An analytic solution to the relativistic Boltzmann equation and its hydrodynamical limit

Mauricio Martinez Guerrero

Collaborators: *G. Denicol, U. Heinz, J. Noronha and M. Strickland*

Based on: **PRL 113 202301 (2014), PRD 90 125026 (2014), arXiv:1506.07500**

Equilibration Mechanisms in Weakly and Strongly Coupled Quantum Field Theory

INT University of Washington, Seattle, USA

August 3-28, 2015

The Ohio State University

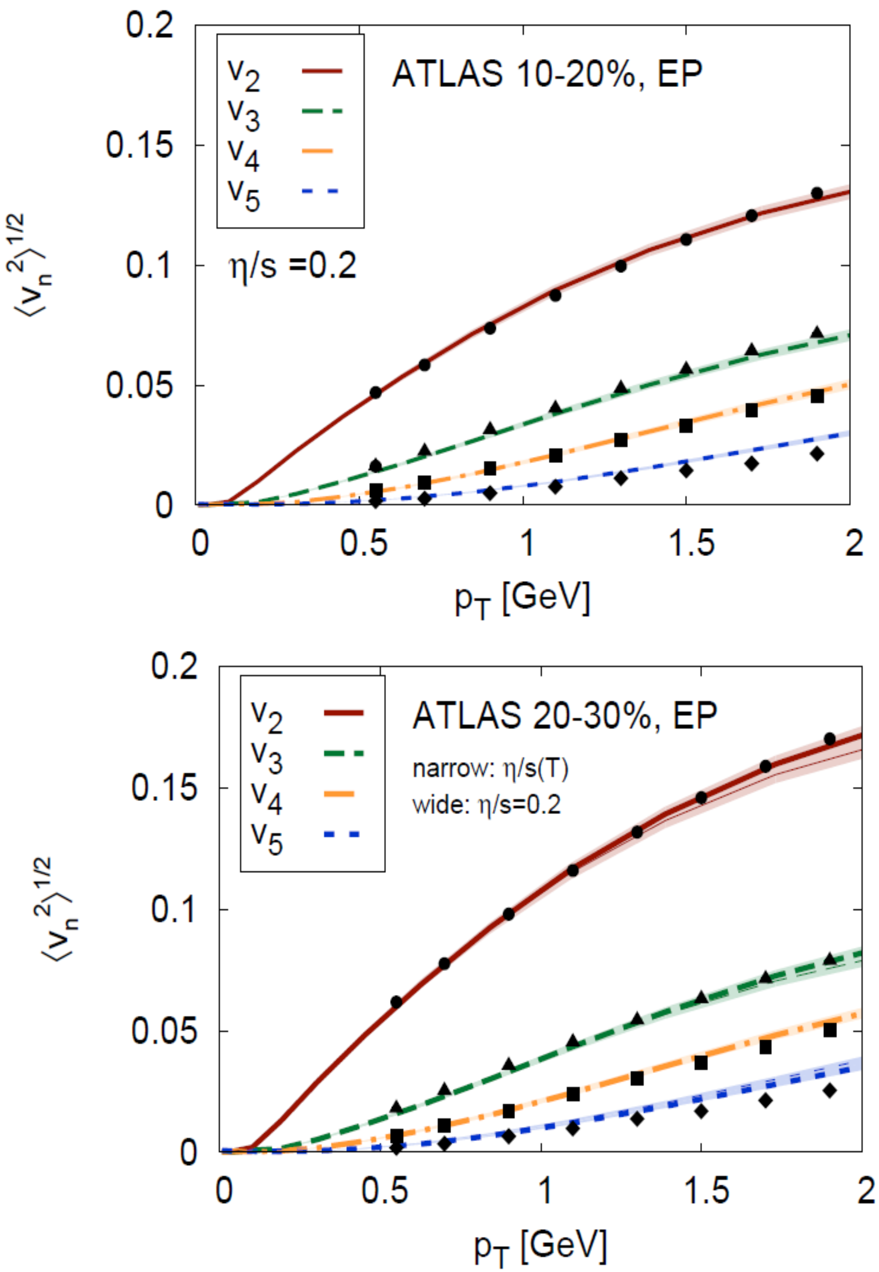
Success of viscous hydrodynamics

Quark gluon plasma: the hottest, tiniest and most perfect fluid ever made on Earth:

$$\frac{\eta}{s} = \frac{2}{4\pi} \pm 50\%$$

Hydro requires as an input:

1. **Initial conditions: CGC, Glauber, etc.**
2. **Evolution for the dissipative fields: 2nd order viscous hydro**
3. **EOS: lattice + hadron resonance gas**
4. **Hadronization and afterburning URQMD, etc.**



Gale et. al, PRL 110, 012302 (2012)

Success of viscous hydrodynamics

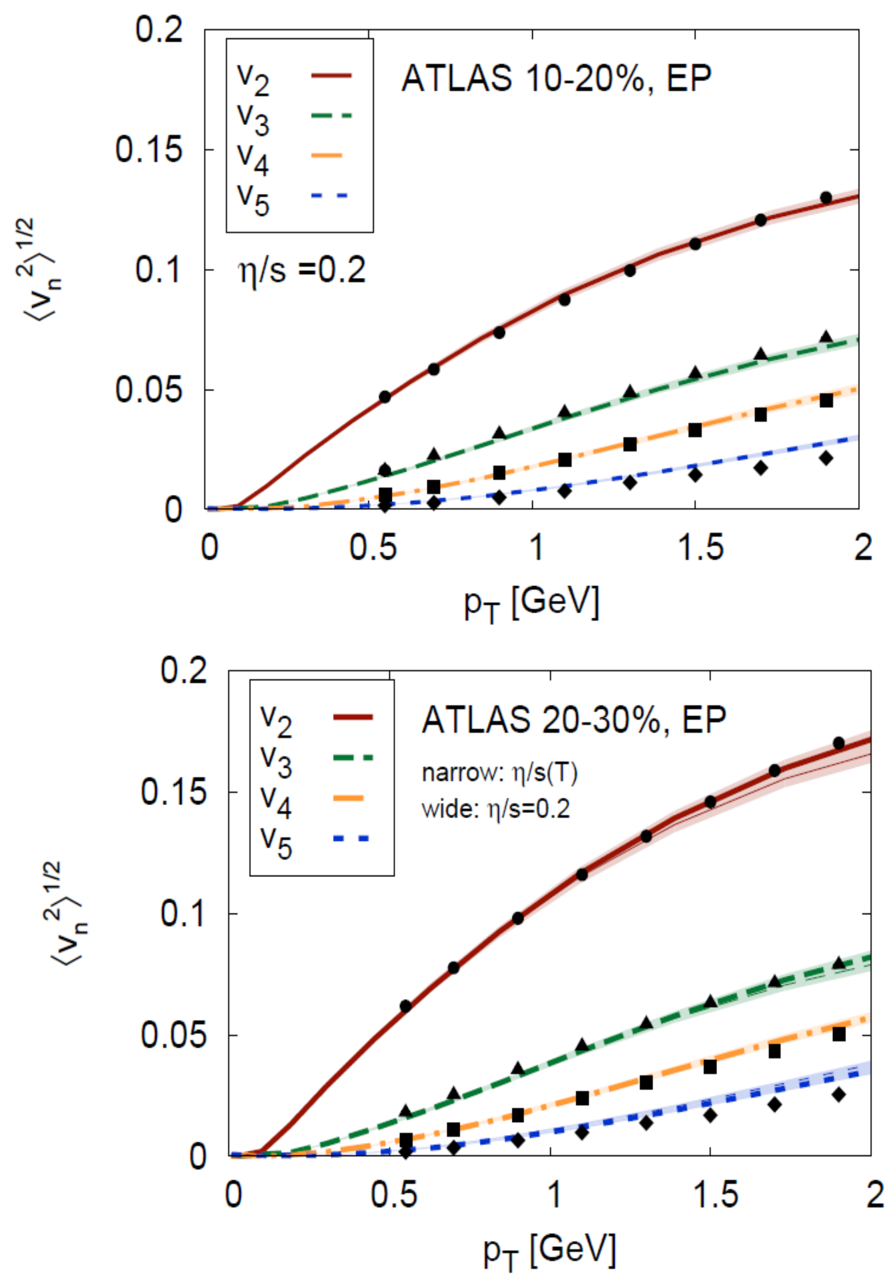
Quark gluon plasma: the hottest, tiniest and most perfect fluid ever made on Earth:

$$\frac{\eta}{s} = \frac{2}{4\pi} \pm 50\%$$

Hydro requires as an input:

1. **Initial conditions: CGC, Glauber, etc.**
2. **Evolution for the dissipative fields: 2nd order viscous hydro**
3. **EOS: lattice + hadron resonance gas**
4. **Hadronization and afterburning URQMD, etc.**

What is the best hydrodynamical description that describes the QGP?



Gale et. al, PRL 110, 012302 (2012)

Our goal

We are interested to solve **exactly** the relativistic Bolzmann equation for massless particles within the relaxation time approximation (RTA)

$$p^t\,\partial_t f \,+\, p_x\,\partial_x f \,+\, p_y\,\partial_y f + p^z \partial_z f = \frac{p \cdot u}{\tau_{rel}}\,(f - f_{eq})$$

$$\mathrm{p}^t = \sqrt{p_x^2 + p_y^2 + p_z^2}$$

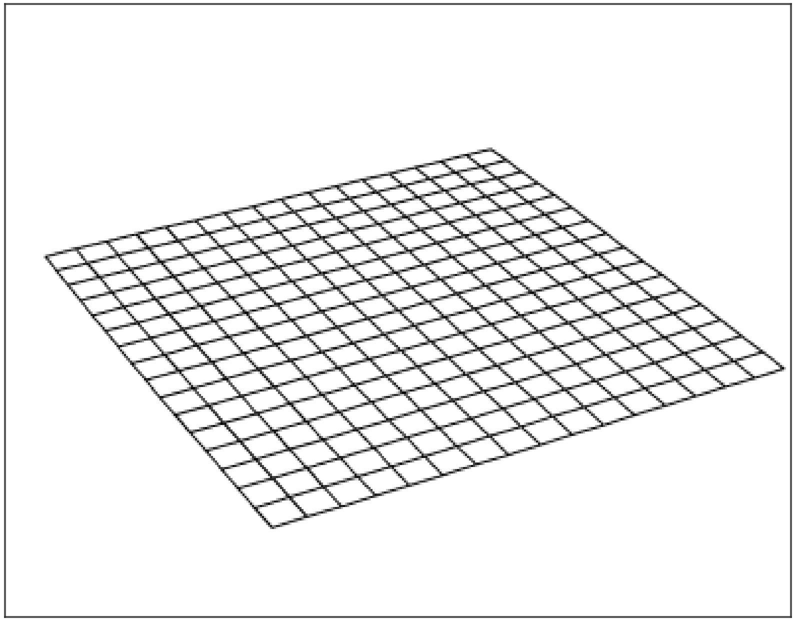
We find an exact solution of the RTA Boltzmann equation for the Gubser flow by understanding the constraints imposed by the symmetries

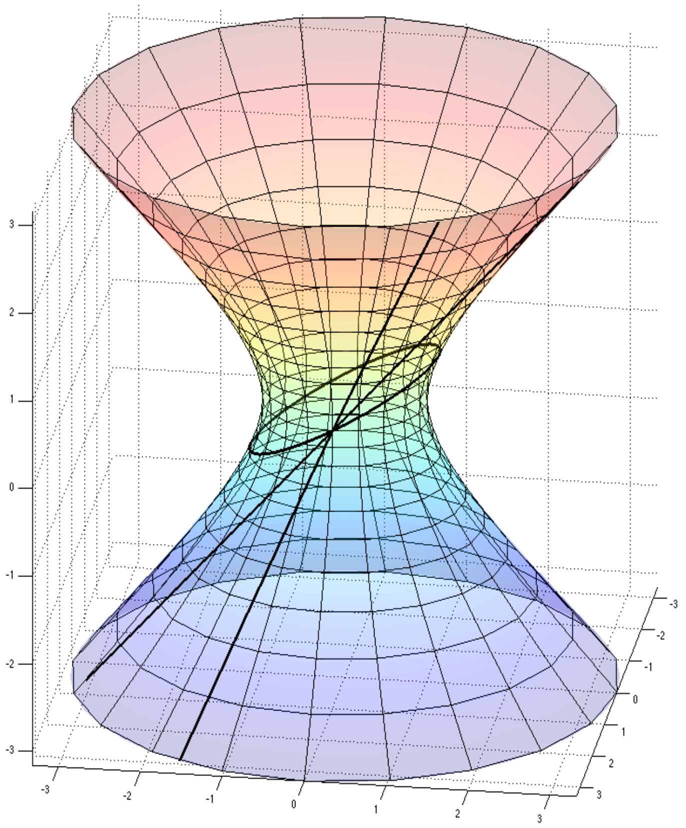
The Gubser flow (2010)

Conformal map

Expanding plasma
In Minkowski space

Static fluid in
a curved space





Symmetries of the Bjorken flow

$$ISO(2) \otimes SO(1,1) \otimes Z_2$$

Z_2 → **Reflections along the beam line**

$SO(1,1)$ → **Longitudinal Boost invariance**

$ISO(2)$ → **Translations in the transverse plane + rotation along the longitudinal z direction**

Generalization of Bjorken's idea: Gubser flow

- However, Bjorken flow does not have transverse expansion.
- One can generalize it by considering symmetry arguments. Gubser (2010)
- Modifying the ISO(2) group allows us to have transverse dynamics (Gubser)

$$\boxed{ISO(2)} \otimes SO(1,1) \otimes Z_2$$

$$\Downarrow$$

$$\boxed{SO(3)_q} \otimes SO(1,1) \otimes Z_2$$

Symmetries of the Gubser flow

$$SO(3)_q \otimes SO(1,1) \otimes Z_2$$

Special Conformal transformations + rotation along the beam line

Boost invariance

Reflections along the beam line

Weyl rescaling + Coordinate transformation

SO(3) is associated with rotations. What are we rotating?
Conformal map provides the answer

Minkowski metric (Milne coordinates)

$$ds^2 = -d\tau^2 + \tau^2 d\varsigma^2 + dr^2 + r^2 d\phi^2$$

↓ **Weyl rescaling**

$$d\hat{s}^2 = \frac{ds^2}{\tau^2}$$

↓ **Coordinate transformation**

$$\rho = -\sinh^{-1}\left(\frac{1 - q^2\tau^2 + q^2 r^2}{2q\tau}\right)$$

$$\theta = \tanh^{-1}\left(\frac{2qr}{1 + q^2\tau^2 - q^2 r^2}\right)$$

$$d\hat{s}^2 = \underbrace{-d\rho^2 + \cosh^2\rho\,(d\theta^2 + \sin^2\theta\, d\phi^2)}_{dS_3} + \underbrace{d\varsigma^2}_{R}$$

Gubser's flow velocity profile

Symmetries in this case are better understood after a Weyl rescaling + Coordinate transformation

In the de Sitter space, the generators of $SO(3)_q$ are

$$\xi_2 = 2q\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$\xi_3 = 2q\left(\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$\xi_4 = \frac{\partial}{\partial\phi}$$

→ SO(3) symmetry is manifest and it corresponds to rotations in the (θ,ϕ) subspace.

- So the only invariant flow compatible with the symmetries is

$$[\xi_i, \hat{u}] = 0 \Rightarrow \hat{u}^\mu = (1,0,0,0)$$

→ **Static flow in de Sitter space**

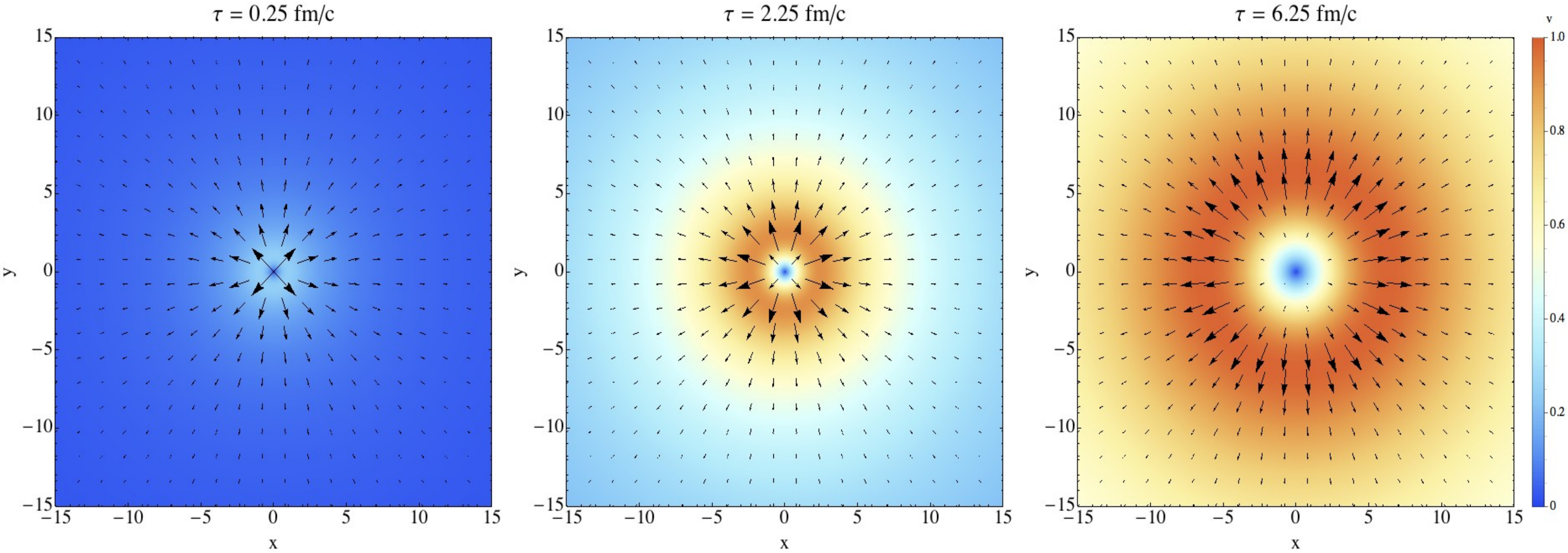
Gubser's flow velocity profile

The flow velocity in Minkowski space is easily calculated:

$$u_\mu = \tau \frac{\partial \hat{x}^\nu}{\partial x^\mu} \hat{u}_\nu \quad \Rightarrow \quad u^\mu = (\cosh \kappa(\tau, r), \sinh \kappa(\tau, r), 0, 0)$$

Non trivial radial flow

$$\kappa(\tau, r) = tanh^{-1}\left(\frac{2q^2\tau r}{1 + q^2\tau^2 + q^2 r^2}\right)$$



Our solution to the RTA Boltzmann equation

Exact solution to the RTA Boltzmann equation

We construct a solution which is invariant under the group
$SO(3)_q \otimes SO(1,1) \otimes Z_2$ → **work in the de Sitter space**

Exact solution to the RTA Boltzmann equation

We construct a solution which is invariant under the group $SO(3)_q \otimes SO(1,1) \otimes Z_2$ → **work in the de Sitter space**

- **In principle**

$$f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \theta, \phi, \varsigma, \hat{p}_\theta, \hat{p}_\phi, \hat{p}_\varsigma)$$

Exact solution to the RTA Boltzmann equation

We construct a solution which is invariant under the group $SO(3)_q \otimes SO(1,1) \otimes Z_2$ → **work in the de Sitter space**

- **In principle**

$$f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \theta, \phi, \varsigma, \hat{p}_\theta, \hat{p}_\phi, \hat{p}_\varsigma)$$

- **Symmetries imposes the following restrictions on the functional dependence of the distribution function**

$$SO(1,1) \longrightarrow f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \theta, \phi, \cancel{\varsigma}, \hat{p}_\theta, \hat{p}_\phi, \hat{p}_\varsigma)$$

Exact solution to the RTA Boltzmann equation

We construct a solution which is invariant under the group
$SO(3)_q \otimes SO(1,1) \otimes Z_2$ → **work in the de Sitter space**

- **In principle**

$$f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \theta, \phi, \varsigma, \hat{p}_\theta, \hat{p}_\phi, \hat{p}_\varsigma)$$

- **Symmetries imposes the following restrictions on the functional dependence of the distribution function**

$$SO(1,1) \longrightarrow f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \theta, \phi, \cancel{\varsigma}, \hat{p}_\theta, \hat{p}_\phi, \hat{p}_\varsigma)$$

$$Z_2 \longrightarrow \hat{p}_\varsigma \to -\hat{p}_\varsigma$$

Exact solution to the RTA Boltzmann equation

We construct a solution which is invariant under the group
$SO(3)_q \otimes SO(1,1) \otimes Z_2 \longrightarrow$ **work in the de Sitter space**

- **In principle** $$f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \theta, \phi, \varsigma, \hat{p}_\theta, \hat{p}_\phi, \hat{p}_\varsigma)$$
- **Symmetries imposes the following restrictions on the functional dependence of the distribution function**

$$SO(1,1) \longrightarrow f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \theta, \phi, \cancel{\varsigma}, \hat{p}_\theta, \hat{p}_\phi, \hat{p}_\varsigma)$$

$$Z_2 \longrightarrow \hat{p}_\varsigma \to -\hat{p}_\varsigma$$

$$SO(3)_q \longrightarrow f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \cancel{\theta}, \cancel{\phi}, \varsigma, \underbrace{\hat{p}_\theta, \hat{p}_\phi}, \hat{p}_\varsigma)$$

$$\hat{p}_\Omega^2 = \hat{p}_\theta^2 + \frac{\hat{p}_\phi^2}{\sin^2\theta}$$

Exact solution to the RTA Boltzmann equation

Thus the symmetries of the Gubser flow imply

$$f(\hat{x}^\mu, \hat{p}_i) = f(\rho, \theta, \phi, \varsigma, \hat{p}_\theta, \hat{p}_\phi, \hat{p}_\varsigma)$$

$$\Big\downarrow \quad SO(3)_q \otimes SO(1,1) \otimes Z_2$$

$$f(\hat{x}^\mu, \hat{p}_i) = f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\varsigma\right)$$

The RTA Boltzmann equation gets reduced to

$$\frac{\partial}{\partial \rho} f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\varsigma\right) = -\frac{1}{\hat{\tau}_{rel}} \left(f\left(\rho, \hat{p}_\Omega^2, \hat{p}_\varsigma\right) - f_{eq}\left(\hat{p}^\rho / \hat{T}(\rho)\right)\right)$$

Due to Weyl invariance $\hat{\tau}_{rel} = c \,/\, \hat{T}(\rho)$

$$c = 5\frac{\eta}{\mathcal{S}} \Longleftrightarrow \frac{\eta}{\mathcal{S}} = \frac{1}{5}\hat{\tau}_{rel}\hat{T}$$

Denicol et. al, PRL105 (2010) 162501,
Denicol et. al,PRD83 (2011) 074019,
Florkowski et. al, PRC88 (2013) 024903

Exact solution to the RTA Boltzmann equation

The exact solution to the RTA Boltzmann equation is

$$f(\rho, \hat{p}_\Omega^2, \hat{p}_\varsigma) = D(\rho, \rho_0) f_0(\rho_0, \hat{p}_\Omega^2, \hat{p}_\varsigma) + \frac{1}{c} \int_{\rho_0}^{\rho} d\rho' \, D(\rho, \rho') \, \hat{T}(\rho') \, f_{eq}(\rho', \hat{p}_\Omega^2, \hat{p}_\varsigma)$$

Damping function:

$$D(\rho, \rho_0) = \exp\left\{ - \int_{\rho_0}^{\rho} d\rho' \, \frac{\hat{T}(\rho')}{c} \right\}$$

Equilibrium distribution function

$$f_0 = f_{eq} = e^{\hat{u} \cdot \hat{p} / \hat{T}}$$

Exact solution to the RTA Boltzmann equation

The exact solution to the RTA Boltzmann equation is

$$f(\rho, \hat{p}_\Omega^2, \hat{p}_\varsigma) = D(\rho, \rho_0) f_0(\rho_0, \hat{p}_\Omega^2, \hat{p}_\varsigma) + \frac{1}{c}\int_{\rho_0}^{\rho} d\rho' \, D(\rho, \rho') \, \hat{T}(\rho') \, f_{eq}(\rho', \hat{p}_\Omega^2, \hat{p}_\varsigma)$$

Damping function:

$$D(\rho, \rho_0) = \exp\left\{ - \int_{\rho_0}^{\rho} d\rho' \, \frac{\hat{T}(\rho')}{c} \right\}$$

Equilibrium distribution function

$$f_0 = f_{eq} = e^{\hat{u}\cdot\hat{p}/\hat{T}}$$

- We can calculate the moments of the distribution function exactly

Exact solution to the RTA Boltzmann equation

The exact solution to the RTA Boltzmann equation is

$$f(\rho,\hat{p}_\Omega^2,\hat{p}_\varsigma) = D(\rho,\rho_0) f_0(\rho_0,\hat{p}_\Omega^2,\hat{p}_\varsigma) + \frac{1}{c}\int_{\rho_0}^{\rho} d\rho'\, D(\rho,\rho')\,\hat{T}(\rho')\, f_{eq}(\rho',\hat{p}_\Omega^2,\hat{p}_\varsigma)$$

Damping function:

$$D(\rho,\rho_0) = \exp\left\{-\int_{\rho_0}^{\rho} d\rho' \frac{\hat{T}(\rho')}{c}\right\}$$

Equilibrium distribution function

$$f_0 = f_{eq} = e^{\hat{u}\cdot\hat{p}/\hat{T}}$$

- We can calculate the moments of the distribution function exactly
- The Landau matching condition $\hat{\varepsilon}_{eq}(\rho) = \hat{\varepsilon}(\rho)$ determines the temperature in f_{eq}

$$\hat{T}^4(\rho) = D(\rho,\rho_0)\mathcal{H}\left(\frac{\cosh\rho_0}{\cosh\rho}\right)\hat{T}^4(\rho_0) + \frac{1}{c}\int_{\rho_0}^{\rho} d\rho'\, D(\rho,\rho')\,\mathcal{H}\left(\frac{\cosh\rho'}{\cosh\rho}\right)\hat{T}^5(\rho')$$

$$\mathcal{H}(x) = \frac{1}{2}\left\{x^2 + x^4\frac{\tanh^{-1}\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}\right\}$$

Testing the validity of different hydrodynamical approximations

Conformal hydrodynamic theories in dS₃ÊR

Energy momentum conservation

$$\hat{\nabla}_\mu \hat{T}^{\mu\nu} = 0 \quad \Rightarrow \quad \frac{1}{\hat{T}}\frac{d\hat{T}}{d\rho} + \frac{2}{3}\tanh\rho = \frac{1}{3}\bar{\pi}^{\varsigma}_{\varsigma}\tanh\rho$$

2nd. Order viscous hydrodynamics

Israel-Stewart (IS) ⇒

$$\partial_\rho \bar{\pi}^{\varsigma}_{\varsigma} + \frac{\bar{\pi}^{\varsigma}_{\varsigma}}{\hat{\tau}_\pi}\tanh\rho + \frac{4}{3}\left(\bar{\pi}^{\varsigma}_{\varsigma}\right)^2 = \frac{4}{15}\tanh\rho$$

Denicol et. al. (DNMR) ⇒

$$\partial_\rho \bar{\pi}^{\varsigma}_{\varsigma} + \frac{\bar{\pi}^{\varsigma}_{\varsigma}}{\hat{\tau}_\pi}\tanh\rho + \frac{4}{3}\left(\bar{\pi}^{\varsigma}_{\varsigma}\right)^2 = \frac{4}{15}\tanh\rho + \frac{10}{7}\bar{\pi}^{\varsigma}_{\varsigma}\tanh\rho$$

$\hat{\tau}_\pi = 5\eta/(\hat{S}\hat{T})$ $\qquad$ $\bar{\pi}^{\varsigma}_{\varsigma} \equiv \pi^{\varsigma}_{\varsigma}/(\hat{T}\hat{\mathcal{S}})$

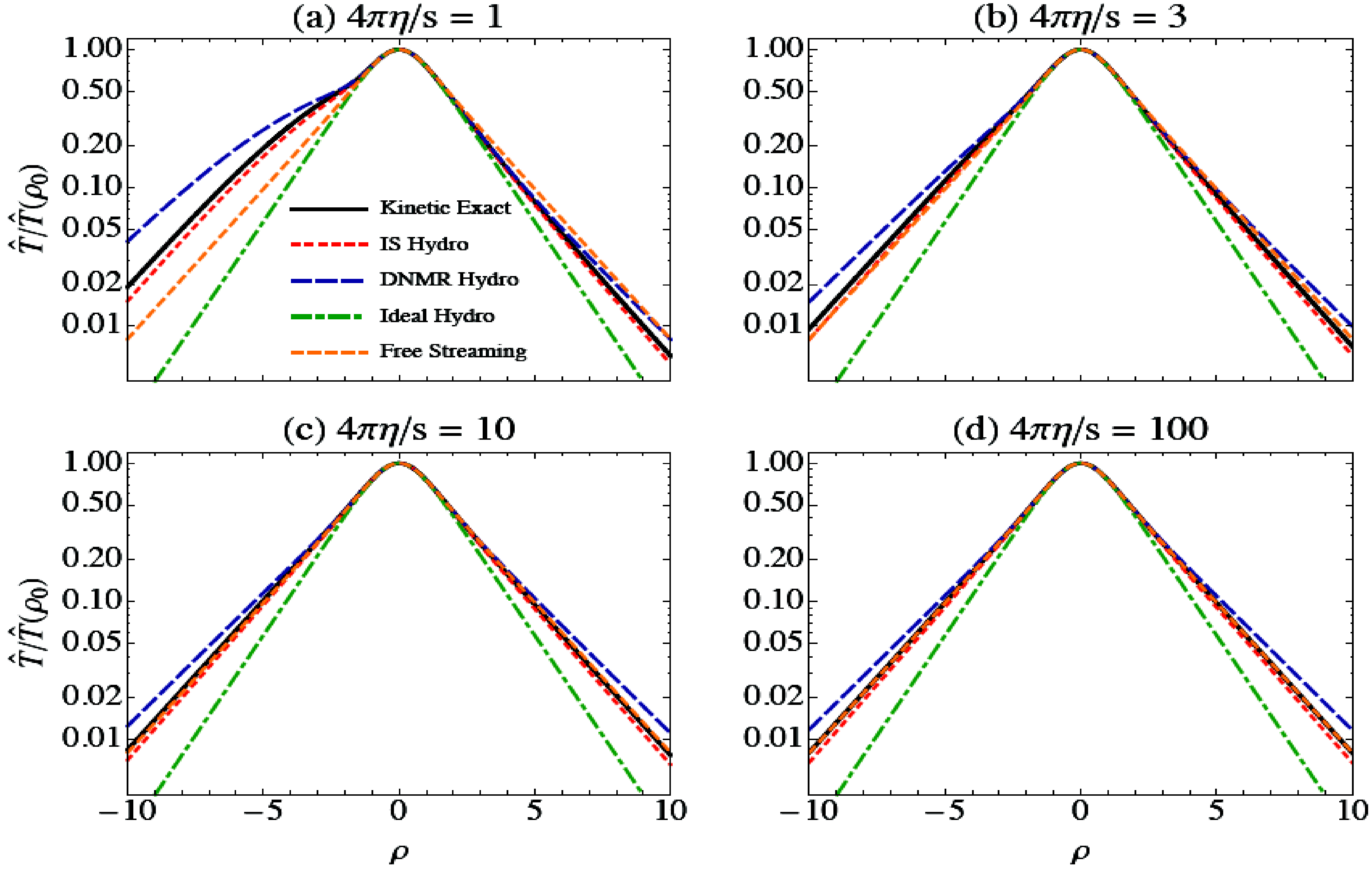
In this work we also consider two interesting limits:

- **Free streaming $\eta/s \to \infty$**

- **Ideal hydrodynamics $\eta/s \to 0$**

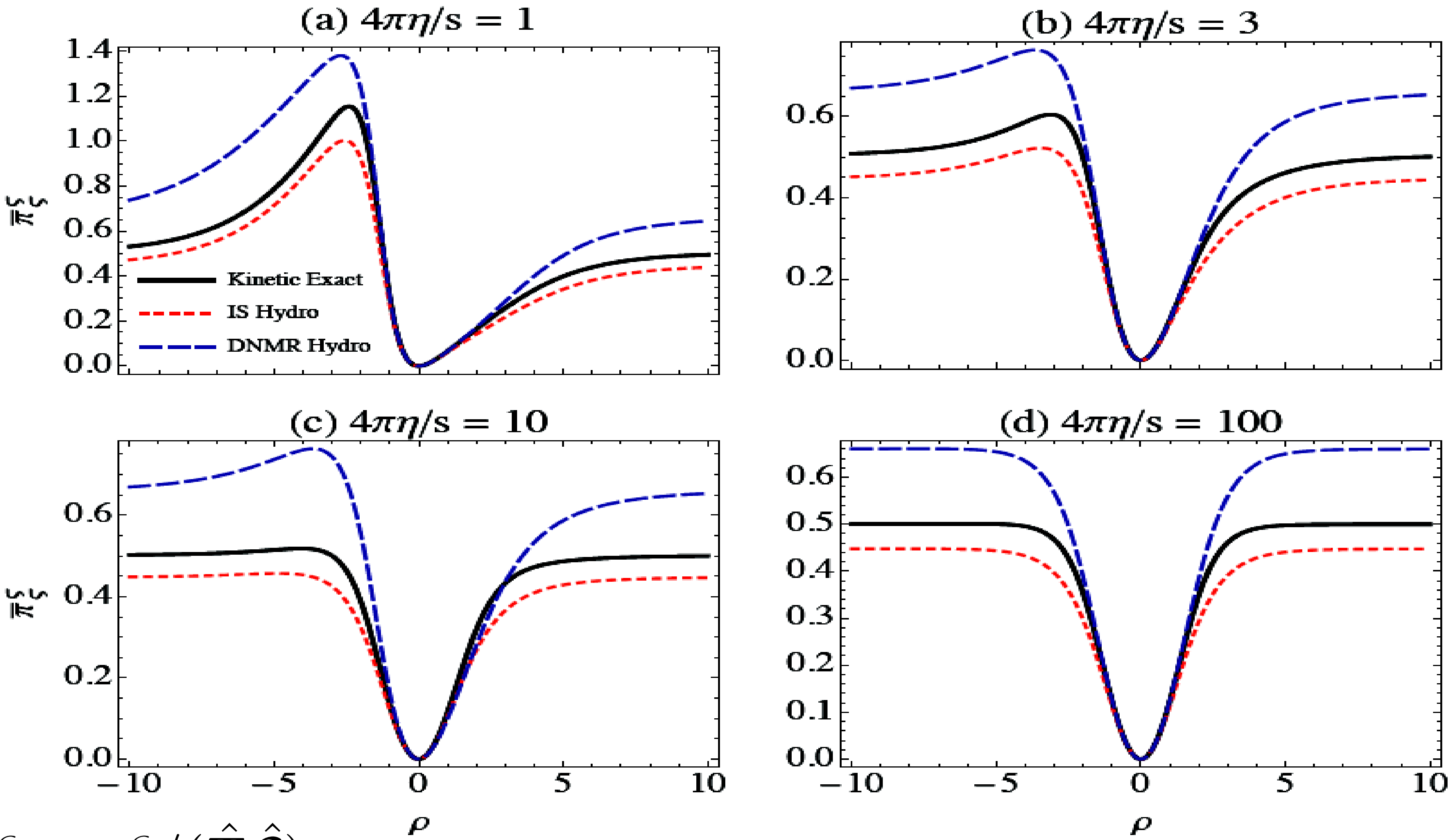
Comparison in de Sitter: Temperature

$\rho_0 = 0 \quad \hat{\mathcal{E}}(\rho_0) = 1$



Denicol et. al. PRL 113 202301 (2014), PRD 90 125026 (2014)

Comparison in de Sitter: Shear viscous

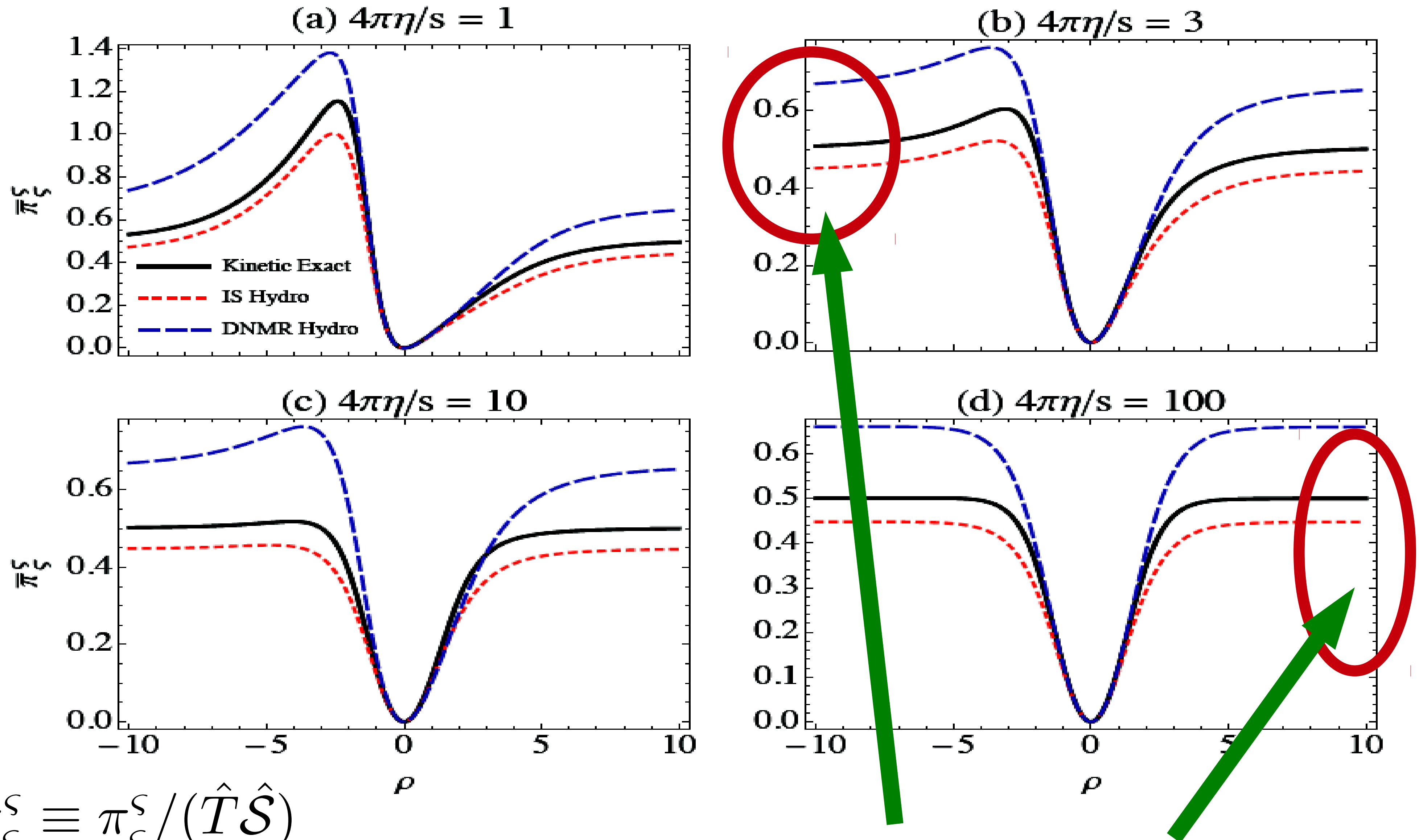


$$\bar{\pi}^{\varsigma}_{\varsigma} \equiv \pi^{\varsigma}_{\varsigma}/(\hat{T}\hat{\mathcal{S}})$$

$$\rho_0 = 0 \quad \hat{\mathcal{E}}(\rho_0) = 1$$

Denicol et. al. PRL 113 202301 (2014), PRD 90 125026 (2014)

Comparison in de Sitter: Shear viscous



$$\bar{\pi}^{\varsigma}_{\varsigma} \equiv \pi^{\varsigma}_{\varsigma}/(\hat{T}\hat{\mathcal{S}})$$

$$\rho_0 = 0 \quad \hat{\mathcal{E}}(\rho_0) = 1$$

Large anisotropies

Denicol et. al. PRL 113 202301 (2014), PRD 90 125026 (2014)

Knudsen number in de Sitter

Deviations between 2nd. Order viscous hydro and the exact solution are ~ 30 %. Why?

$$\mathrm{Kn} = \hat{\tau}_{rel}|\hat{\nabla}\cdot\hat{u}| = 2c\frac{\tanh\rho}{\hat{T}(\rho)}$$

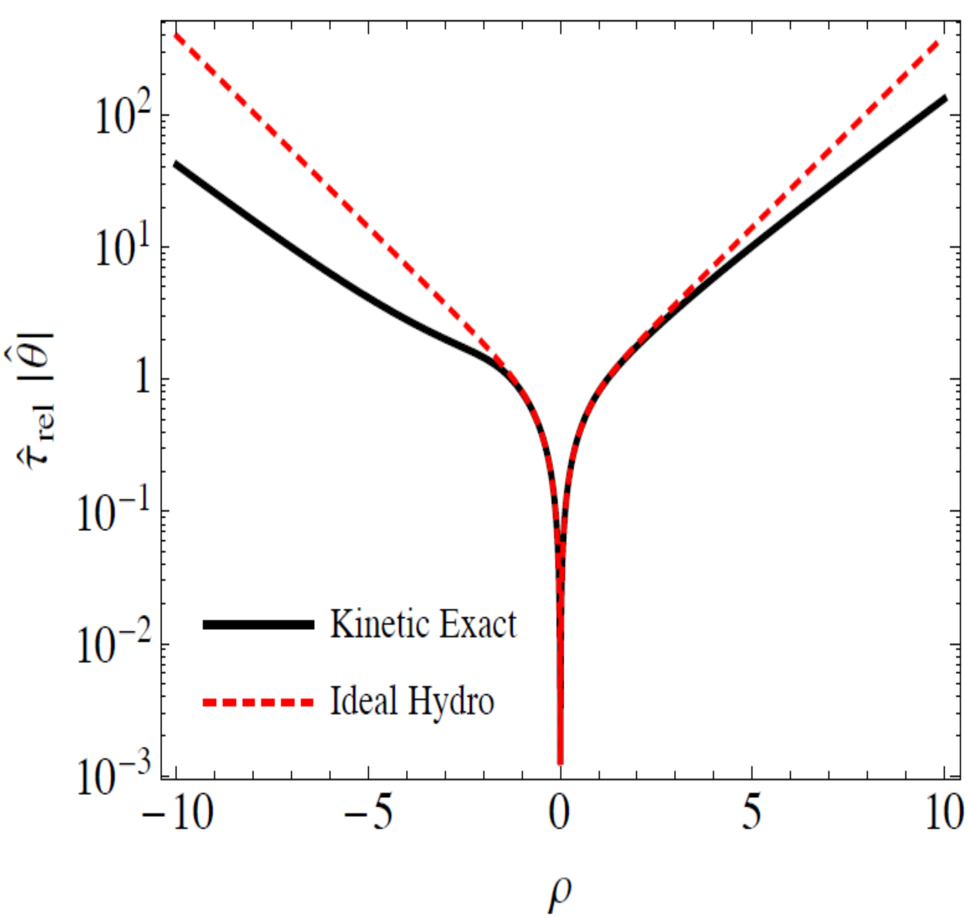
Ideal Hydro:

$$\mathrm{Kn}_{ideal} = 2\frac{c}{\hat{T}_0}|\tanh^{1/3}(\rho)\,\sinh^{2/3}(\rho)|$$

$$\lim_{\rho\pm\infty}\mathrm{Kn}_{ideal} \sim e^{\rho}$$

Do we really need an isotropic state when we have hydrodynamical behavior?

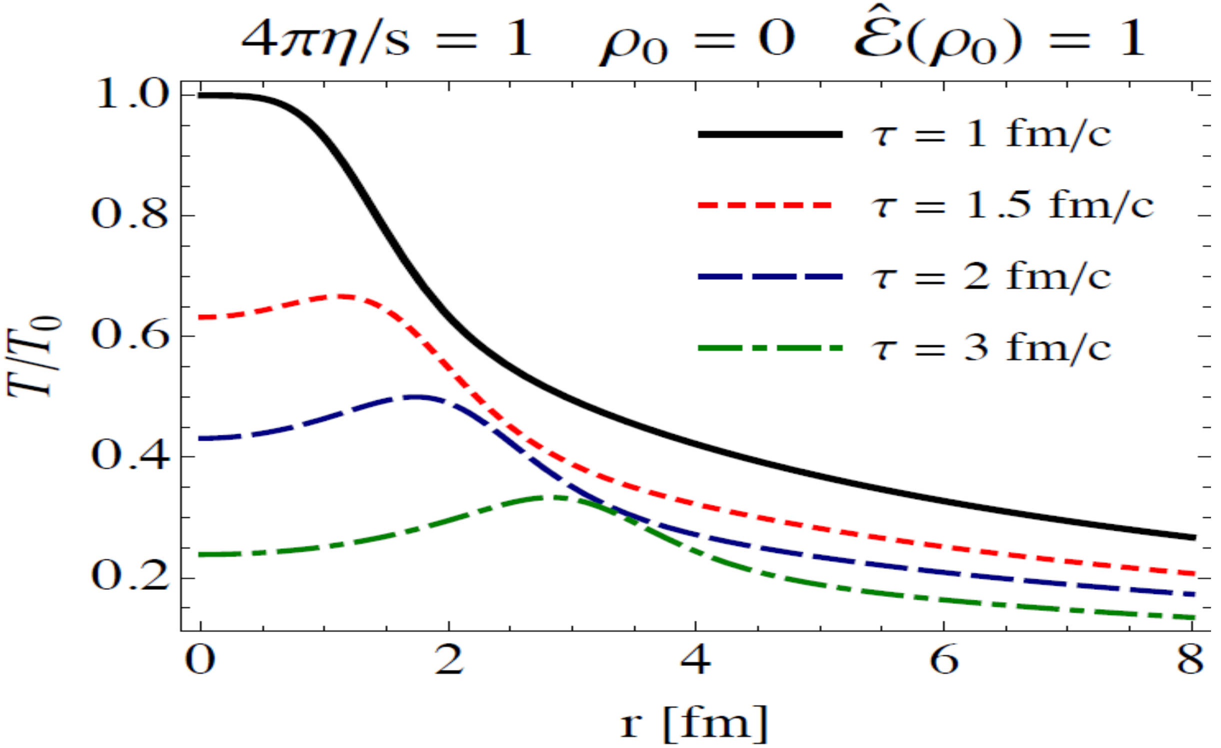
$4\pi\eta/s = 1 \quad \rho_0 = 0 \quad \hat{\mathcal{E}}(\rho_0) = 1$

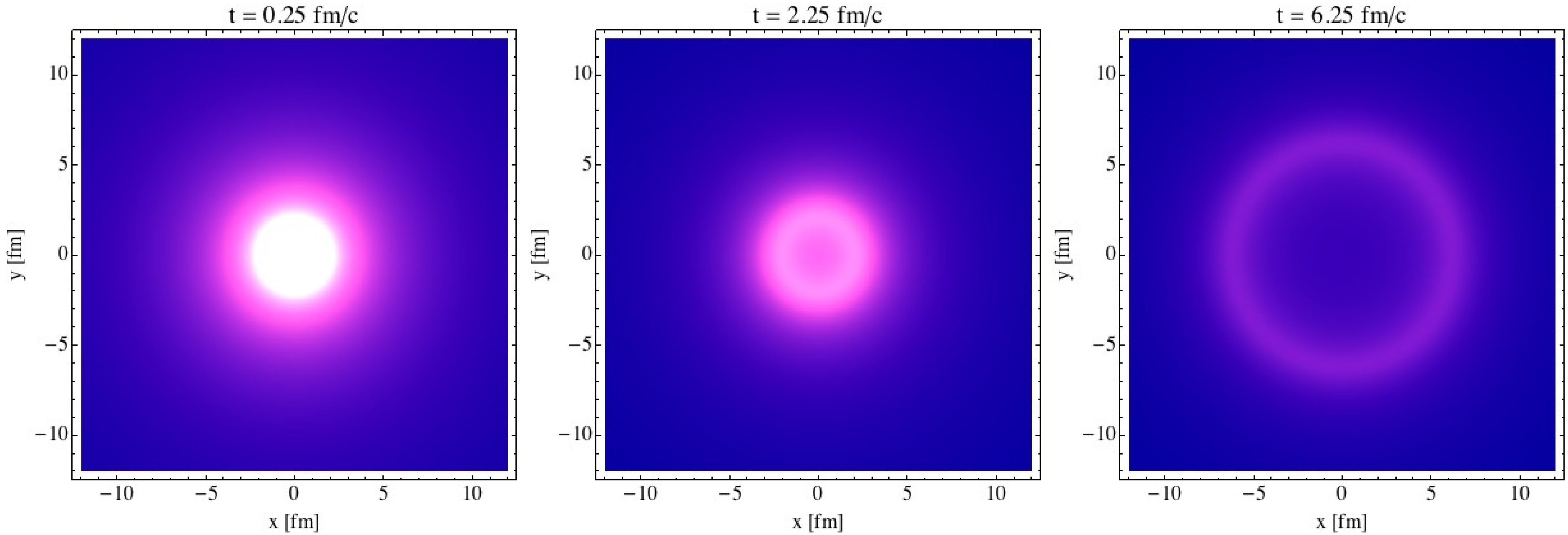


Denicol et. al. PRL 113 202301 (2014), PRD 90 125026 (2014)

Temperature in Minkowski space

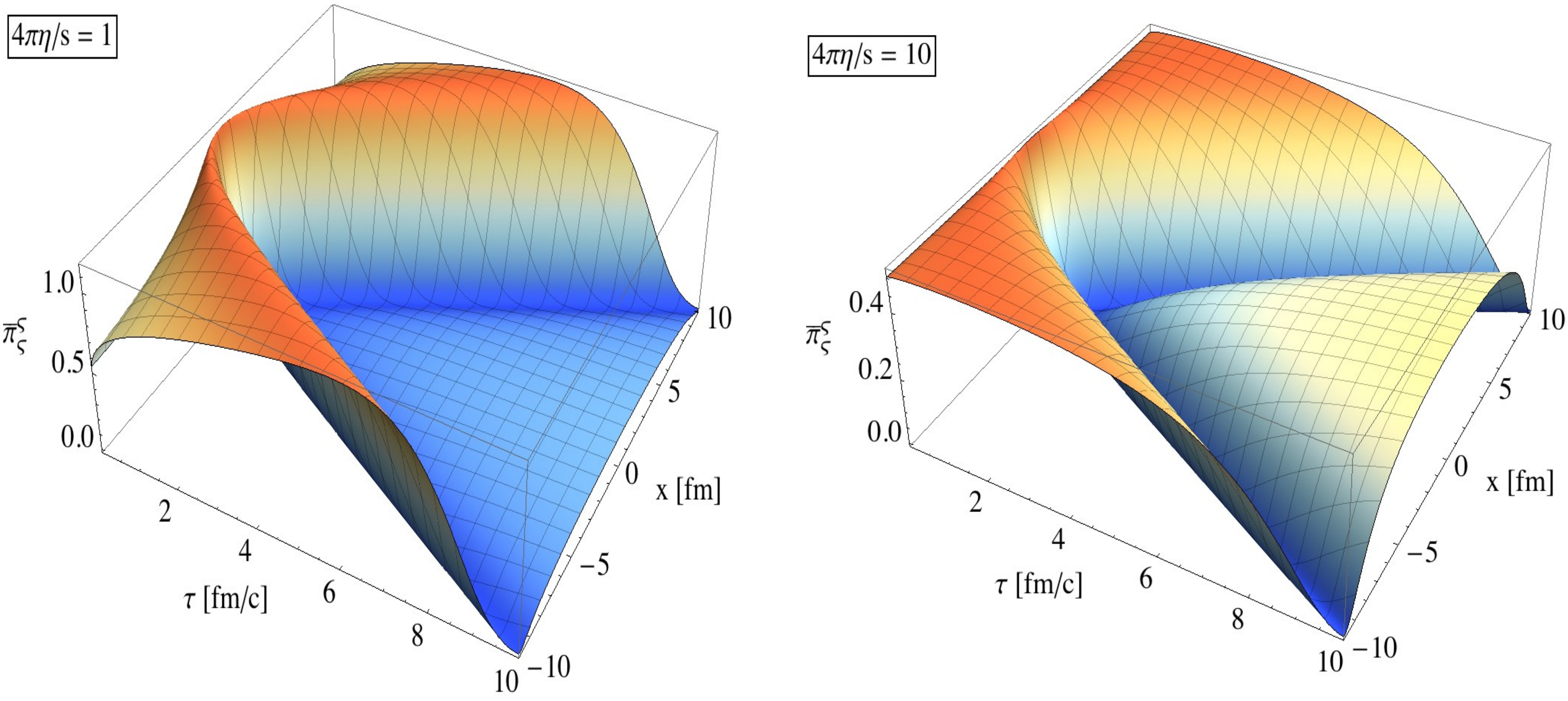
$$T(\tau, r) = \frac{\hat{T}(\rho(\tau, r))}{\tau}$$





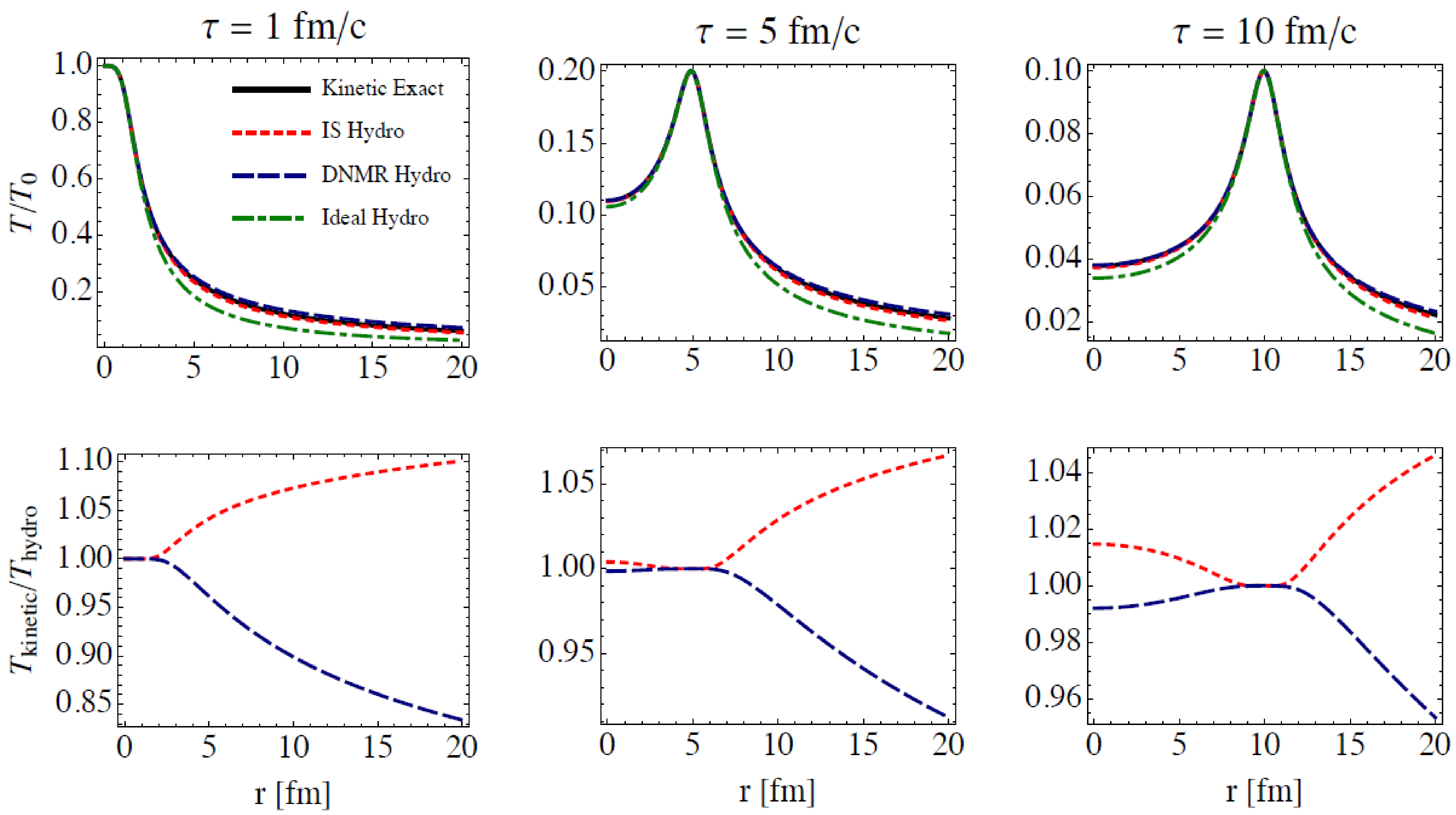
Shear viscous tensor in Minkowski space

$$\bar{\pi}^{\varsigma}_{\varsigma} \equiv \pi^{\varsigma}_{\varsigma}/(\hat{T}\hat{\mathcal{S}})$$



Denicol et. al. PRL 113 202301 (2014), PRD 90 125026 (2014)

Comparisons in Minkowski space: Temperature

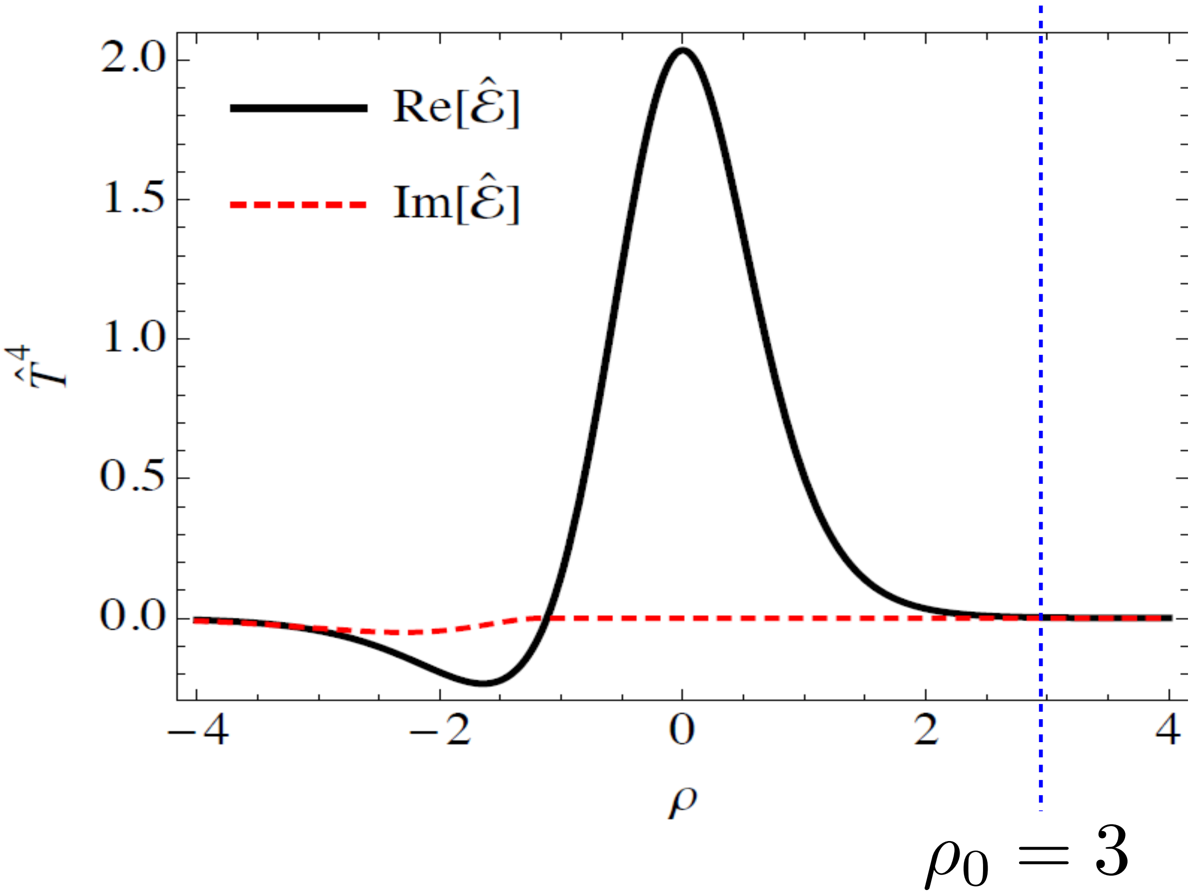


$$\frac{\eta}{S} = \frac{1}{4\pi}$$

Denicol et. al. PRL 113 202301 (2014), PRD 90 125026 (2014)

Restrictions of the Gubser solution to the Boltzmann equation

Unphysical results for moments of f(x,p)



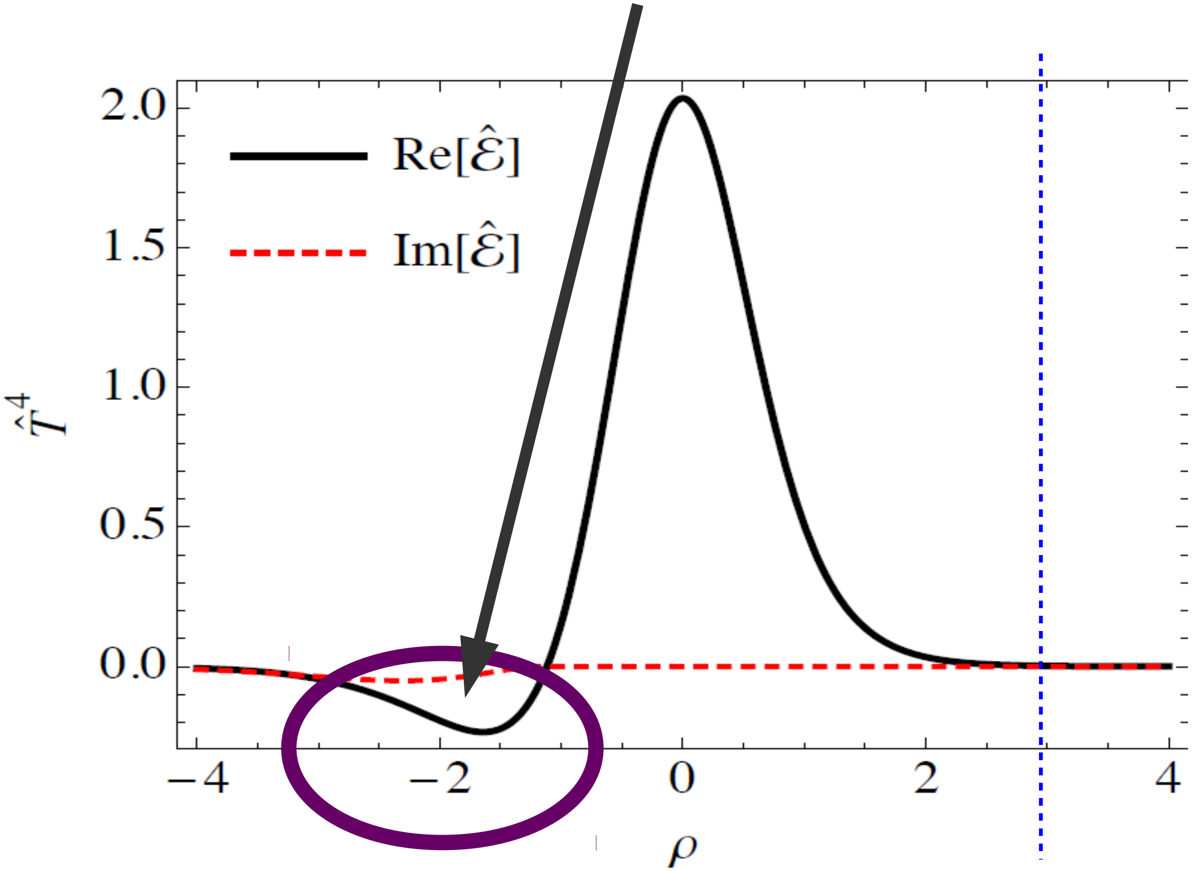
$(4\pi)\eta/S = 3$

$\hat{T}(\rho_0) = 0.21$

Denicol et. al. PRL 113 202301 (2014), PRD 90 125026 (2014)

Unphysical results for moments of f(x,p)

Some initial conditions in de Sitter space lead to unphysical behaviour of the temperature/energy density



$(4\pi)\eta/S = 3$

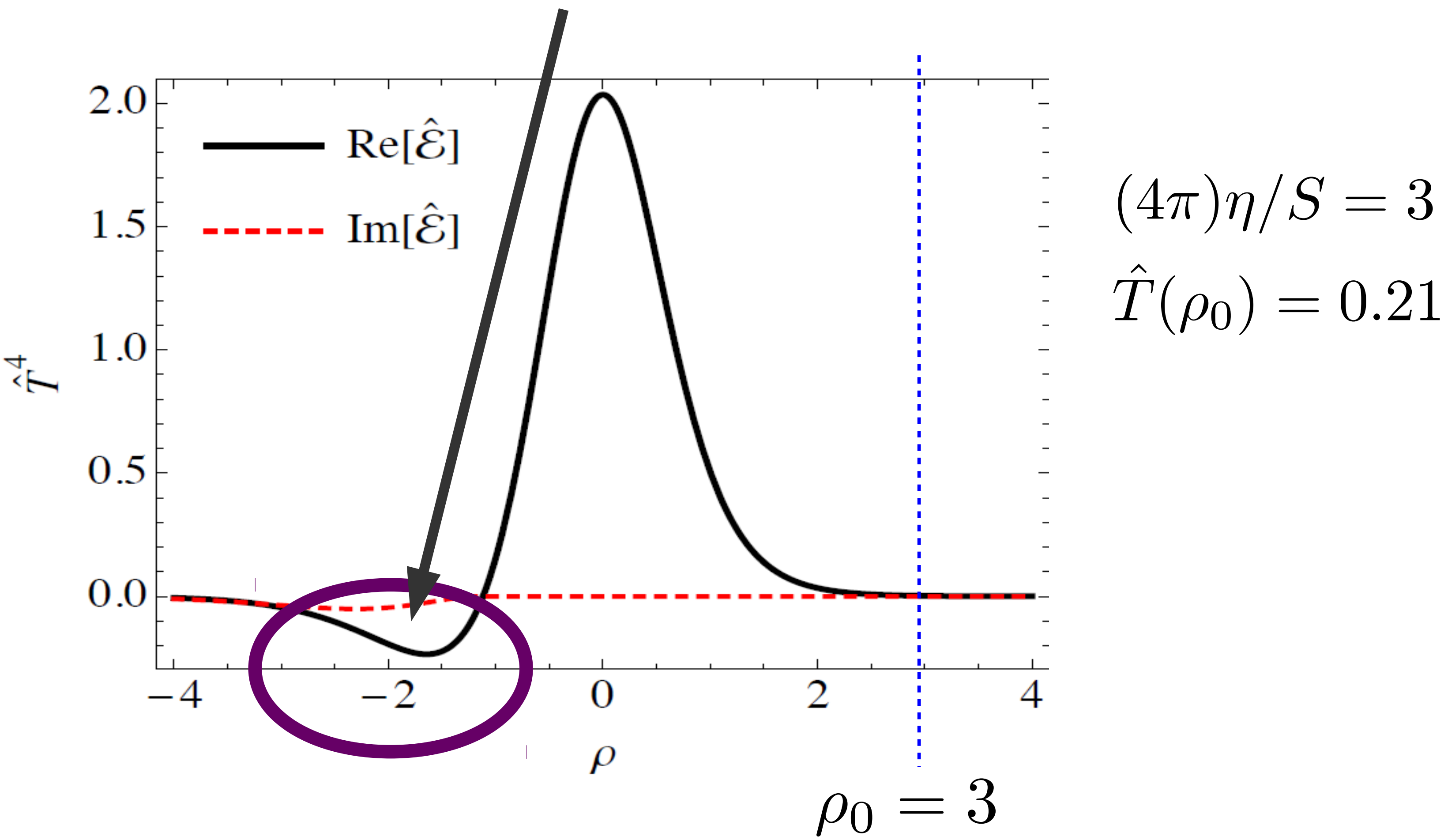
$\hat{T}(\rho_0) = 0.21$

$\rho_0 = 3$

Denicol et. al. PRL 113 202301 (2014), PRD 90 125026 (2014)

Unphysical results for moments of f(x,p)

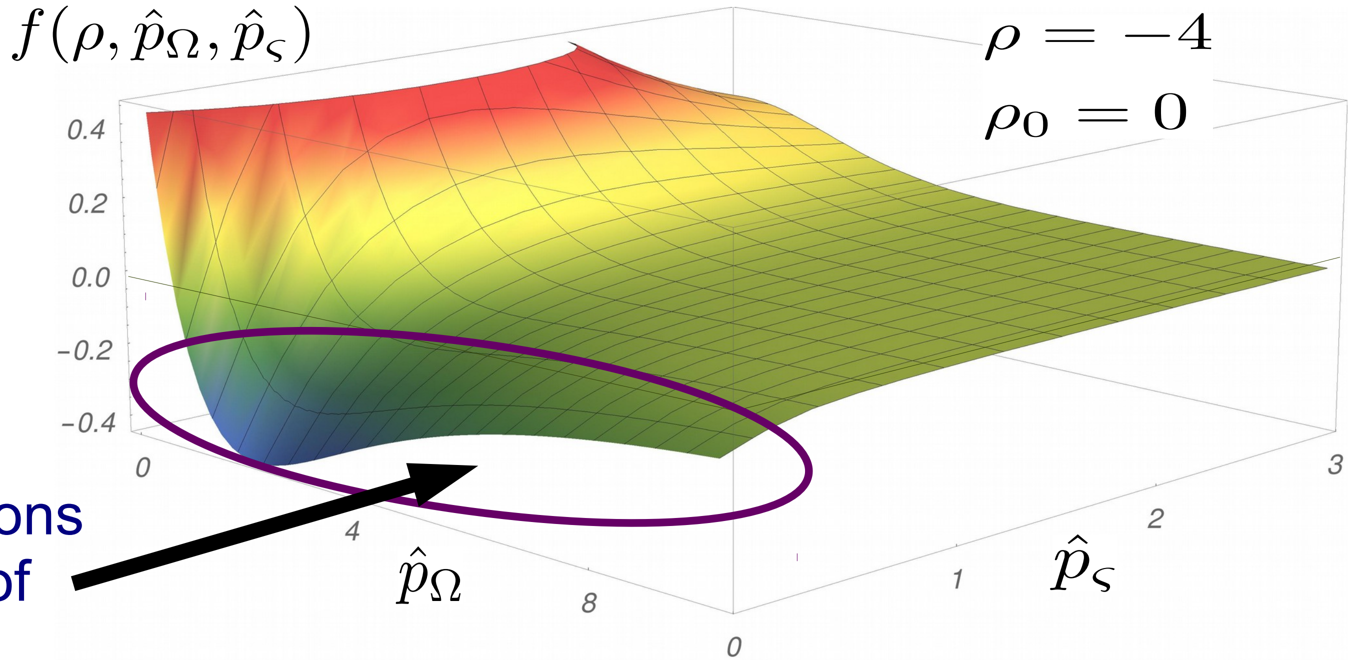
Some initial conditions in de Sitter space lead to unphysical behaviour of the temperature/energy density



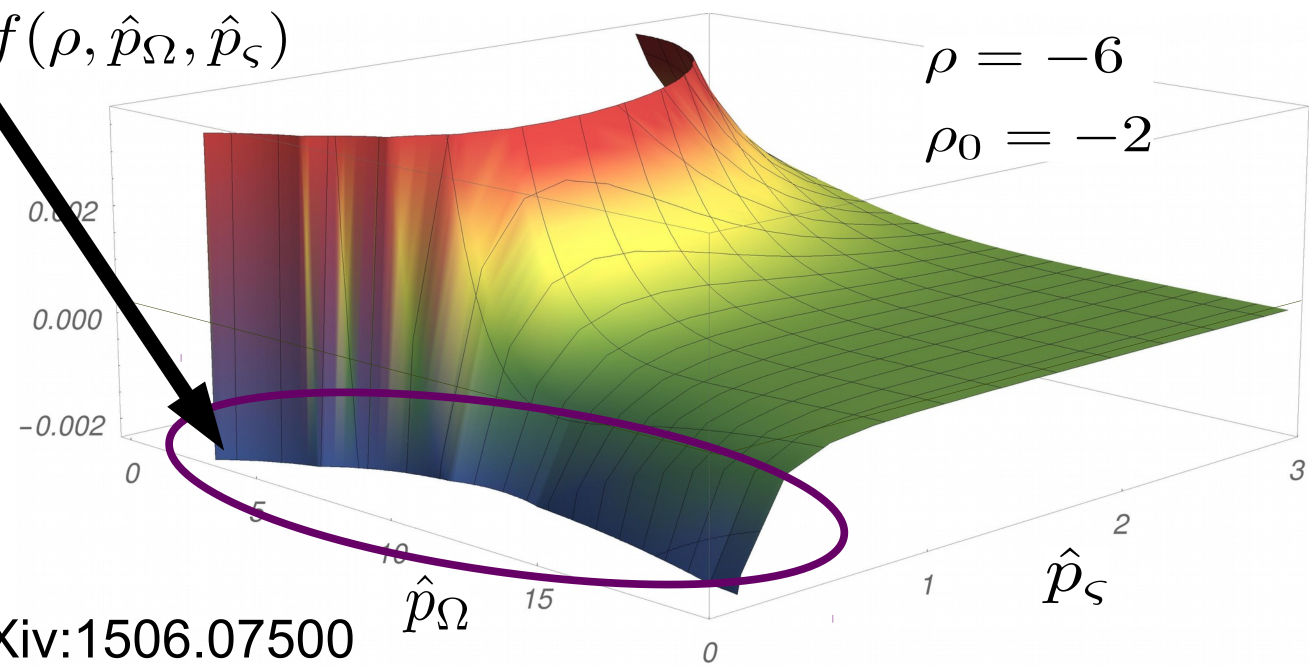
Instead of analyzing moments of the distribution function we study its evolution in the phase space

Denicol et. al. PRL 113 202301 (2014), PRD 90 125026 (2014)

Negative contributions to the distribution function



- For certain initial conditions f Ô 0 in certain regions of momentum space
- The system is not translationally invariant

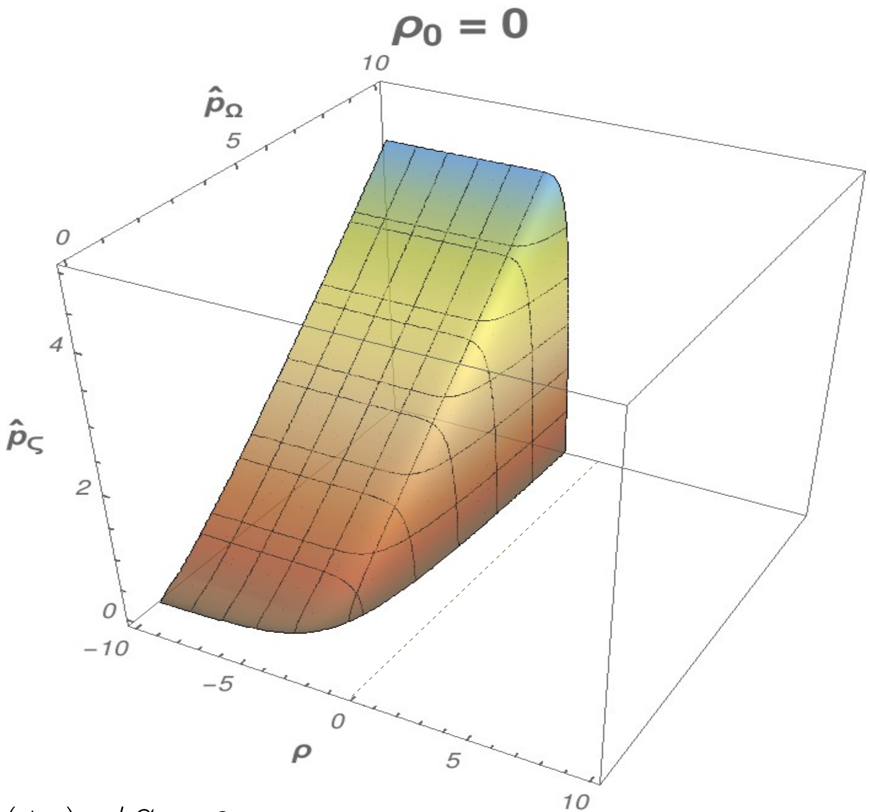


$$(4\pi)\eta/S = 3$$

U. Heinz and M. Martinez, arXiv:1506.07500

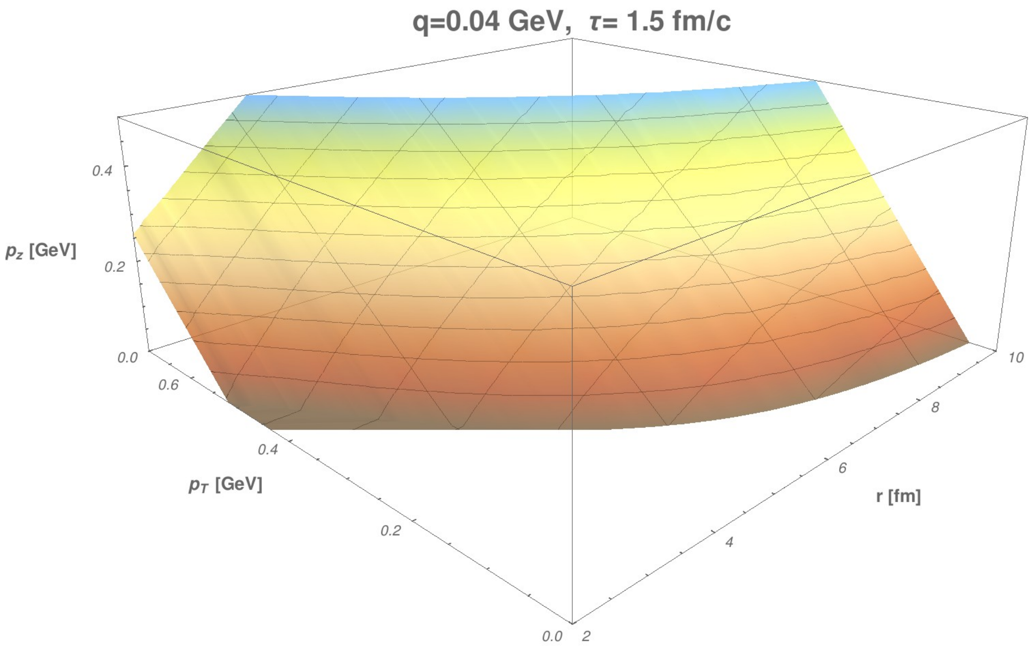
Determining the physical boundary

In de Sitter



$(4\pi)\eta/S = 3$

In Minkowski



The surface where f = 0 determines the boundary that separates the “ill” from the physically valid phase space regions

U. Heinz and M. Martinez, arXiv:1506.07500

Interpretation of the results

We have some important issues

- The expansion rate of the Gubser flow grows exponentially at infinity

$$\lim_{\rho\to\pm\infty} \hat{D}_\mu \hat{u} = \pm e^{\rho}$$

Any initial configuration **never** reaches thermal equilibrium

- The distribution function becomes negative in certain regions **only** when $\rho - \rho_0 \leq 0$

Interpretation of the results

- If the initial condition f_0 is fixed at $\rho_0 = -\infty$ the system always evolves without a problem in the forward ρ region

  $\Rightarrow$ f increases everywhere in momentum space and the distribution function does not have negative values.

- If the initial condition f_0 is fixed at finite ρ_0 the system evolves in both forward and backward ρ regions

  $\Rightarrow$ f increases when ρ increases but f decreases when ρ decreases.

Conclusions and outlook

Conclusions

- We find a new solution to the RTA Boltzmann equation undergoing simultaneously longitudinal and transverse expansion.
- We use this kinetic solution to test the validity and accuracy of different viscous hydrodynamical approaches.
- 2^{nd} order viscous hydro provides a reasonable description when compared with the exact solution.
- This solution opens novel ways to test the accuracy of different hydro approaches

Conclusions

- The observed sick behavior of the moments of the exact solution is related with unphysical behavior of the distribution function in certain regions of the phase space.
- For equilibrium initial conditions, the distribution function can become negative in certain regions of the available phase space when $\rho - \rho_0 \leq 0$
- The non-physical behavior is qualitatively independent of the value for η / s.
- We have fully determined the boundary in phase space where the distribution function is always positive.

Closely related works

- More solutions to the Boltzmann equation (perfect fluid with dissipation and non-hydro modes, unorthodox Bjorken flow, etc)

  3 dim Expanding plasma In Minkowski space ⟷ 1 dim Hydrostatic fluid in a curved space

  Hatta, Martinez and Xiao, PRD 91 (2015) 8, 085024.

  Noronha and Denicol, arXiv:1502.05892

- Gubser exact solution for highly anisotropic systems (see Mike's talk)

  Nopoush, Ryblewski, Strickland, PRD91 (2015) 4, 045007

- Exact analytical solution to the full non-linear Boltzmann equation for a rapidly expanding system

  Bazow, Denicol, Heinz, Martinez and Noronha, arXiv:1507.07834

Outlook

We can learn and get physical insights about isotropization/thermalization problem by using symmetries...

Backup slides

Emergent conformal symmetry of the Boltzmann Eqn.

A tensor (m,n) of canonical dimension Á transforms under a conformal transformation as

$$Q^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n}(x) \rightarrow e^{(\Delta + m - n)\Omega(x)} Q^{\mu_1 \dots \mu_m}_{\nu_1 \dots \nu_n}(x)$$

Ê is an arbitrary function.

The Boltzmann equation for massless particles is invariant under a conformal transformation (Baier et. al. JHEP 0804 (2008) 100)

$$p^\mu \partial_\mu f + \Gamma^\lambda_{\mu i} p_\lambda p^\mu \frac{\partial f}{\partial p_i} - \mathcal{C}[f] = 0$$

$$e^{2\Omega} \left(p^\mu \partial_\mu f + \Gamma^\lambda_{\mu i} p_\lambda p^\mu \frac{\partial f}{\partial p_i} - \mathcal{C}[f] \right) = 0$$

Symmetries of the Bjorken flow

$$ISO(2) \otimes SO(1,1) \otimes Z_2$$

Z_2 → **Reflections along the beam line** $z \to -z$

$SO(1,1)$ → **Longitudinal Boost invariance** $\xi_1 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}$

$ISO(2)$ → **Translations in the transverse plane + rotation along the longitudinal z direction**

$$\xi_2 = \frac{\partial}{\partial x} \quad , \xi_3 = \frac{\partial}{\partial y}$$

$$\xi_4 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

Symmetries of the Gubser flow

$$SO(3)_q \otimes SO(1,1) \otimes Z_2$$

Reflections along the beam line →

$$Z_2$$
$$z \to -z$$

Boost invariance →

$$SO(1,1)$$
$$\xi_1 = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}$$

Special Conformal transformations + rotation along the beam line →

$$SO(3)_q$$
$$\xi_i = \frac{\partial}{\partial x^i} + q^2\left(2\,x^i\,x^\mu\frac{\partial}{\partial x^\mu} - x^\mu x_\mu\frac{\partial}{\partial x^i}\right) \quad i = 2,3$$
$$\xi_4 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

Weyl rescaling + Coordinate transformation

$$\rho = -\sinh^{-1}\left(\frac{1 - q^2\tau^2 + q^2r^2}{2q\tau}\right)$$

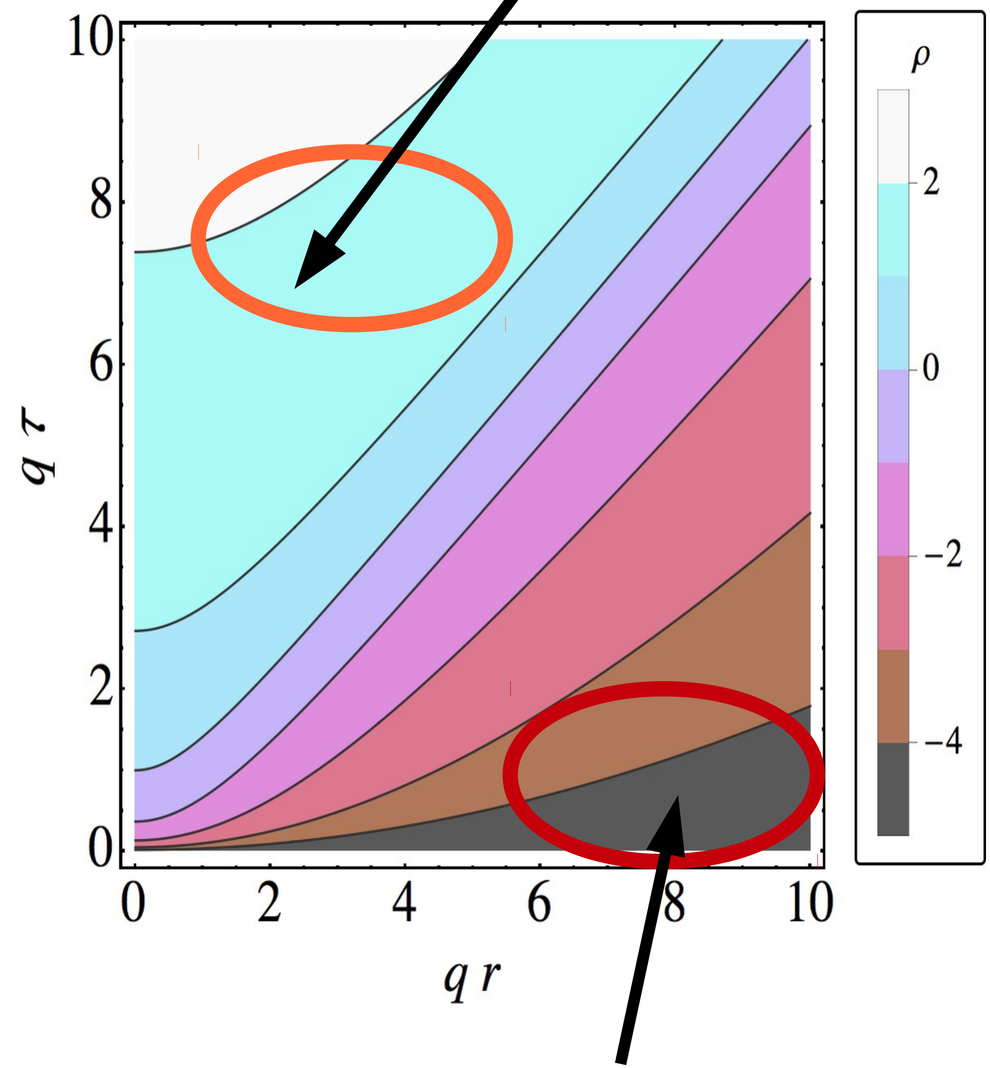
$$\theta = \tanh^{-1}\left(\frac{2qr}{1 + q^2\tau^2 - q^2r^2}\right)$$

$$\rho \in (-\infty, \infty)$$

$$0 < \theta < 2\pi$$

ρ is the affine parameter (e.g.“time”)

$\rho > 0 \Rightarrow \tau \gg r$



$\rho < 0 \Rightarrow r \gg \tau$

Transforming the momentum coordinates

When going from de Sitter to Minkowski

$$p^\tau = \frac{\gamma}{\tau}\left(\hat{p}^\rho + v(\tau,r)\,\cosh\rho\,\hat{p}^\theta\right)$$

$$p^r = \frac{\gamma}{\tau}\left(\cosh\rho\,\hat{p}^\theta + v(\tau,r)\,\hat{p}^\rho\right)$$

$$p_\phi = r^2\,p^\phi = \frac{r^2}{\tau^2}\,\hat{p}^\phi\,,$$

$$p_\varsigma = \hat{p}^\varsigma\,,$$

For $z=0$ and $\hat{p}_\phi = 0$ we can write the $SO(3)_q$ invariant $\hat{p}^2_\Omega$ as

$$\begin{aligned}\hat{p}^2_\Omega &= \hat{p}^2_\theta + \frac{\hat{p}^2_\phi}{\sin^2\theta}\,,\\ &= \left(\hat{p}^\theta\cosh^2\rho(\tau,r)\right)^2\\ &= \cosh^2\rho(\tau,r)\,\tau^2\,\left[\gamma(\,p_T - v(\tau,r)\,p^\tau\,)\right]^2\end{aligned}$$

U. Heinz and M. Martinez, arXiv:1506.07500

Gubser solution's for conformal hydrodynamics

The energy-momentum tensor of a conformal fluid

$$T^{\mu\nu} = u^\mu u^\nu (\varepsilon + \mathcal{P}) + g^{\mu\nu}\mathcal{P} + \pi^{\mu\nu}$$

From the energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$

$$\frac{d\hat{\varepsilon}(\rho)}{d\rho} + \frac{8}{3}\hat{\varepsilon}\tanh\rho - \hat{\pi}^{\eta\eta}\tanh\rho = 0$$

In IS theory the equation of motion of the shear viscous tensor $\pi^{\mu\nu}$

$$\tau_{rel}\partial_\rho \hat{\pi}_{\langle\mu\nu\rangle} + \hat{\pi}_{\mu\nu} = -2\eta\sigma_{\mu\nu} - \frac{4}{3}\hat{\pi}_{\mu\nu}\theta$$

$$\theta = \partial_\mu u^\mu \qquad \hat{\sigma}^{\mu\nu} = \hat{\Delta}^{\mu\nu}_{\alpha\beta}\partial^\alpha u^\beta$$

Ideal and NS solution (2010): Gubser, PRD82 (2010)085027, NPB846 (2011)469 Conformal IS theory (2013): Denicol et. al. arXiv:1308.0785

Gubser solution for ideal hydrodynamics

From the E-M conservation law + ideal EOS + no viscous terms

$$\nabla_\mu T^{\mu\nu} = 0 \qquad\qquad p = \frac{\epsilon}{3} \quad \eta = \zeta = 0$$

It follows this equation in the $(\rho, \theta, \phi, \eta)$ coordinates

$$\frac{d}{d\rho}\left(\hat{\epsilon}^{3/4}\cosh^2\rho\right) = 0$$

The solution is easy to find

$$\hat{\epsilon} = \hat{\epsilon}_0(\cosh\rho)^{-8/3}$$

To go back to Minkowski space

$$\epsilon = \frac{\hat{\epsilon}}{\tau^4} = \frac{\hat{\epsilon}_0}{\tau^{4/3}} \frac{(2q)^{8/3}}{\left[1 + 2q^2(\tau^2 + r^2) + q^4(\tau^2 - r^2)^2\right]^{4/3}}$$

S. Gubser, PRD 82 (2010),085027
S. Gubser, A. Yarom, NPB 846 (2011), 469

Free streaming limit of the Gubser solution to the Boltzmann equation

In the limit when η/s $\longrightarrow\infty$ one can obtain the free streaming limit of the exact solution of the Boltzmann equation for the Gubser flow

$$\hat{T}_{\text{free streaming}}(\rho) = \mathcal{H}^{1/4}\left(\frac{\cosh\rho_0}{\cosh\rho}\right)\hat{T}_0(\rho_0)$$

$$\hat{\pi}^{\varsigma\varsigma}_{\text{free streaming}}(\rho) = \mathcal{A}\left(\frac{\cosh\rho}{\cosh\rho_0}\right)\frac{\hat{T}_0^4}{\pi^2}$$

where

$$\mathcal{H}(x) = \frac{1}{2}\left(x^2 + x^4\frac{\tanh^{-1}\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}\right)$$

$$\mathcal{A}(x) = \frac{x\sqrt{x^2-1}(1+2x^2) + (1-4x^2)\coth^{-1}\left(x/\sqrt{x^2-1}\right)}{2x^3(x^2-1)^{3/2}}$$

Gubser solution for the Navier-Stokes equations

Let´s preserve the conformal invariance of the theory

$$p = \frac{\epsilon}{3} \qquad \eta = \mathrm{H}_0 \epsilon^{3/4} \qquad \zeta = 0$$

The temperature and the energy are related by

$$\hat{\epsilon} = \hat{T}^4$$

So from the EM conservation one obtains a solution for the temperature

$$\hat{T}(\rho) = \frac{\hat{T}_0}{(\cosh\rho)^{2/3}} \left[1 + \frac{\mathrm{H}_0}{9\hat{T}_0} \sinh^3\rho \; {}_2F_1\left(\frac{3}{2}, \frac{7}{6}, \frac{5}{2}, -\sinh^2\rho\right)\right]$$

These solutions predict NEGATIVE temperatures

S. Gubser, PRD 82 (2010),085027
S. Gubser, A. Yarom, NPB 846 (2011), 469

Conformal IS solution

In the de Sitter space the equations of motion are

$$\frac{1}{\hat{T}}\frac{d\hat{T}}{d\rho} + \frac{2}{3}\tanh\rho = \frac{1}{3}\bar{\pi}^{\xi}_{\xi}(\rho)\ \tanh\rho\,,$$

$$\frac{c}{\hat{T}}\frac{\eta}{s}\left[\frac{d\bar{\pi}^{\xi}_{\xi}}{d\rho} + \frac{4}{3}\left(\bar{\pi}^{\xi}_{\xi}\right)^2\tanh\rho\right] + \bar{\pi}^{\xi}_{\xi} = \frac{4}{3}\frac{\eta}{s\hat{T}}\tanh\rho\,,$$

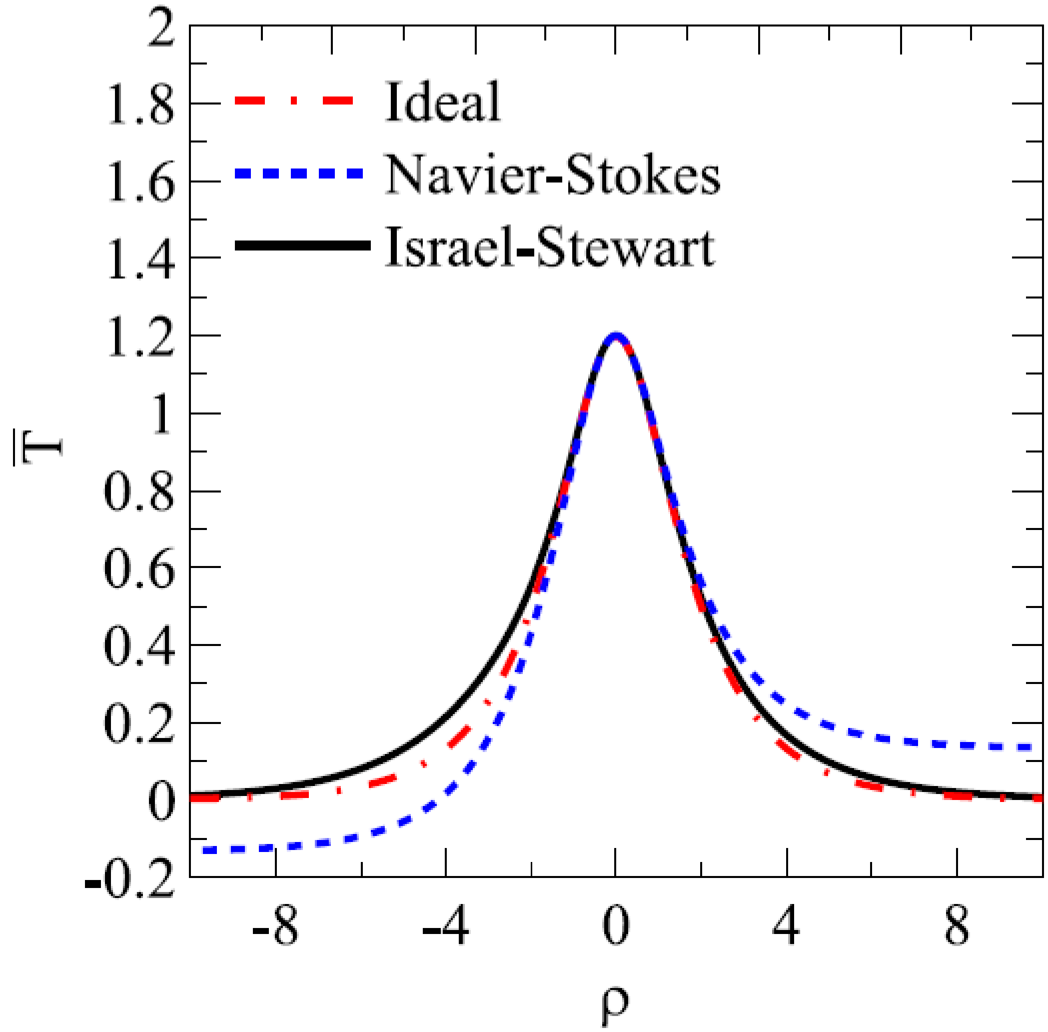
where in order to have conformal symmetry one assumes

$$p = \frac{\epsilon}{3} \qquad s \sim T^3 \qquad \zeta = 0$$

$$\eta \sim s \qquad \tau_R = c\,\eta/(Ts)$$

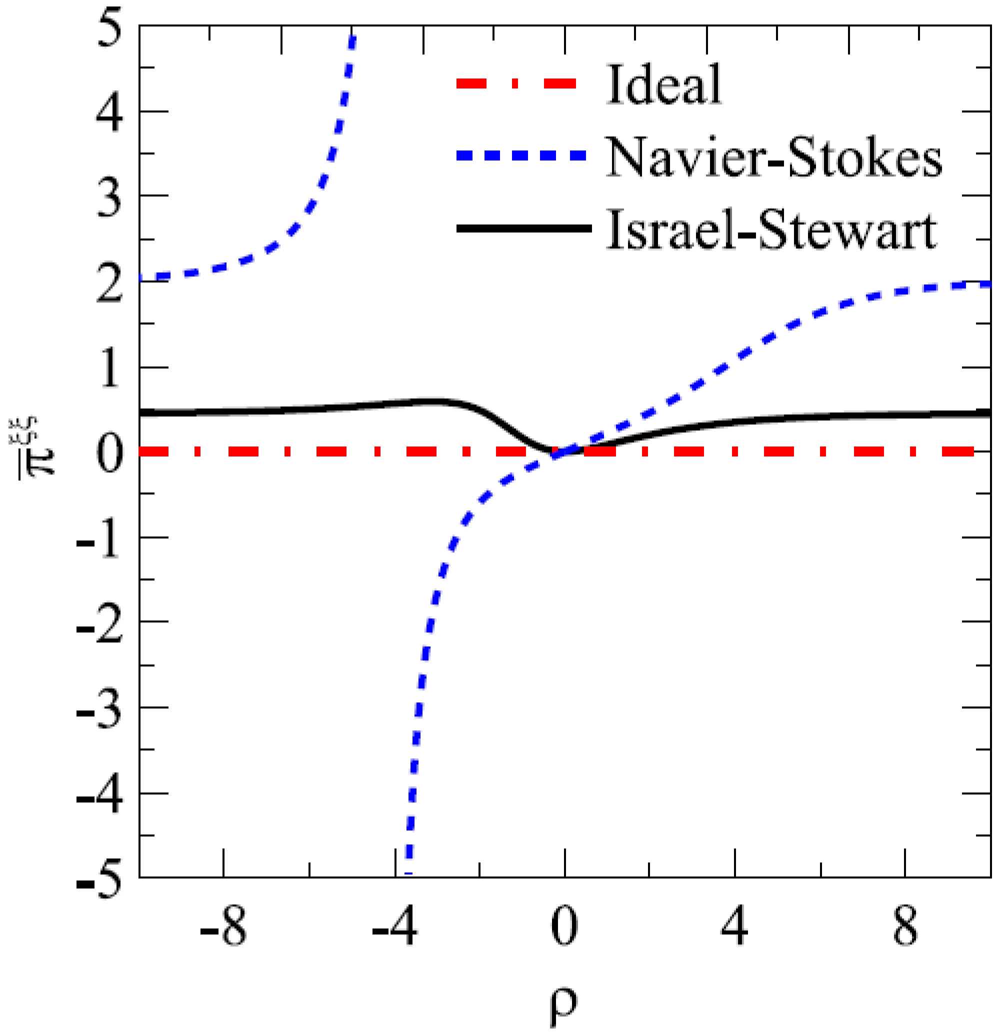
Marrochio, Noronha, Denicol, Luzum, Jeon, Gale, arXiv:1307.6130

Comparing Conformal Solutions



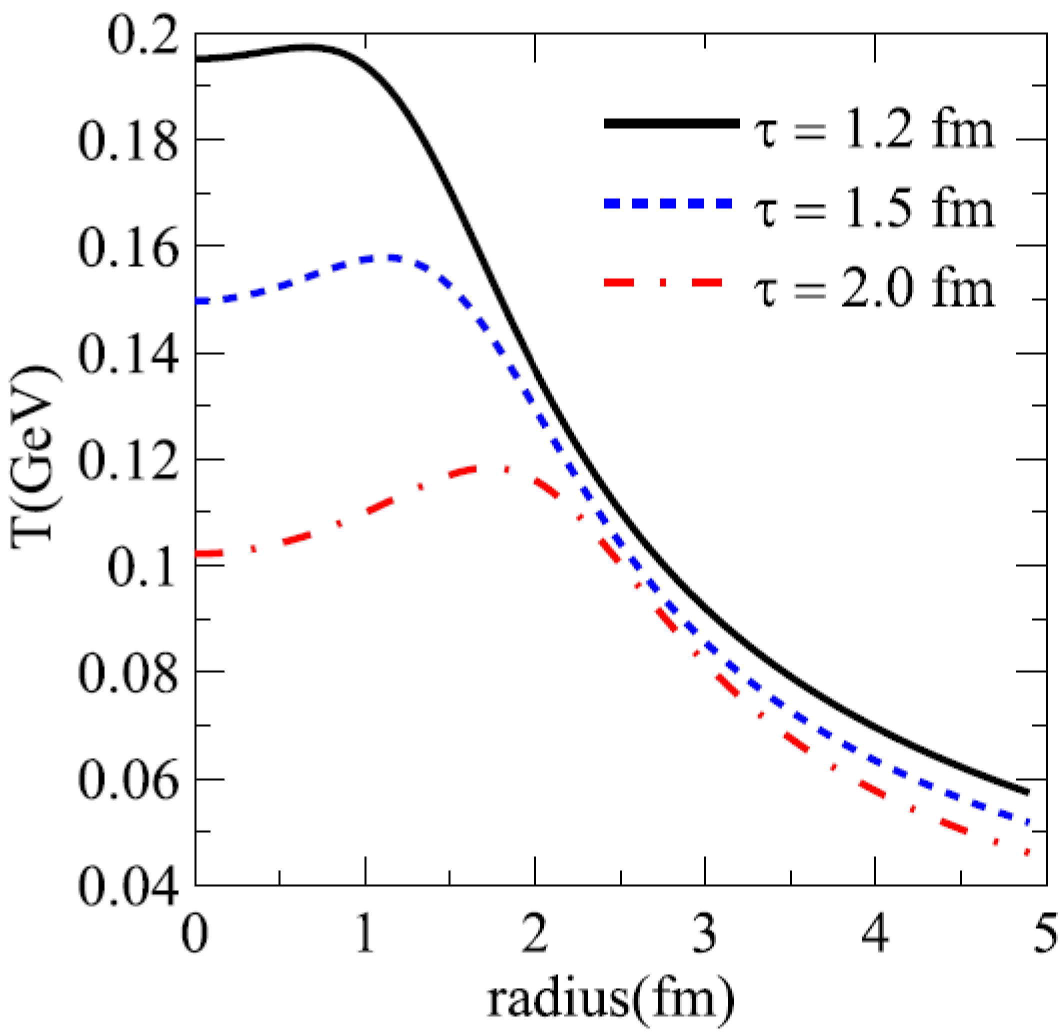
Marrochio, Noronha, Denicol, Luzum, Jeon, Gale, arXiv:1307.6130

Comparing Conformal Solutions



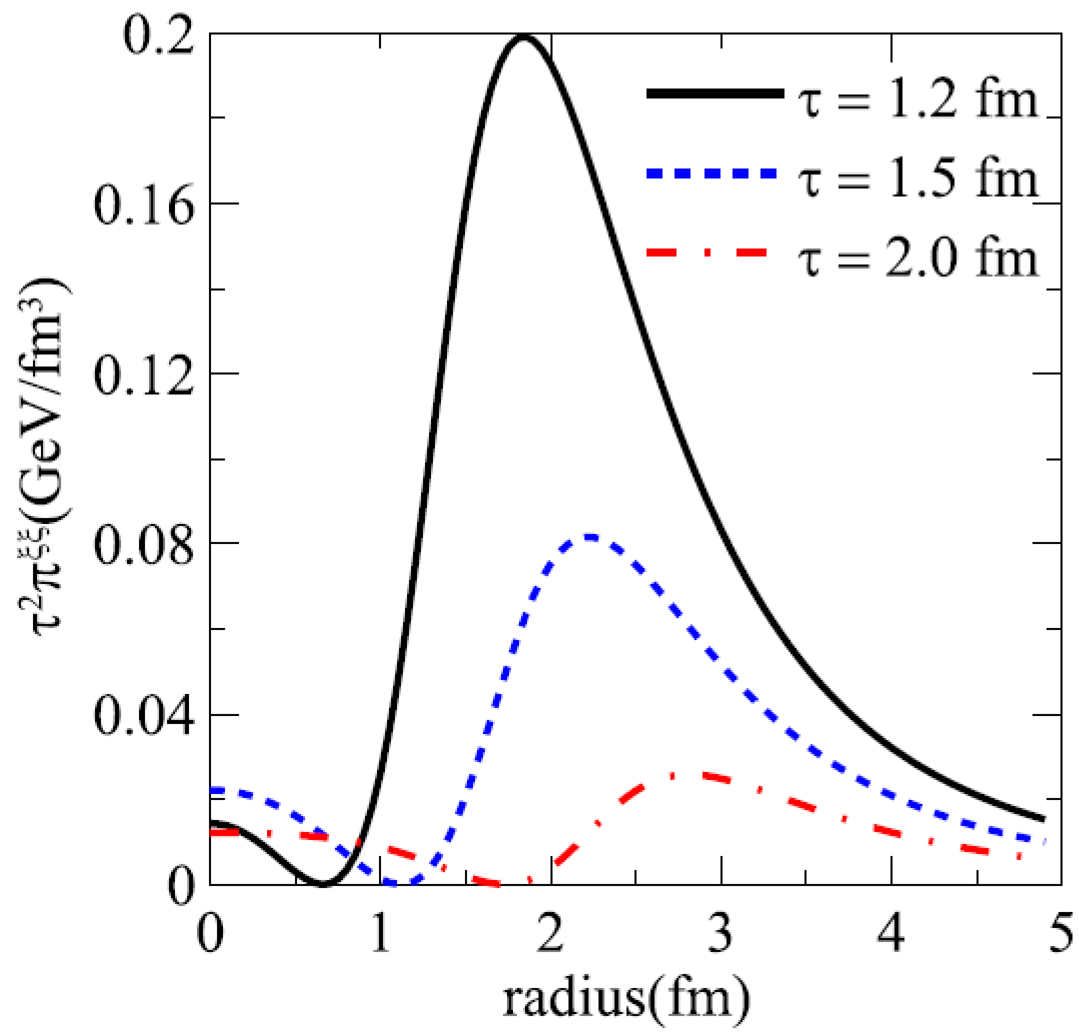
Marrochio, Noronha, Denicol, Luzum, Jeon, Gale, arXiv:1307.6130

Comparing Conformal Solutions



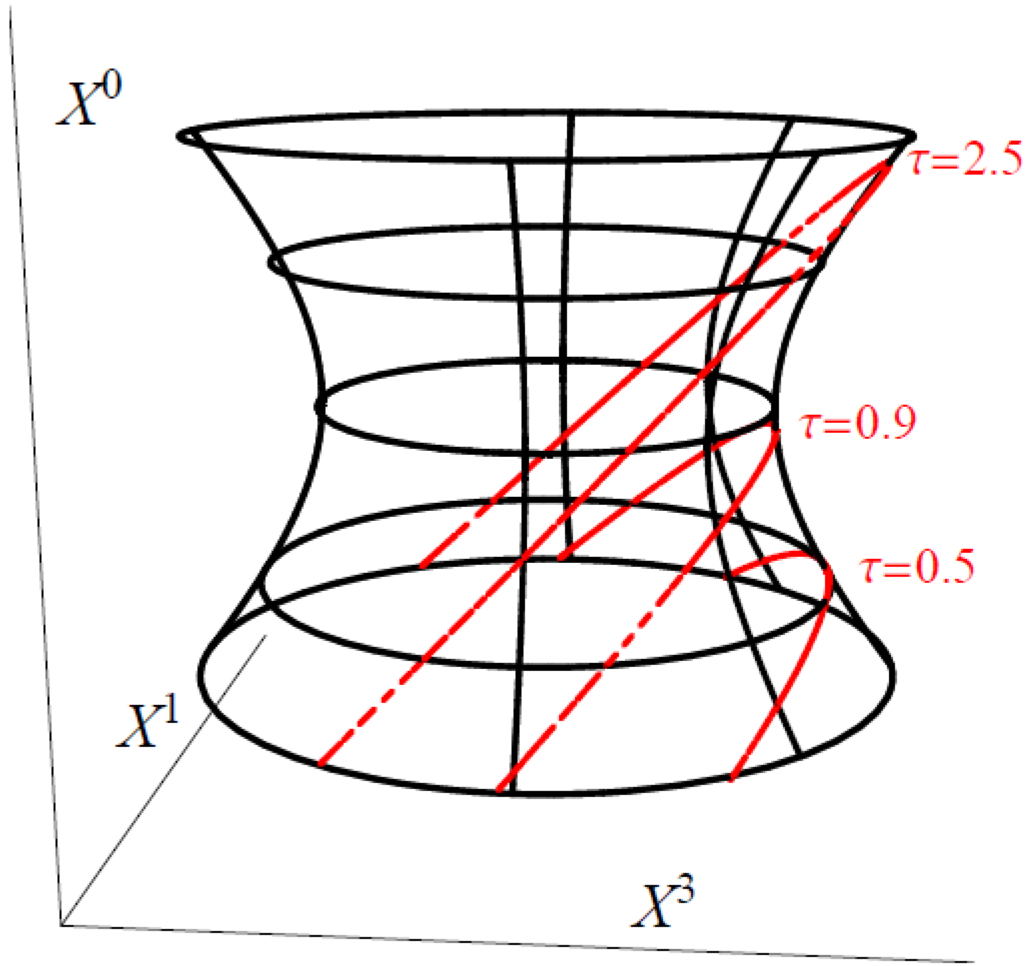
Marrochio, Noronha, Denicol, Luzum, Jeon, Gale, arXiv:1307.6130

Comparing Conformal Solutions



Marrochio, Noronha, Denicol, Luzum, Jeon, Gale, arXiv:1307.6130

A quick look to the de Sitter geometry



S. Gubser, A. Yarom, NPB 846 (2011), 469