On the effective action in hydrodynamics

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arXiv: 1502.03076

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Outline

Introduction

Effective action: Phenomenology

Effective action: Bottom-up

Effective action: Top-down

Questions

What is hydrodynamics?

all here

Basic hydrodynamics

Symmetry ⇒ conservation laws

conserved density

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

spatial current

ρ = number density, momentum density, energy density

Cooking ingredients

Variables: velocity, temperature, B fields, superfluids, liquid crystals, ...

Symmetries: Galilean, Lorentz, anomalies, external sources...

Derivative expansion: $J=J(u,\partial u,\partial^2 u,...)$

Constraints: Onsager relations, entropy current, ...

Predictions of hydro can be tested

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Decay of the Velocity Autocorrelation Function*

B. J. Alder and T. E. Wainwright
Lawrence Radiation Laboratory, University of California, Livermore, California 94550
(Received 10 July 1969)

Molecular-dynamic studies of the behavior of the diffusion coefficient after a long time s have shown that the velocity autocorrelation function decays as s^{-1} for hard disks and as $s^{-3/2}$ for hard spheres, at least at intermediate fluid densities. A hydrodynamic similarity solution

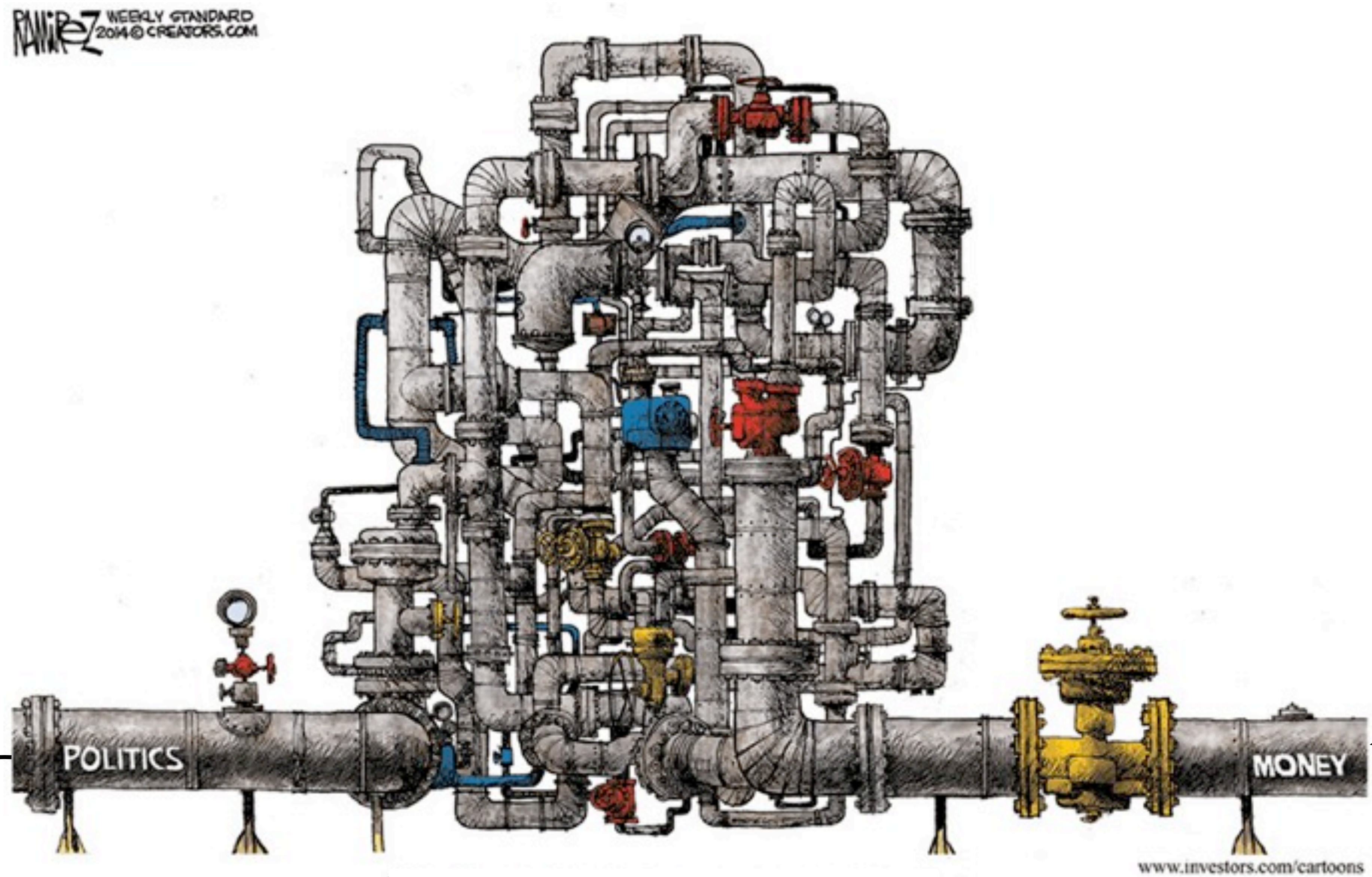
$$\langle v(t)v(0)\rangle \sim \frac{1}{t^{d/2}}$$

hydrodynamics fails at *long* times

$$D = \lim_{\omega\to 0}(D_0 + \text{const}\ \omega^{1/2})\,, \qquad d = 3$$

$$D = \lim_{\omega\to 0}(D_0 + \text{const}\ \ln\omega)\,, \qquad d = 2$$

Want an incredible machine



state

$\langle T_{ab}(x_1)\, T_{cd}(x_2)\, \ldots \rangle$

- all orderings: partially retarded etc
- knows about FDT
- good for ω,k→0
- knows about long-time tails

Want an incredible machine

$$W[g] = \int D\Phi \, \exp(iS_{eff}[\Phi, g])$$

Q: what are Φ, g, and S_{eff} ?

state

$$\langle T_{ab}(x_1)\, T_{cd}(x_2) \ldots \rangle$$

- all orderings: partially retarded etc
- knows about FDT
- good for $\omega, k \to 0$
- knows about long-time tails

Main question

How do you build the incredible machine?

The point of the action is to provide a weight in the path integral.

The point of the path integral is to build the generating functional.

In a classical thermal system, path integral = thermal fluctuations.

What is the action for that path integral?

Main question

How do you do low-energy effective field theory in a thermal state?

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Linear hydro fluctuations

all here

Gaussian noise $\langle \tau_{ij}(x)\tau_{kl}(y)\rangle = 2TG_{ijkl}\,\delta(x-y)$

$$T_{ij} = T^{\rm cl}_{ij} + \tau_{ij}$$

Solve $\partial_\mu T^{\mu\nu} = 0$ $\Rightarrow$ $\delta v^i = \delta v^i[\tau],\ \delta T = \delta T[\tau]$

Get $\langle \delta T\, \delta T\rangle$, $\langle \delta v^i\, \delta v^j\rangle$, $\langle T^{ij} T^{kl}\rangle$, Kubo formulas...

Non-linear hydro fluctuations

$$T^{ij} = \cdots + (\epsilon{+}p)v^i v^j + \cdots + \tau^{ij}$$

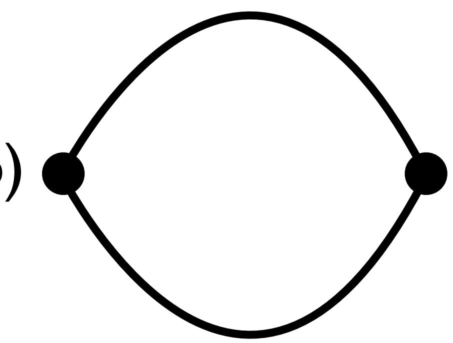
linear

leading non-linear,
no derivatives term

higher order

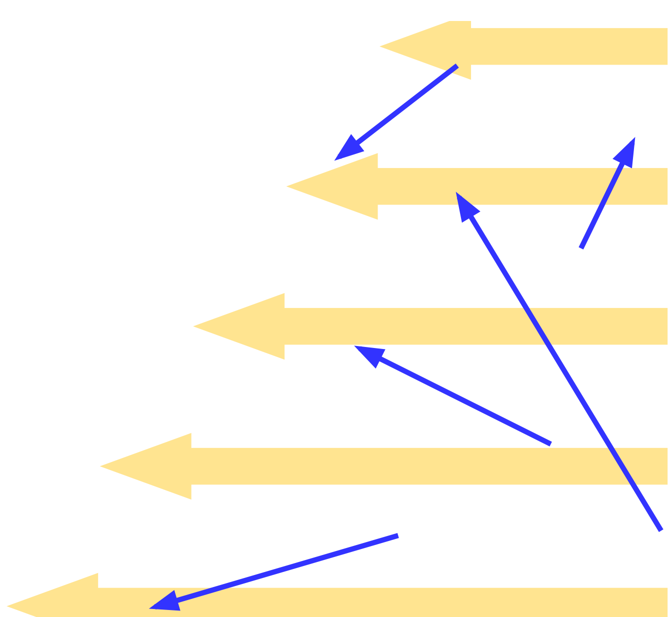
$\langle T_{xy} T_{xy} \rangle$ = $2T\eta$ + $(\varepsilon{+}p)$

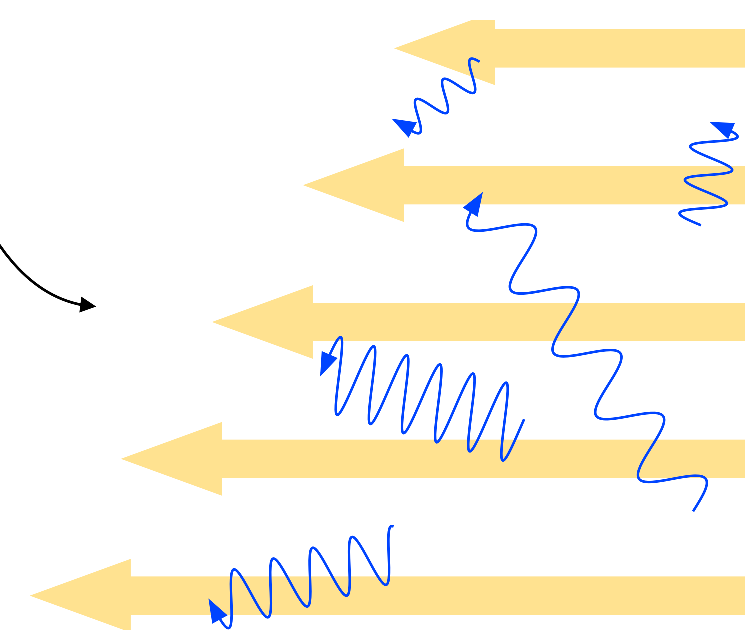
⟨δv δv⟩



⟨δv δv⟩

$(\varepsilon{+}p)$





Example: viscosity in 3+1 dim

$$\langle T_{xy} T_{xy} \rangle^R = p + O(\Lambda^3 T) - i\omega \left(\eta + \frac{17 T^2 \Lambda}{120 \pi^2 \eta / s} \right) + O\left(\frac{\omega^{3/2}}{(\eta/s)^{3/2}} \right) + O(\omega^2)$$

0-th order classical

correction to p

1-st order classical

correction to η

cutoff-independent

2-nd order classical

Total physical viscosity includes all such corrections

Function $\eta + 1/\eta$ has a minimum, lower bound on η_{tot}?

Small viscosity implies large corrections

Quark-gluon plasma at $T \gtrsim T_c$: corrections are large for $\eta/s = 0.08$

Analogy with gravity

Classical hydro: $T^{\mu\nu}_{\rm cl} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + \ldots$

$$G_{xy,xy}(\omega) = O(1) + O(\omega) + O(\omega^{3/2}) + O(\omega^2) + \ldots$$

Classical gravity: $S = \int d^4x \left[\frac{1}{16\pi G} R + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \ldots \right]$

$$V(r) = -\frac{G m_1 m_2}{r} \left[1 + O\left(\frac{Gm}{r}\right) + O\left(\frac{G\hbar}{r^2}\right) + O(e^{-m_0 r}) \right]$$

$$m_0 \sim (c_i G)^{-1/2}$$

Classical: Stelle 1978
Quantum: Bjerrum-Bohr, Donoghue, Holstein, 2002

Example: charge conductivity in 2+1 dim

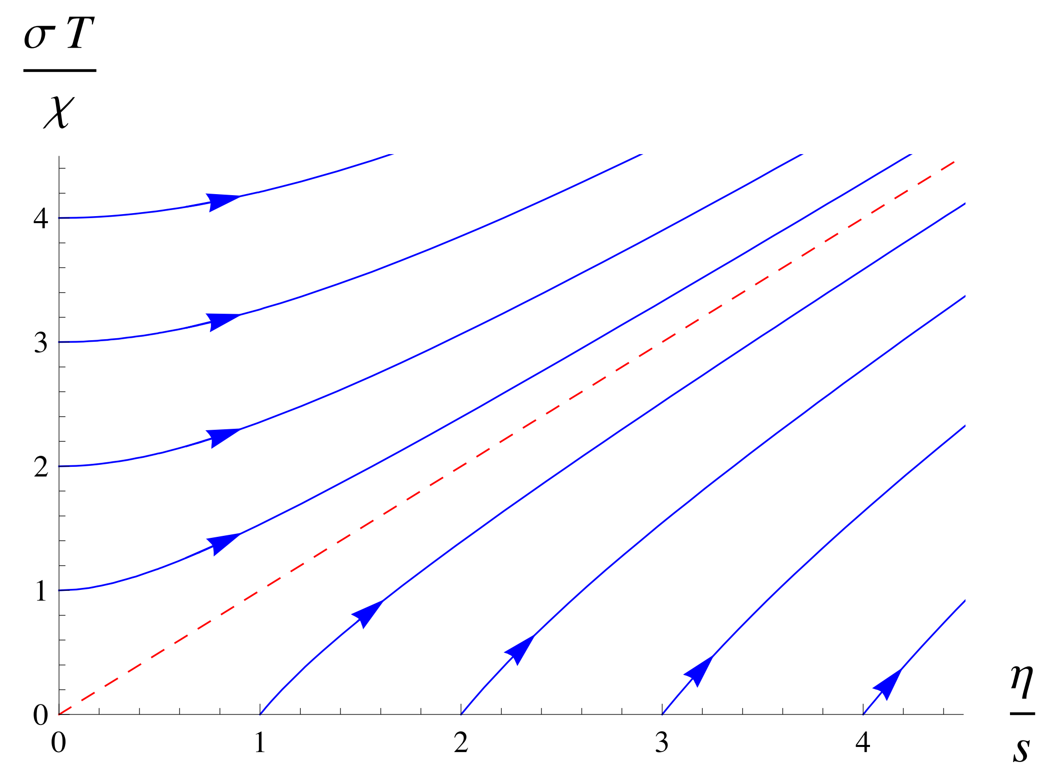
2+1 dim: $\eta=\eta_0+O(\ln\omega)$, $\sigma=\sigma_0+O(\ln\omega)$

Running viscosity: $\eta(\lambda) \equiv \eta(\omega{=}\lambda)\,,\ \sigma(\lambda) \equiv \sigma(\omega{=}\lambda)\,,\ g_\eta \equiv \eta/s\,,\ g_\sigma \equiv \sigma T/\chi$

$$\lambda\frac{\partial g_\eta}{\partial\lambda} = -\frac{1}{16\pi c}\frac{1}{g_\eta}\,,$$

$$\lambda\frac{\partial g_\sigma}{\partial\lambda} = -\frac{1}{8\pi c}\frac{1}{g_\sigma + g_\eta}$$

s/T^2, counts d.o.f.



IR "fixed line" as $\omega\to 0$:

$$\boxed{\frac{\eta}{s} = \frac{\sigma T}{\chi}}$$

These were bits and pieces of the machine

How do we build the full incredible machine?

Building machine: just add the noise

$$\partial_\mu \left(T^{\mu\nu}_{\rm cl} + \tau^{\mu\nu}\right) = 0$$

$$T^{\mu\nu}_{\rm cl} = \epsilon u^\mu u^\nu + p\Delta^{\mu\nu} - G^{\mu\nu\alpha\beta}\partial_\alpha u_\beta$$

viscosities & projectors,
standard Landau-Lifshitz hydro

$$\langle \tau_{\mu\nu}(x)\tau_{\alpha\beta}(y)\rangle = 2TG_{\mu\nu\alpha\beta}\,\delta(x-y)$$

same $G_{\mu\nu\alpha\beta}$ for FDT in equilibrium

Get stochastic differential eq-s with multiplicative noise

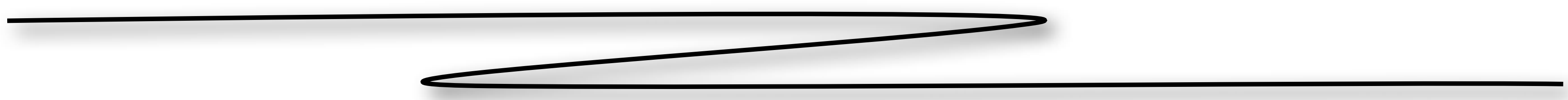
Discretization ambiguities...

Effective action for dissipative relativistic fluids

Integrate out the noise:

$$S_{\text{eff}} = \int dt\, d^d x \left[i\partial_\mu \tilde{\phi}_\nu\, T^{\mu\nu}_{\text{cl}} + T\, \partial_\mu \tilde{\phi}_\nu\, G^{\mu\nu\lambda\sigma}\, \partial_\lambda \tilde{\phi}_\sigma + \quad \right]$$

exponentiating the Jacobian



Where is the derivative expansion?

Where is the "frame" invariance?

How to get all types of correlation functions?

Why is the noise Gaussian?

UV divergences?

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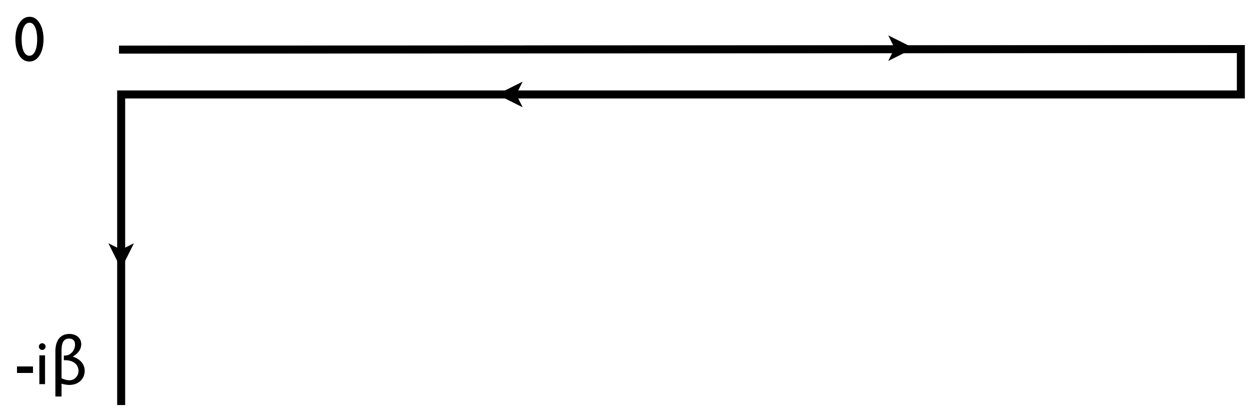
FInite-temperature field theory



$$\varphi_r = \frac{1}{2}(\varphi_1 + \varphi_2)$$

$$\varphi_a = \varphi_1 - \varphi_2$$

$$G^{\mathrm{Ret}} = -i\langle \varphi_r \varphi_a \rangle$$

$$G^{\mathrm{Adv}} = -i\langle \varphi_a \varphi_r \rangle$$

$$G^{\mathrm{Symm}} = 2\langle \varphi_r \varphi_r \rangle$$

Example: diffusion

$$G^{\text{Ret}}_{nn} = \frac{D\chi\mathbf{k}^2}{i\omega - D\mathbf{k}^2}\,, \quad G^{\text{Adv}}_{nn} = \frac{-D\chi\mathbf{k}^2}{i\omega + D\mathbf{k}^2}\,, \quad G^{\text{Symm}}_{nn} = \frac{4TD\chi\mathbf{k}^2}{\omega^2 + (D\mathbf{k}^2)^2}$$

$i\omega - Dk^2$ from diffusion eqn $\partial_t n - D\nabla^2 n = 0$

$\chi = (\partial n/\partial \mu)_{\mu=0}$ susceptibility, from coupling to the source

T=temperature, from FDT

Q: What is $S_{eff}[\varphi_r, \varphi_a]$ which gives in $\int D\varphi_r D\varphi_a \exp[iS_{eff}]\ldots$?

$$G^{\text{Ret}}_{nn} = -i\langle\varphi_r\varphi_a\rangle$$

$$G^{\text{Adv}}_{nn} = -i\langle\varphi_a\varphi_r\rangle$$

$$G^{\text{Symm}}_{nn} = 2\langle\varphi_r\varphi_r\rangle$$

A: Such S_{eff} is easy to find, but it is non-local.

Diffusion: effective action

Add auxiliary fields

Add external sources

Make Lorentz-invariant

Make r- and a-gauge invariant

$$Z[A_r, A_a] = \int D\phi_r D\phi_a \; e^{iS[\phi_r, \phi_a, A_r, A_a]}$$

$$S = \int dt\, d^d x \;\; (J^\mu_{\rm cl}[\phi_r, A_r] D_\mu \phi_a + iT\sigma \Delta^{\mu\nu} D_\mu \phi_a D_\nu \phi_a)$$

hydro current

n or μ

physical gauge field

$\partial_\mu \phi_a + A^a_\mu$

set to zero at the end

This gives all <$J_\mu J_\nu$>^{R,A,S} in equilibrium

Linear hydro: effective action

$$Z[h_r, h_a] = \int D\phi^r_\mu D\phi^a_\mu \; e^{iS[\phi^r,\phi^a,h_r,h_a]}$$

$$S = \int dt\, d^d x\, \sqrt{-g_r}\, \left(T^{\mu\nu}_{\rm cl}[\phi_r, g_r]\mathcal{D}_\mu \phi^a_\nu + iT\, \mathcal{D}_\mu \phi^a_\nu\, G^{\mu\nu\alpha\beta}\mathcal{D}_\alpha \phi^a_\beta\right)$$

hydro E-M tensor

β^μ

physical metric

$\frac{1}{2}\left(h^a_{\mu\nu} - \nabla^r_\mu \phi^a_\nu - \nabla^r_\nu \phi^a_\mu\right)$

set to zero at the end

This gives all $<T_{\mu\nu}\, T_{\alpha\beta}>^{R,A,S}$ in equilibrium

Whatever $S_{\rm eff}$ is, it must reduce to this near equilibrium

Full hydro effective action?

$$S = \int dt\, d^d x \sqrt{-g_r}\left(T^{\mu\nu}_{\rm cl}[\phi_r, g_r]\mathcal{D}_\mu \phi^a_\nu + iT\, \mathcal{D}_\mu \phi^a_\nu\, G^{\mu\nu\alpha\beta}\mathcal{D}_\alpha \phi^a_\beta\right)$$

Small fluctuations only (2-point functions)

Derivative expansion not obvious

Need to understand the symmetries

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Effective action: first pass

Near equilibrium $\beta^\mu = \bar{\beta}^\mu + \beta'^\mu$

Choose $T=1/(-\beta\cdot\beta)^{1/2}$, $u^\mu=T\beta^\mu$ as variables

Derivative expansion:

$$\Gamma = \int dt\, d^d x\, \sqrt{-g}\, \Big(p(T) + a(T)\nabla\cdot u + b(T)\dot{T} + \ldots \Big)$$

Vary w.r.t. g, keep β fixed:

$$T^{\mu\nu} = \underbrace{p g^{\mu\nu} + T p' u^\mu u^\nu}_{\text{normal perfect fluid}} + \underbrace{(b-a')\Big(\Delta^{\mu\nu}\dot{T} - T u^\mu u^\nu \nabla\cdot u\Big)}_{\text{can be set to zero by a frame choice}}$$

No dissipation here

Symmetries

CTP generating functional:

$$Z_\rho[j_1, j_2] = \int d\tilde{q}_1\, d\tilde{q}_2\, dq_f\, \langle \tilde{q}_1 | \rho | \tilde{q}_2 \rangle \int_{\tilde{q}_1}^{q_f} Dq_1 \int_{\tilde{q}_2}^{q_f} Dq_2\; e^{i \int_{t_0}^{t_f} L(q_1, j_1) - i \int_{t_0}^{t_f} L(q_2, j_2)}$$

1. Local symmetries:

$$Z_\rho[j_1, j_2] = Z_{\rho'}[j_1 + \delta_{\xi_1} j_1, j_2 + \delta_{\xi_2} j_2]$$

2. Can use r,a variables: $q_r=(q_1+q_2)/2$, $q_a=q_1-q_2$

$$S[q_1, j_1] - S[q_2, j_2] = \int q_a\, EOM(q_r, j_r) + O(j_a, q_a^2)$$

3. Normalization:

$$Z_\rho[j_1, j_1] = 1$$

Conservation laws: 1 and 2

$$\delta_g W[g^1, g^2] = \int \tfrac{1}{2}\sqrt{-g^1}\,\langle T_1^{\mu\nu}\rangle \delta g^1_{\mu\nu} - \int \tfrac{1}{2}\sqrt{-g^2}\,\langle T_2^{\mu\nu}\rangle \delta g^2_{\mu\nu}$$

$$\delta g^1_{\mu\nu} = g^1_{\mu\lambda}\partial_\nu \xi_1^\lambda + g^1_{\nu\lambda}\partial_\mu \xi_1^\lambda + \partial_\lambda g^1_{\mu\nu}\,\xi_1^\lambda\,, \qquad \delta g^2_{\mu\nu} = g^2_{\mu\lambda}\partial_\nu \xi_2^\lambda + g^2_{\nu\lambda}\partial_\mu \xi_2^\lambda + \partial_\lambda g^2_{\mu\nu}\,\xi_2^\lambda$$

Diffeo invariance of $W[g^1,g^2]$ gives

$$\nabla^1_\mu \langle T_1^{\mu\nu}\rangle = 0\,, \quad \nabla^2_\mu \langle T_2^{\mu\nu}\rangle = 0$$

not useful b/c hydro
variables are r-fields

Conservation laws: r and a

Define r- and a-sources: $g^r \equiv \frac{1}{2}\left(g^1 + g^2\right)\,, \qquad g^a \equiv g^1 - g^2$

physical metric — set to zero at the end

$$\sqrt{-g^r}\,\langle T_r^{\mu\nu}\rangle \equiv \tfrac{1}{2}\sqrt{-g^1}\,\langle T_1^{\mu\nu}\rangle + \tfrac{1}{2}\sqrt{-g^2}\,\langle T_2^{\mu\nu}\rangle$$

physical stress tensor

$$\sqrt{-g^r}\,\langle T_a^{\mu\nu}\rangle \equiv \sqrt{-g^1}\,\langle T_1^{\mu\nu}\rangle - \sqrt{-g^2}\,\langle T_2^{\mu\nu}\rangle$$

Diffeo invariance of W[g^1,g^2] gives

$$g^r_{\nu\lambda}\nabla_\mu\langle T_r^{\mu\nu}\rangle + \tfrac{1}{4}\left(g^a_{\nu\lambda}\partial_\mu\langle T_a^{\mu\nu}\rangle + \Gamma^a_{\lambda\mu\nu}\langle T_a^{\mu\nu}\rangle + g^a_{\nu\lambda}\Gamma^{\rho\ r}_{\rho\mu}T_a^{\mu\nu}\right) = 0\,,$$

$$g^r_{\nu\lambda}\nabla_\mu\langle T_a^{\mu\nu}\rangle + g^a_{\nu\lambda}\partial_\mu\langle T_r^{\mu\nu}\rangle + \Gamma^a_{\lambda\mu\nu}\langle T_r^{\mu\nu}\rangle + g^a_{\nu\lambda}\Gamma^{\rho\ r}_{\rho\mu}\langle T_r^{\mu\nu}\rangle = 0$$

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})\,, \quad \nabla_\mu \equiv \nabla^r_\mu$$

Main problem

To manifest the symmetries, need 1- and 2-fields

To write down hydro e.o.m., need r- and a-fields

A proposal

Realize the symmetries as if $D_1 \times D_2 \rightarrow D_r$

D_r manifest in hydro

a-type d.o.f. $\xi_a \rightarrow \varphi_a$ look like Goldstone bosons

Structure of the effective action

$$S_{\text{eff}} = \mathcal{I}_r + \mathcal{J}_r\,\mathcal{D}\varphi_a + \mathcal{K}_r\,(\mathcal{D}\varphi_a)^2 + \ldots$$

$O(a^0)$ $O(a)$ $O(a^2)$

I_r: contains thermodynamics

J_r: classical hydro equations

K_r: responsible for FDT

Different orders in the a-expansion are not independent

Effective action for the neutral fluid

Up to O(a) can work with r- and a-diffeos instead of 1- and 2-diffeos

Variables:

β^μ vector under r, singlet under a

$\varphi_a{}^\mu$ vector under r, shifts under a

Data to construct S_{eff} :

$$T = 1/\sqrt{-\beta\cdot\beta}$$

$$u^\mu = \beta^\mu/\sqrt{-\beta\cdot\beta}$$

$$g^r_{\mu\nu}$$

$$\xi^a_{\mu\nu} \equiv g^a_{\mu\nu} - \nabla_\mu \varphi^a_\nu - \nabla_\nu \varphi^a_\mu$$

Effective action for the neutral fluid

Scalars up to O(a) and O(∂^0):

$$\alpha_1 \equiv T,\ \alpha_2 \equiv u^\mu u^\nu \xi^a_{\mu\nu},\ \alpha_3 \equiv g_r^{\mu\nu}\xi^a_{\mu\nu}$$

Scalars up to O(a) and O(∂):

$$\dot\alpha_i\ ,\ \nabla_\mu u^\mu\ ,\ u^\mu \xi^a_{\mu\nu}\nabla^\nu T\ ,\ \xi^a_{\mu\nu}\nabla^\mu u^\nu\ ,\ u^\mu \xi^a_{\mu\nu}\dot u^\nu\ ,\ u^\mu u^\nu \dot\xi^a_{\mu\nu}\ ,\ u^\mu \nabla^\nu \xi^a_{\mu\nu}$$

integrate by parts

Effective action up to O(a) and up to O(∂):

$$S = \int \sqrt{-g_r}\left[F + f_1 \dot T + f_2 \nabla_\mu u^\mu + f_3\, \xi^a_{\mu\nu} u^{(\mu}\nabla^{\nu)} T + f_4 \xi^a_{\mu\nu}\nabla^{(\mu}u^{\nu)} + f_5 \xi^a_{\mu\nu} u^{(\mu}\dot u^{\nu)}\right] + O(a^2)$$

This does describe the most general dissipative fluid

Energy-momentum tensor from the effective action

$$S_{\rm eff} = O(a^0) + \int \tfrac{1}{2}\sqrt{-g_r}\, T_r^{\mu\nu}\left(g^a_{\mu\nu} - \nabla_\mu \varphi^a_\nu - \nabla_\nu \varphi^a_\mu\right) + O(a^2)$$

$$T_r^{\mu\nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) + t^{\mu\nu}$$

$\mathcal{E}$, $\mathcal{P}$, q^μ, $t^{\mu\nu}$ made out of $F(\alpha_j)$, $f_i(\alpha_j)$

This is the standard dissipative fluid in a general frame:

$$\epsilon(T) = 2\frac{\partial F}{\partial \alpha_2} - 2\frac{\partial F}{\partial \alpha_3}\,, \quad p(T) = 2\frac{\partial F}{\partial \alpha_3}$$

$\eta = -f_4$, ς=(combination of f_1',f_2',f_3,f_4)

To sum up

$$S_{\text{eff}} = \mathcal{I}_r + \mathcal{J}_r\, \mathcal{D}\varphi_a + \mathcal{K}_r\, (\mathcal{D}\varphi_a)^2 + \ldots$$

$\mathcal{I}_r$: $O(a^0)$ $\qquad$ $\mathcal{J}_r\,\mathcal{D}\varphi_a$: $O(a)$ $\qquad$ $\mathcal{K}_r\,(\mathcal{D}\varphi_a)^2$: $O(a^2)$

S_{eff} up to $O(a)$, up to $O(\partial)$ gives the standard dissipative hydro

Can generalize to charged fluids, with external gauge fields

Can presumably repeat up to $O(\partial^2)$, get 2nd order hydro

ta-da!!!

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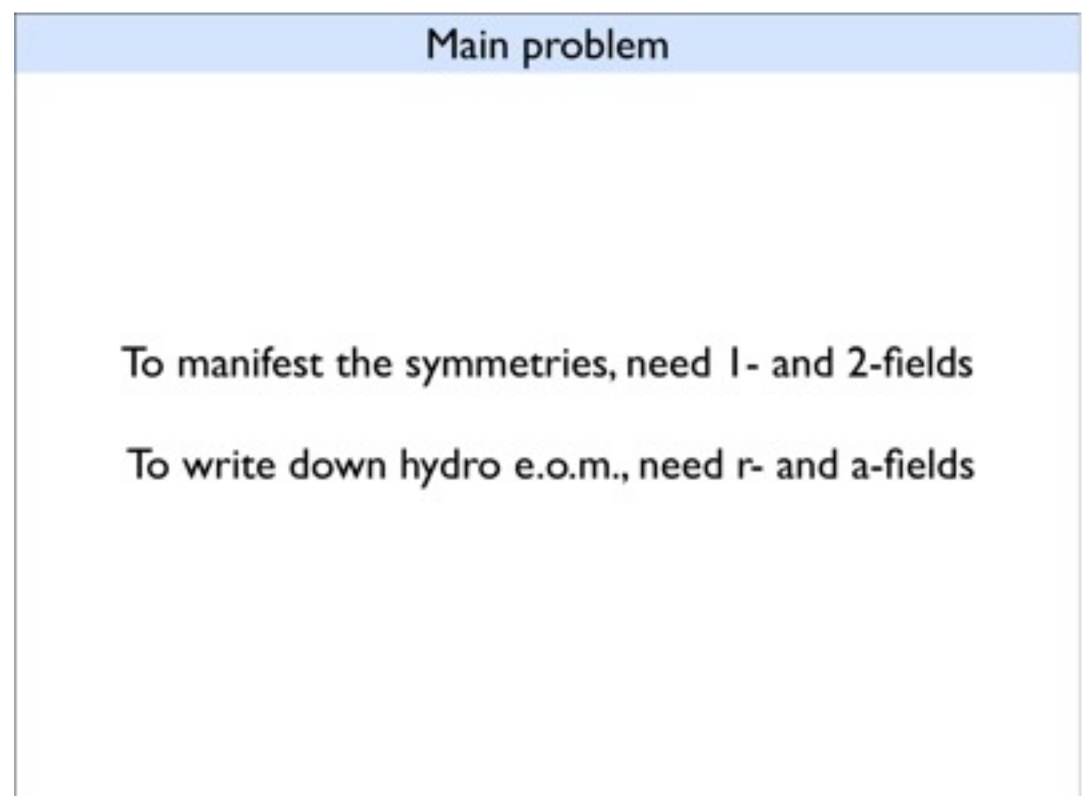
Where is the FDT?

We only went to O(a), using r- and a- diffeo invariance

FDT appears at $O(a^2)$

But it's not legit to use r- and a-diffeo invariance beyond O(a)

So need S_{eff} in terms of 1- and 2-variables... what are they?



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To write down hydro e.o.m., need r- and a-fields

Example: $U(1)_1 \times U(1)_2 \to U(1)_r$

Break to the diagonal subgroup, one Goldstone:

$$\mathcal{L} = v^2(\partial_\mu \varphi + Z_\mu)(\partial^\mu \varphi + Z^\mu) + (\text{massive})$$

$$Z_\mu \to A^1_\mu - A^2_\mu$$

Naive derivative expansion for φ does not know about $U(1)\times U(1)$

Realizing $U(1)\times U(1)$ requires the "Higgs", a "non-hydro" mode

Other questions

$$S_{\text{eff}} = \mathcal{I}_r + \mathcal{J}_r\,\mathcal{D}\varphi_a + \mathcal{K}_r\,(\mathcal{D}\varphi_a)^2 + \ldots$$

I_r, J_r, K_r appear at different orders in the derivative expansion, but they are not independent

Constraints coming from the entropy current or from the existence of equilibrium are not obvious

Integration measure for r-fields is not clear: remember  ?

Related work

The "eightfold" way

Haehl, Loganayagam, Rangamani, arXiv:1502.00636

Classify transport coefficients in *classical* hydro

Non-dissipative transport follows from an effective action

Impose extra $U(1)_T$ gauge invariance for adiabaticity and 2nd law

Double the d.o.f. similar to the CTP formalism

Coupling between 1- and 2-sectors unclear

Conclusions

Effective action for near-equilibrium states is not obvious: are thermal fluctuations really harder than quantum fluctuations?

Double the d.o.f.

Non-linear realization of the doubled symmetry

May need “high-energy” fields (1 and 2) in the effective theory

If you know how to construct the effective theory,
please tell me!

Thank you!