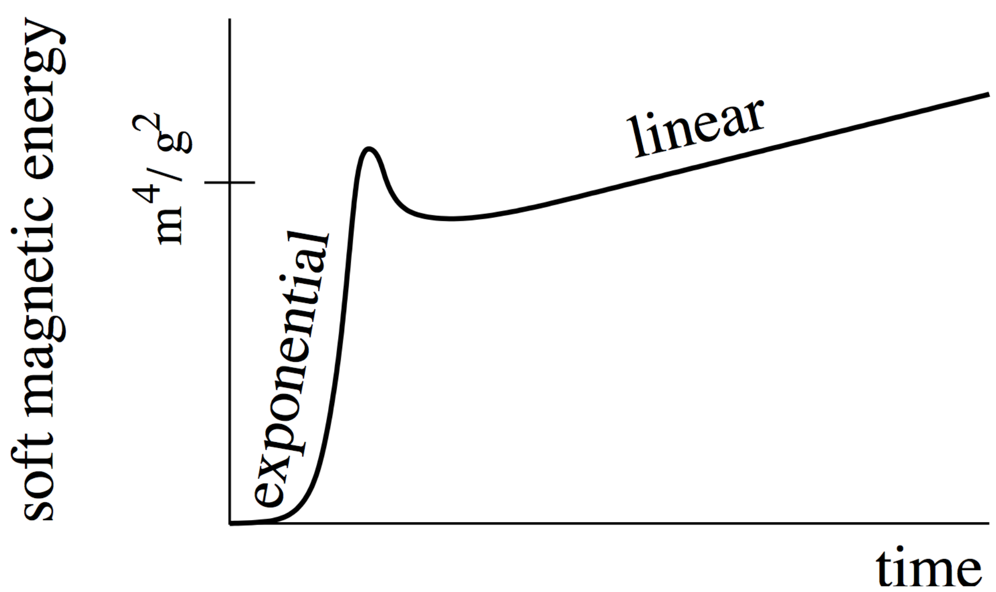
V. E. Zakharov, V. S. L'vov, G. Falkovich (Springer- Verlag, 1992)

Turbulence in early stages of Heavy Ion Collisions

- Far-from equilibrium evolution of anisotropic particle distribution in momentum space (Initial conditions in HIC)

- Chromo-Weibel Instabilities : Anisotropy (hard modes) induces exponentially growing soft modes - transverse magnetic and electric fields which help restoring isotropy- (early stage: as in abelian plasmas) turning to a linear growth due to nonlinear interactions inherent to non-abelian plasmas

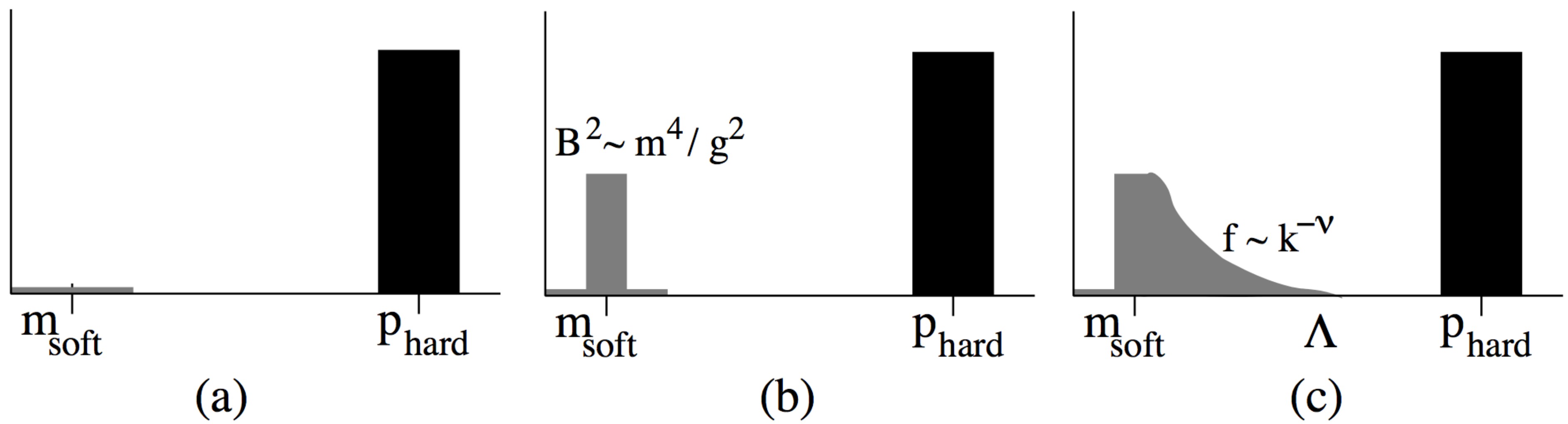
E. S. Weibel (1959)
S. Mrowczynski (1993)
P. Arnold, J. Lenaghan, G. Moore, L. Yaffe (2005)
A. Rebhan, P. Romatschke, and M. Strickland (2005)
D. Bödeker, K. Rummukainen (2005)



P. Arnold, G. D. Moore (2005)

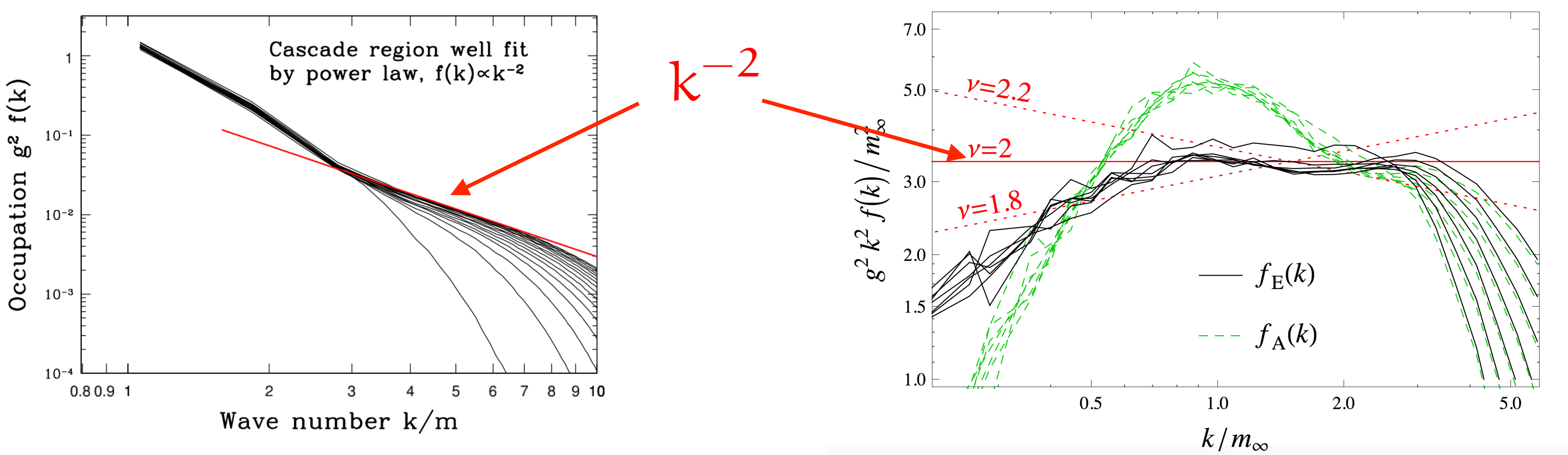
Turbulence in early stages of Heavy Ion Collisions

- Hard-loop simulations (large scale separation between hard modes and soft excitations) : Nonlinear interactions develop a turbulent cascade in the UV with exponent 2



P.Arnold, G. D. Moore (2005)

A. Ipp, A. Rebhan, M. Strickland (2011)



Weak Turbulence in Kinetic Theory

- Can one understand this power spectrum k^{-2} from first principles?

- Note: from A. H. Mueller, A. I. Shoshi, S. M. H. Wong (2006):

  Turbulence in QCD is nonlocal $\Rightarrow$ $n(k) \sim k^{-1}$

- Caveats (in this work):

  - Homogeneous and isotropic system of gluons
  - Energy injection with constant rate P at $k_f \gg m$: Dispersion relation $\omega(k) \equiv |k|$
  - Weak nonlinearities in the classical limit (high occupancy):

$$g^2 \ll 1 \qquad \text{and} \qquad 1 \ll n(k) \ll \frac{1}{g^2}$$

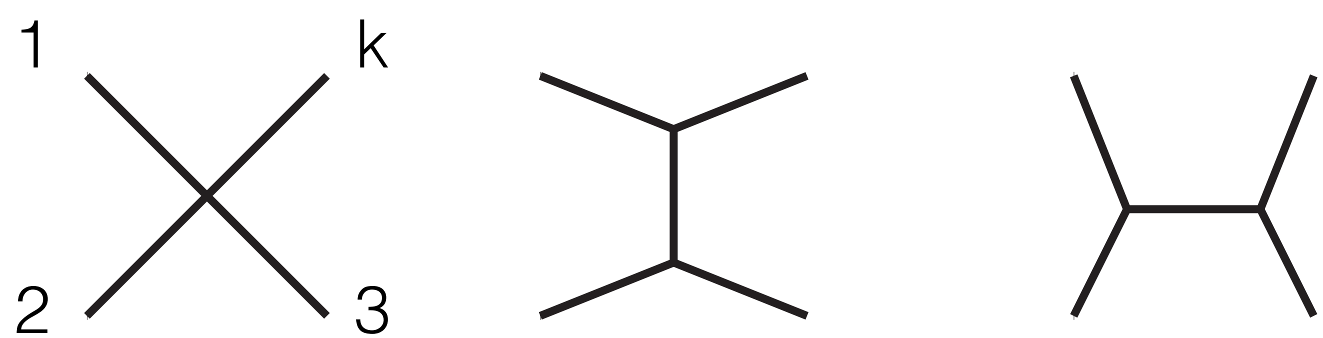
Elastic 2 to 2 process (4-waves interactions)

- Elastic gluon-gluon scattering

$$\frac{\partial}{\partial t} n_k = \frac{1}{2} \int_{k_1,k_2,k_3} \frac{1}{2\omega(k)} |\mathcal{M}_{12\to 3k}|^2 \, \delta(\sum_i k_i)\delta(\sum_i \omega_i) \; F[n]$$

$$F[n] \equiv [n_{k_1} n_{k_2} n_k + n_{k_1} n_{k_2} n_{k_3} - n_{k_1} n_{k_3} n_k - n_{k_2} n_{k_3} n_k] \sim n^3$$

$\mathcal{M}_{12\to 3k} \equiv$



- Two constant of motion: particle number and energy $\Rightarrow$ Two fluxes

$$Q \equiv \dot{N} = \int d^3k \, \dot{n}(k) \qquad\qquad P \equiv \dot{E} = \int d^3k \, |k| \, \dot{n}(k)$$

Kolmogorov-Zakharov (KZ) Spectra

- From collision integral: nonlinear 4-wave interactions

$$P \sim Q \sim \dot{n} \sim n^3 \quad \Rightarrow n \sim P^{1/3} \sim Q^{1/3}$$

- Dimensional analysis determines uniquely the out-of-equilibrium steady state (KZ) power spectra if the interactions are local in momentum space

$$n(k) \sim \frac{Q^{1/3}}{k^{4/3}} \qquad\qquad n(k) \sim \frac{P^{1/3}}{k^{5/3}}$$

particle cascade　　　　energy cascade

- Same exponents for scalar theories in the absence of condensation

Dual cascade: Fjørthoft argument

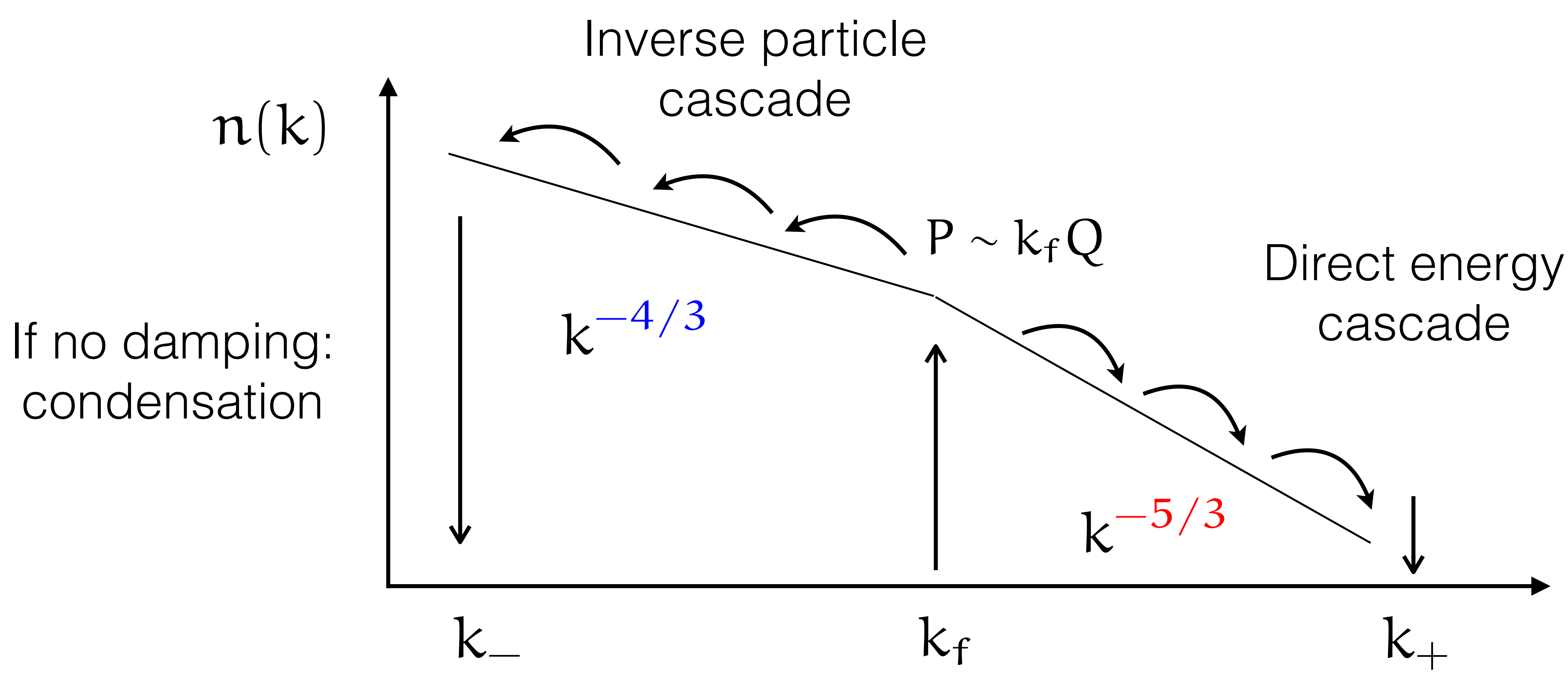
- Direction of fluxes: Injection of energy at k_f and dissipating at

$$k_- \ll k_f \ll k_+$$

- Direct energy cascade: If energy was dissipating at low momenta then particles would dissipate at a faster rate than the pumping rate!

$$Q_- \sim \frac{P}{k_-} \gg \frac{P}{k_f} \sim Q$$

Fjørthoft (1953)

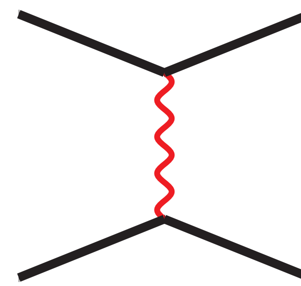


Are KZ spectra in QCD physically relevant?

Small angle approximation

- Coulomb interaction is singular at small momentum transfer $k \gg q \geq m$

$$|\mathcal{M}_{k1\to 23}|^2 \sim \alpha^2 \frac{s^2}{t^2}$$



$$t \sim q^2 \ll s$$

- Fokker-Plank equation:
  Diffusion and drag

$$\frac{\partial}{\partial t} n_k \equiv \frac{\hat{q}}{4k^2} \frac{\partial}{\partial k} k^2 \left[\frac{\partial}{\partial k} n_k + \frac{n_k^2}{T_*} \right]$$

L. D. Landau (1937) B. Svetitski (1988)

Diffusion coefficient

$$\hat{q} \equiv \sim \alpha^2 \int d^3k\, n_k^2$$

Screening mass

$$m^2 \sim \alpha \int \frac{d^3k}{|k|} n_k$$

Effective temperature

$$T_* \sim \frac{\hat{q}}{\alpha m^2}$$

- KZ spectra are not stationary solutions of the collision integral (contrary to non-relativistic Coulomb scattering! A. V. Kats, V. M. Kontorovich, S. S. Moiseev, and V. E. Novikov (1975))

- $\hat{q}$ diverges in the IR for $n \sim k^{-5/3}$ and for $n \sim k^{-4/3}$ in the UV

Steady state solutions

$$\frac{\partial}{\partial t} n_k \equiv \frac{\hat{q}}{4k^2} \frac{\partial}{\partial k} k^2 \left[\frac{\partial}{\partial k} n_k + \frac{n_k^2}{T_*} \right] + F - D$$

Forcing Dumping

- Thermal fixed point: $\frac{T_*}{k-\mu}$
- Non-thermal fixed point (inverse particle cascade): $n(k) \sim \frac{A}{k} > \frac{T_*}{k}$

$$A \equiv \frac{1}{2} T_* \left(1 + \sqrt{1 + \frac{16Q}{\hat{q} T_*}} \right)$$

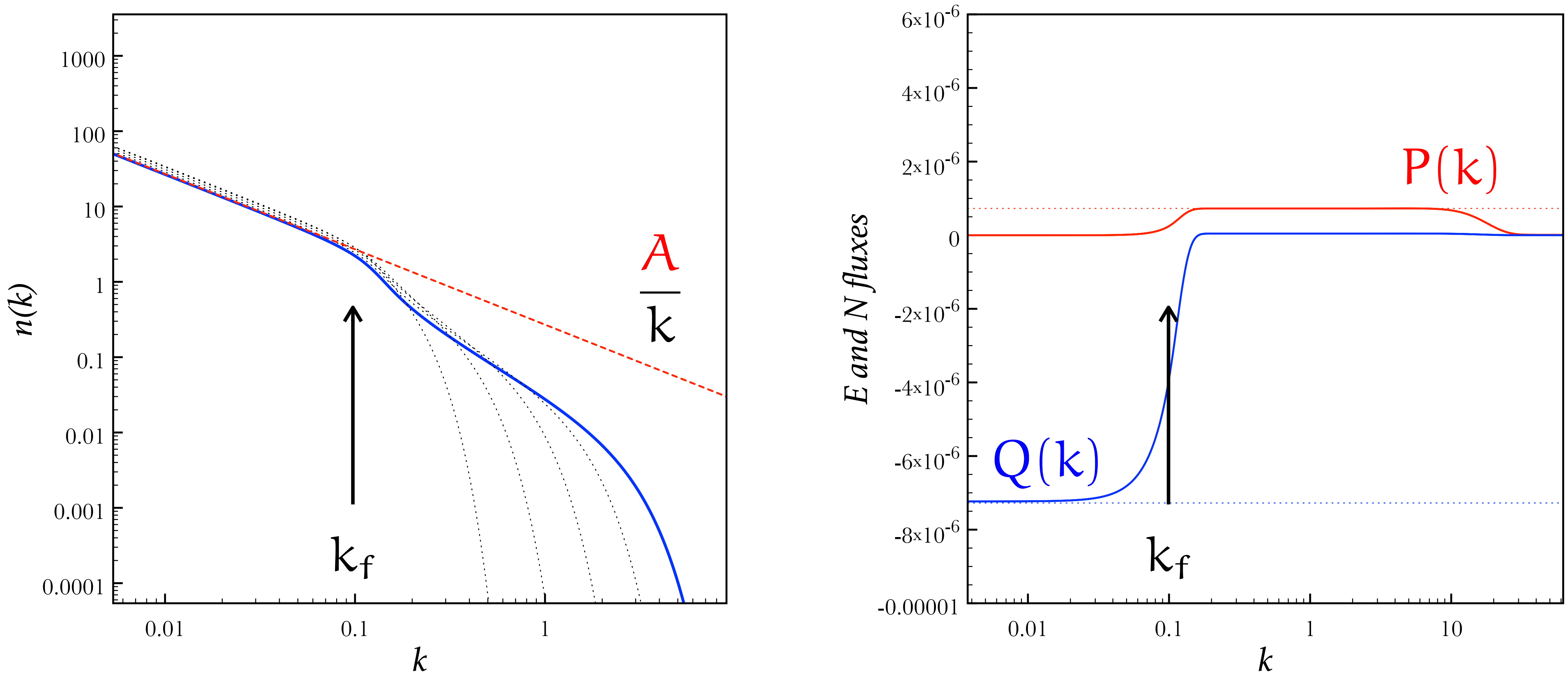
- Warm cascade behavior:

2-D Optical turbulence: S. Dyachenko, A.C. Newell, A. Pushkarev, V.E. Zakharov (1992)

Boltzmann equation: D. Proment, S. Nazarenko, P. Asinari, and M. Onorato (2011)

- No (homogeneous) steady state solution for the energy cascade

Numerical simulation of FK equation with forcing



- The occupation number (left) and, the energy and particle number fluxes (right) at late times

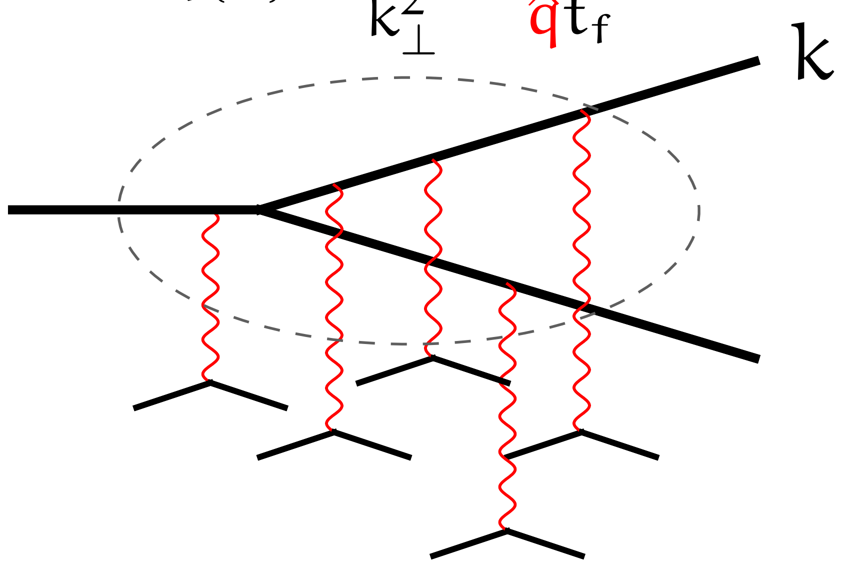
Inelastic processes in the small angle approximation

Effective 3 waves interactions (1 to 2 scatterings)

- LPM regime: many scatterings can cause a gluon to branch with the rate

formation time: $t_f(k) \sim \frac{k}{k_\perp^2} \sim \frac{k}{\hat{q} t_f}$

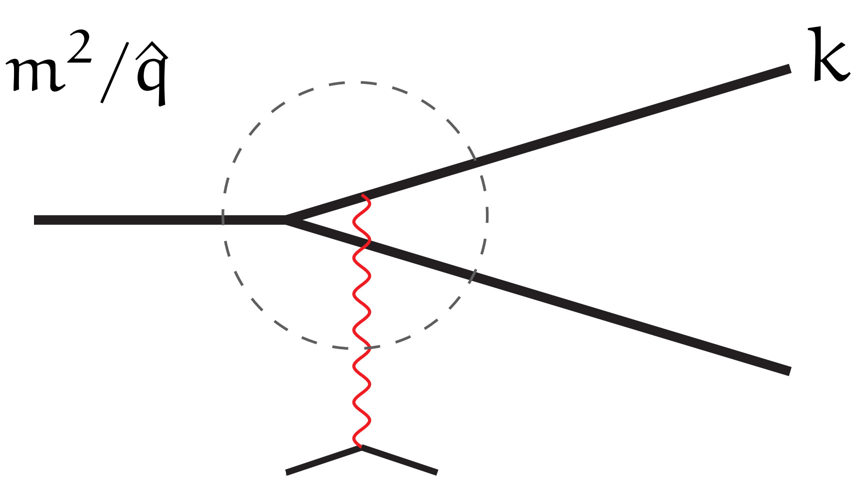
$$k\frac{d\Gamma}{dk} \sim \frac{\alpha}{t_f(k)} \sim \alpha\sqrt{\frac{\hat{q}}{k}}$$



R. Baier, Y. Dokshitzer, A. H. Mueller, S. Peigné, D. Schiff (1995) V. Zakharov (1996)

- Bethe-Heitler regime for $t_f(k) < \ell_{mfp} \sim m^2/\hat{q}$

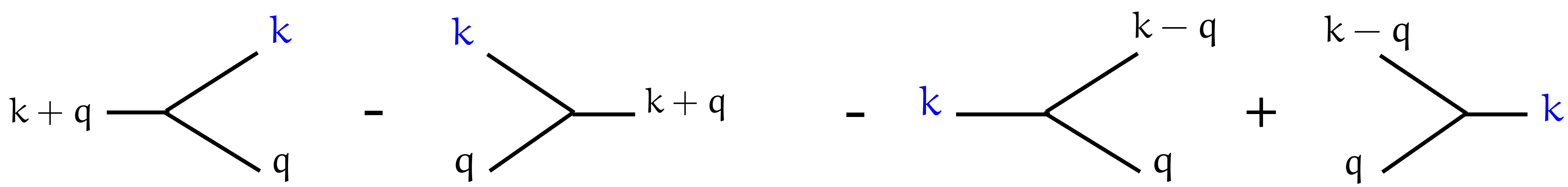
$$k\frac{d\Gamma}{dk} \sim \frac{\alpha}{\ell_{mfp}}$$



J. F. Gunion and G. Bertsch (1982)

Effective 3 waves interaction (1 to 2 scatterings)

$$\frac{\partial}{\partial t} n_k \equiv \frac{1}{k^3}\left[\int_0^\infty dq K(k+q,q)F(k+q,q) - \int_0^k dq K(k,q)F(k,q)\right]$$



$$F(k,q) \equiv n_{k+q}n_k + (n_{k+q} - n_k)n_q \sim n^2$$

$$K(k,q) \equiv \alpha\sqrt{\hat{q}}\,\frac{(k+q)^{7/2}}{k^{1/2}q^{3/2}}$$

R. Baier, Y. Dokshitzer, A. H. Mueller, D. Schiff, D. T. Son (2000)
P. Arnold, G. D. Moore, L. G. Yaffe (2002)

- Direct energy cascade (if interactions are local!)

$$n_k \sim \frac{P^{1/2}}{\hat{q}^{1/4}\,k^{7/4}}$$

P. Arnold, G. D. Moore (2005)

Locality of interactions

- Assume a power spectrum $n \sim k^{-x}$ and require the energy flux to be independent of k

- We obtain $x = 7/4$ and

$$P = \alpha\sqrt{\hat{q}}\int_0^1 dz\,\frac{(1-z)^x + z^x - 1}{z^{x+1/2}(1-z)^{x+3/2}}\,\ln\frac{1}{z}$$

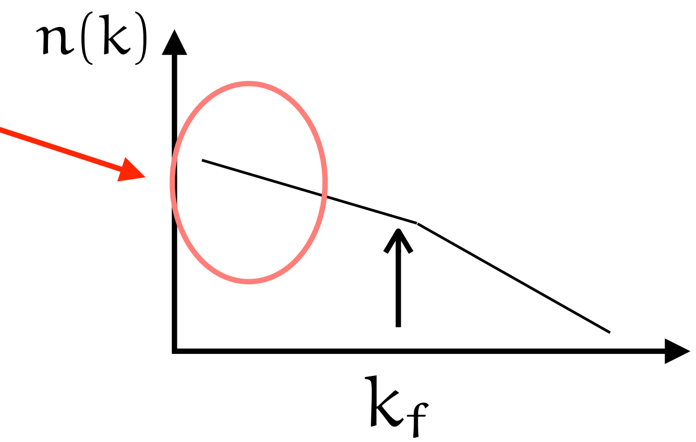
- The above integral diverges: $P = \infty$

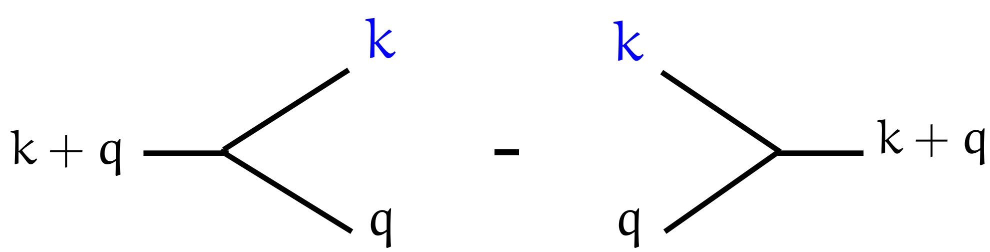
$\Rightarrow$ Effective 3 waves Interaction is nonlocal in momentum space and the KZ spectrum cannot be realized

Gradient expansion of the collision integral ($k \ll k_f$)

- The collision integral is dominated by strongly asymmetric branchings

- In the regime: $k \ll k_f$ To the left of the source





$$\frac{\partial}{\partial t} n_k \simeq \frac{1}{k^3} \left[\int_0^\infty dq K(k+q, q) F(k+q, q) \right] a \simeq \alpha \frac{\sqrt{\hat{q}}}{k^{7/2}} \left[T_* - k\, n(k) \right]$$

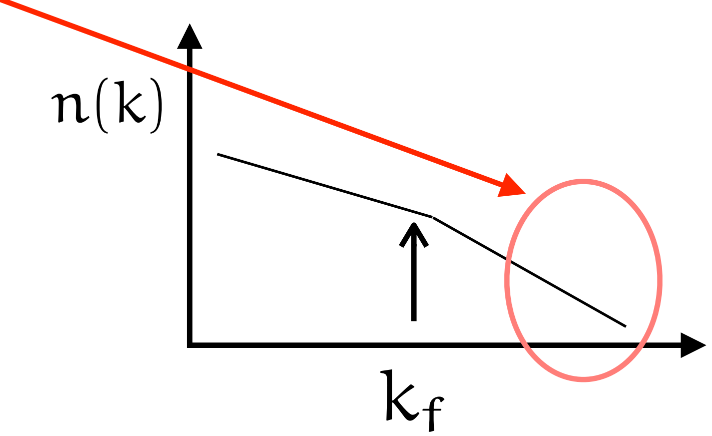
- Late times (steady state) solution is thermal (no fluxes) :

$$n(k) \equiv \frac{T_*}{k}$$

- In agreement with Mueller, Shoshi, Wong (2006) Abraao York, Kurkela, Lu, Moore (2014)

Gradient expansion of the collision integral ($k \gg k_f$)

- In the regime: $k \gg k_f$ To the right of the source. We perform a gradient expansion around $k \gg q$



- We obtain an isotropic diffusion equation in "4-D"

$$\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{inel}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k)$$

- with the inelastic diffusion coefficient (in the LPM regime)

$$\hat{q}_{inel} = \alpha \sqrt{\hat{q}} \int_0^\infty dq \, \sqrt{q} \, n(q)$$

Gradient expansion of the collision integral (II)

$$\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{inel}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k)$$

- Recall that 3-D diffusion conserves number of particles: $N \sim \int dk\, k^2\, n(k)$

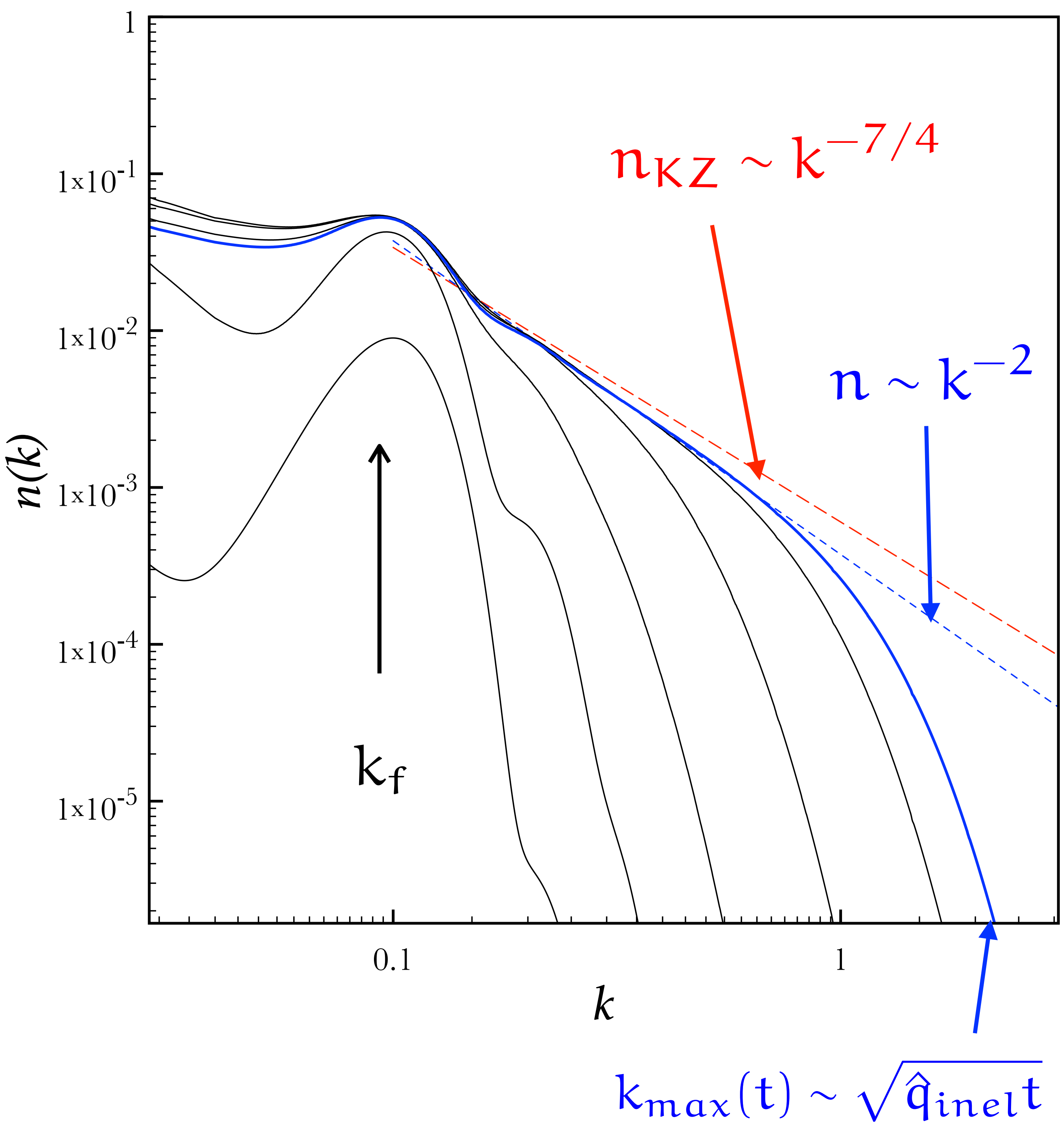
Its fixed point (inverse particle cascade): $n(k) \sim \frac{1}{k}$

- 4-D diffusion conserves energy: $E \sim \int dk\, k^3\, n(k)$

Its fixed point (direct energy cascade):

$$n(k) \sim \frac{1}{k^2}$$

Numerical simulation with forcing (full solution)



Parametric estimate:

$$\hat{q} \sim k_f^3 n^2$$

$$\Downarrow$$

$$n(k) \sim \frac{P}{\hat{q} k^2} \sim \frac{P^{1/3} k_f^{1/3}}{k^2}$$

Nonlocal turbulent spectrum:
hard gluons in the inertial range interact dominantly with gluons at the forcing scale (energy gain)

Interplay between elastic and inelastic processes (I)

- For a spectrum falling faster than ***1/k*** one can neglect the drag term in the elastic part. Then, the collision integral in the UV reads

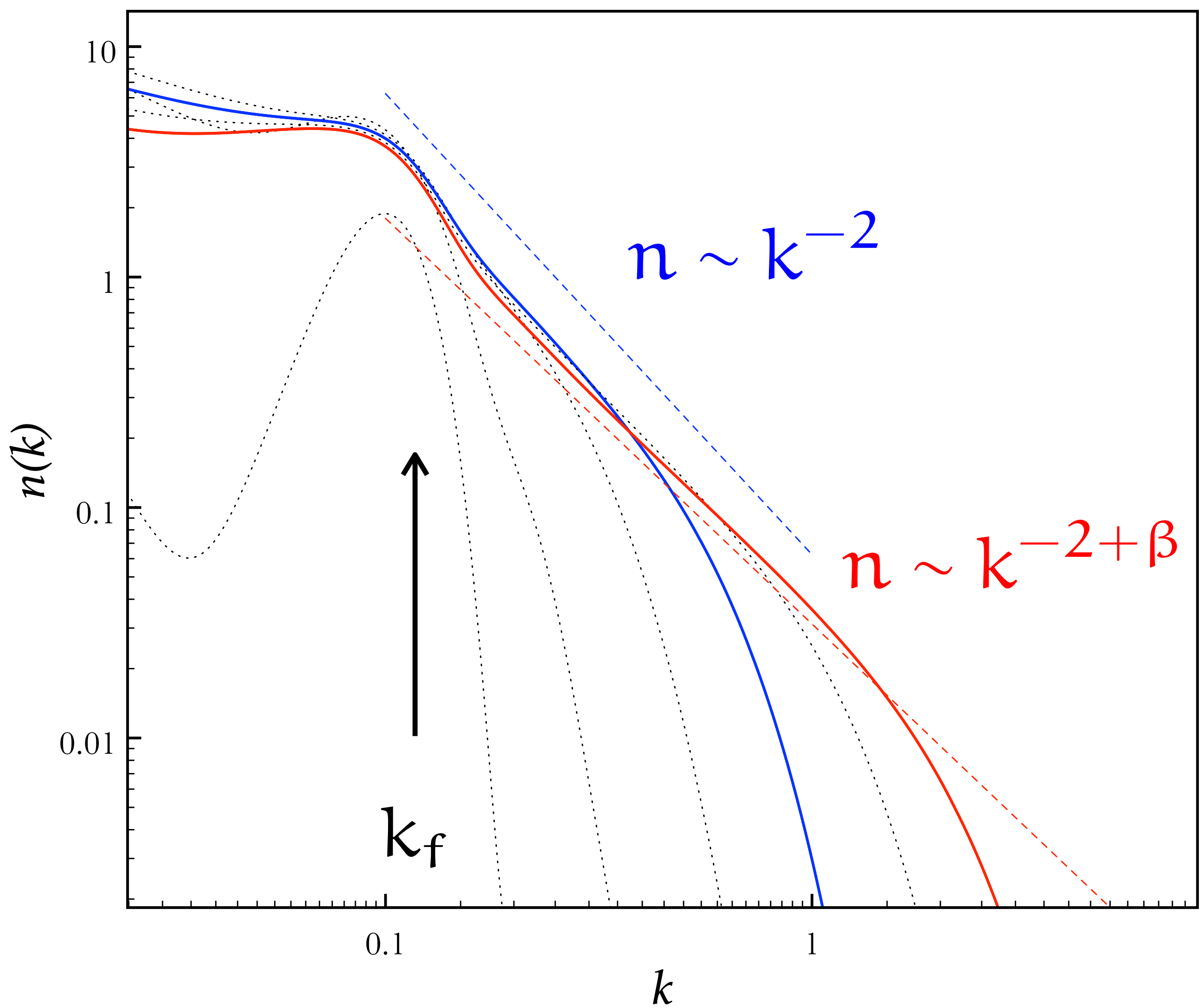
$$\frac{\partial}{\partial t} n(k) \simeq \frac{\hat{q}_{inel}}{4k^3} \frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} n(k) + \frac{\hat{q}_{el}}{4k^2} \frac{\partial}{\partial k} k^2 \frac{\partial}{\partial k} n(k)$$

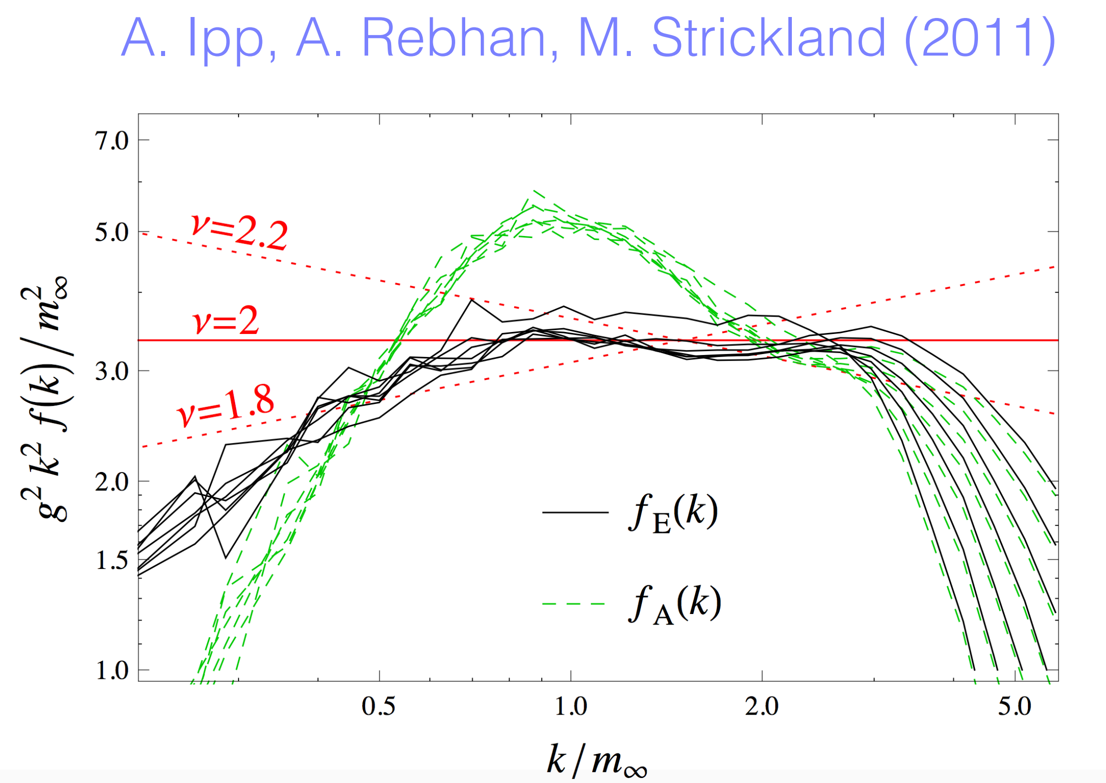
$$\equiv \frac{B}{4k^{3-\beta}} \frac{\partial}{\partial k} k^{3-\beta} \frac{\partial}{\partial k} n(k)$$

where $B = \hat{q}_{inel} + \hat{q}_{el}$ and $\beta = \frac{2}{1 + \hat{q}_{inel}/\hat{q}_{el}}$

- Steady state solution: $n(k) \sim \frac{1}{k^{2-\beta}}$ $\quad 0 < \beta < 1$

Interplay between elastic and inelastic processes (II)





Exponent $\beta \simeq 0.24$ is computed self-consistently

- Elastic processes reduce slightly the exponent at asymptotically late times. At intermediate times ***k*** $^{-2}$ spectrum is observed: balance between the drag and diffusion terms?

Summary

- Wave turbulence in QCD is different from scalar theories. It is characterized by nonlocal interactions in momentum space: Kolmogorov-Zakharov spectra are not physically relevant

- Inelastic processes dominates the dynamics with a direct energy cascade

- To the right of the forcing scale: Kinetic theory predicts a steady state spectrum $\sim k^{-2}$ (in the LPM and BH regimes) in agreement with Hard Loop simulations

- To the left of the forcing scale the system appears to be thermalized: warm cascade

- Outlook: mass corrections, anisotropic fluxes, strong turbulence in the presence of strong fields (on the lattice): different exponents?