Improved hydrodynamic approximations for the early stages of heavy-ion collisions

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References:

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- Exact solutions have been found for weakly interacting systems with highly symmetric flow patterns and density distributions: **Bjorken and Gubser flow**
- Can be used to test different hydrodynamic expansion schemes for the Boltzmann equation in the Relaxation Time Approximation (RTA)

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Boltzmann Equation in Relaxation Time Approximation (RTA):

$$p^\mu \partial_\mu f(x,p) = C(x,p) = \frac{p\cdot u(x)}{\tau_{\rm rel}(x)}\Big(f_{\rm eq}(x,p) - f(x,p)\Big)$$

For conformal systems $\tau_{\rm rel}(x) = c/T(x) = 5\eta/(\mathcal{S}\,T) \equiv 5\bar{\eta}/T(x)$.

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Macroscopic currents:

$$j^\mu(x) = \int_p p^\mu\, f(x,p) \equiv \langle p^\mu \rangle; \quad T^{\mu\nu}(x) = \int_p p^\mu\, p^\nu\, f(x,p) \equiv \langle p^\mu p^\nu \rangle$$

where $$\int_p \cdots \equiv \frac{g}{(2\pi)^3}\int \frac{d^3p}{E_p}\cdots \equiv \langle \dots \rangle$$

Hydrodynamics from kinetic theory (I):

Expand the solution $f(x,p)$ of the Boltzmann equation as

$$f(x,p) = f_0(x,p) + \delta f(x,p) \qquad \Big(\big|\delta f/f_0\big| \ll 1 \Big)$$

where f_0 is parametrized through macroscopic observables as

$$f_0(x,p) = f_0\left(\frac{\sqrt{p_\mu \Xi^{\mu\nu}(x) p_\nu} - \tilde{\mu}(x)}{\tilde{T}(x)}\right)$$

where $\quad \Xi^{\mu\nu}(x) = u^\mu(x) u^\nu(x) - \Phi(x)\Delta^{\mu\nu}(x) + \xi^{\mu\nu}(x).$

$u^\mu(x)$ defines the local fluid rest frame (LRF).
$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the spatial projector in the LRF.
$\tilde{T}(x)$, $\tilde{\mu}(x)$ are the effective temperature and chem. potential in the LRF.
$\Phi(x)$ partially accounts for bulk viscous effects in expanding systems.
$\xi^{\mu\nu}(x)$ describes deviations from local momentum isotropy in anisotropically expanding systems due to shear viscosity.

Hydrodynamics from kinetic theory (II):

$u^\mu(x),\ \tilde{T}(x),\ \tilde{\mu}(x)$ are fixed by the Landau matching conditions:

$$T^\mu_{\ \nu} u^\nu = \mathcal{E}(\tilde{T},\tilde{\mu};\xi,\Phi)u^\mu; \qquad \Big\langle u{\cdot}p \Big\rangle_{\delta f} = \Big\langle (u{\cdot}p)^2 \Big\rangle_{\delta f} = 0$$

$\mathcal{E}$ is the LRF energy density. We introduce the true local temperature $T(\tilde{T},\tilde{\mu};\xi,\Phi)$ and chemical potential $\mu(\tilde{T},\tilde{\mu};\xi,\Phi)$ by demanding $\mathcal{E}(\tilde{T},\tilde{\mu};\xi,\Phi){=}\mathcal{E}_{\rm eq}(T,\mu)$ and $\mathcal{N}(\tilde{T},\tilde{\mu};\xi,\Phi){\equiv}\langle u{\cdot}p\rangle_{f_0}{=}\mathcal{R}_0(\xi,\Phi)\mathcal{N}_{\rm eq}(T,\mu)$ (see cited literature for $\mathcal{R}$ functions).
Writing

$$T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu} \equiv T_0^{\mu\nu} + \Pi^{\mu\nu}, \qquad j^\mu = j_0^\mu + \delta j^\mu \equiv j_0^\mu + V^\mu,$$

the conservation laws

$$\partial_\mu T^{\mu\nu}(x) = 0, \qquad \partial_\mu j^\mu(x) = \frac{\mathcal{N}(x) - \mathcal{N}_{\rm eq}(x)}{\tau_{\rm rel}(x)}$$

are sufficient to determine $u^\mu(x)$, $T(x)$, $\mu(x)$, but not the dissipative corrections $\xi^{\mu\nu}$, Φ, $\Pi^{\mu\nu}$, and V^μ whose evolution is controlled by microscopic physics.

Hydrodynamics from kinetic theory (III):

Different hydrodynamic approaches can be characterized by the different assumptions they make about the dissipative corrections and/or the different approximations they use to derive their dynamics from the underlying Boltzmann equation:

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- **Viscous anisotropic hydrodynamics (vaHydro):** improved **aHydro** that additionally evolves residual dissipative flows $\Pi^{\mu\nu}$, V^μ with **IS** or **DNMR theory**.

Overview

BE for systems with highly symmetric flows: I. Bjorken flow

- Longitudinal boost invariance, transverse homogeneity ("physics on the light cone", no transverse flow) $\Longrightarrow$ $u^\mu = (1,0,0,0)$ in Milne coordinates (τ, r, ϕ, η) where $\tau = (t^2 - z^2)^{1/2}$ and $\eta = \frac{1}{2}\ln[(t-z)/(t+z)]$ $\Longrightarrow$ $v_z = z/t$

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- Metric: $ds^2 = d\tau^2 - dr^2 - r^2 d\phi^2 - \tau^2 d\eta^2$, $\quad g_{\mu\nu} = \mathrm{diag}(1, -1, -r^2, -\tau^2)$

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- Symmetry restricts possible dependence of distribution function $f(x,p)$ (Baym '84, Florkowski et al. '13, '14):

$$f(x,p) = f(\tau; p_\perp, w) \qquad \text{where} \qquad w = t p_z - zE = \tau m_\perp \sinh(y - \eta).$$

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- **Solution:**

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where $\qquad D(\tau_2, \tau_1) = \exp\left(- \int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\mathrm{rel}}(\tau'')} \right).$

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  $\Longrightarrow$ $u^\mu = (1,0,0,0)$ in de Sitter coordinates $(\rho, \theta, \phi, \eta)$ where
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- With $T(\tau,r) = \hat{T}(\rho(\tau,r))/\tau$ RTA BE simplifies to the ODE

$$\frac{\partial}{\partial\rho} f(\rho;\hat{p}_\Omega^2,\hat{p}_\varsigma) = -\frac{\hat{T}(\rho)}{c}\left[f\left(\rho;\hat{p}_\Omega^2,\hat{p}_\varsigma\right) - f_{\rm eq}\left(\hat{p}^\rho/\hat{T}(\rho)\right)\right].$$

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Hydrodynamic equations for systems with Gubser flow*:

- The exact solution for f can be worked out for any "initial" condition $f_0(\hat{p}_\Omega^2, w) \equiv f(\rho_0; \hat{p}_\Omega^2, w)$. We here use equilibrium initial conditions, $f_0 = f_{\mathrm{eq}}$.

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- By taking hydrodynamic moments, the exact f yields the exact evolution of all components of $T^{\mu\nu}$. Here, $\Pi^{\mu\nu}$ has only one independent component, $\pi^{\eta\eta}$.
- This exact solution of the BE can be compared to solutions of the various hydrodynamic equations in de Sitter coordinates, using identical initial conditions.
  - **Ideal:** $\hat{T}_{\rm ideal}(\rho) = \frac{\hat{T}_0}{\cosh^{2/3}(\rho)}$
  - **NS:** $\frac{1}{\hat{T}}\frac{d\hat{T}}{d\rho} + \frac{2}{3}\tanh\rho = \frac{1}{3}\bar{\pi}^\eta_\eta(\rho)\tanh\rho$ (viscous T-evolution) with $\bar{\pi}^\eta_\eta \equiv \hat{\pi}^\eta_\eta/(\hat{T}\hat{\mathcal{S}})$ and $\hat{\pi}^{\eta\eta}_{NS} = \frac{4}{3}\hat{\eta}\tanh\rho$ where $\frac{\hat{\eta}}{\hat{\mathcal{S}}} \equiv \bar{\eta} = \frac{1}{5}\hat{T}\hat{\tau}_{\rm rel}$
  - **IS:** $\frac{d\bar{\pi}^\eta_\eta}{d\rho} + \frac{4}{3}\left(\bar{\pi}^\eta_\eta\right)^2\tanh\rho + \frac{\bar{\pi}^\eta_\eta}{\hat{\tau}_{\rm rel}} = \frac{4}{15}\tanh\rho$
  - **DNMR:** $\frac{d\bar{\pi}^\eta_\eta}{d\rho} + \frac{4}{3}\left(\bar{\pi}^\eta_\eta\right)^2\tanh\rho + \frac{\bar{\pi}^\eta_\eta}{\hat{\tau}_{\rm rel}} = \frac{4}{15}\tanh\rho + \frac{10}{21}\bar{\pi}^\eta_\eta\tanh\rho$
  - **aHydro:** see M. Nopoush et al., PRD 91 (2015) 045007
  - **vaHydro:** not yet available

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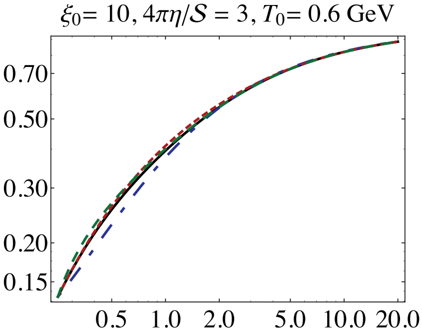
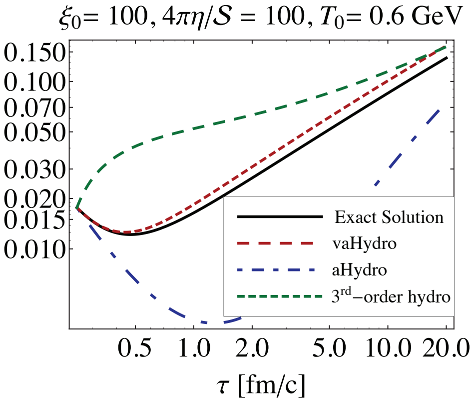
*For Bjorken flow, including **(0+1)-d vaHydro**, see UH@QM14

Overview

Bjorken flow (I)

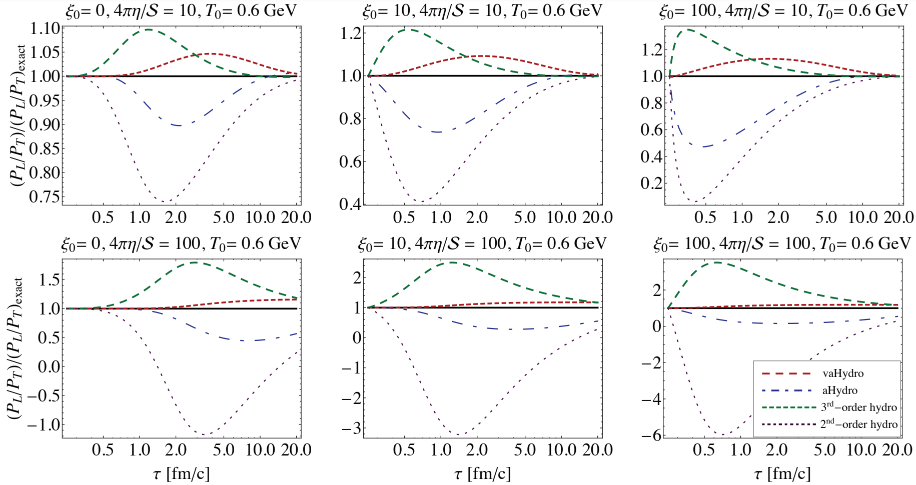
Pressure anisotropy P_L/P_T vs. τ:





In the right plot, IS theory yields negative $P_L/P_T < 0$!

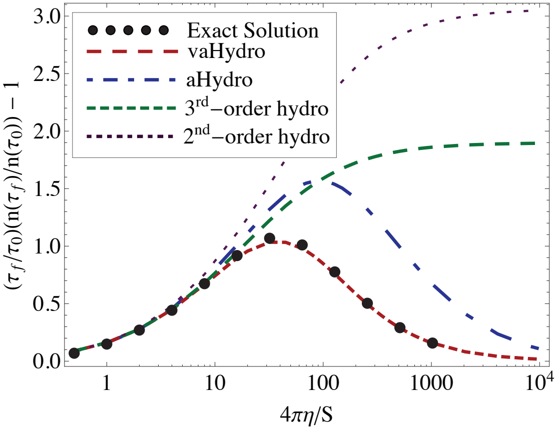
Bjorken flow (II)



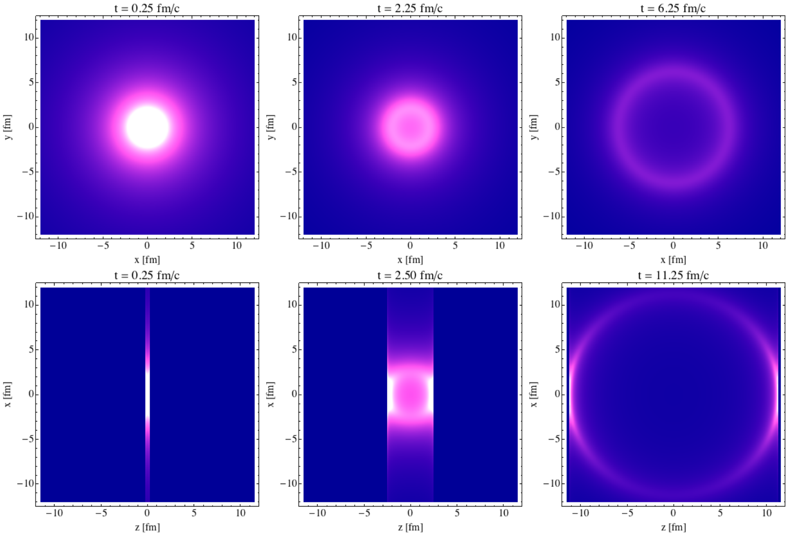
vaHydro agrees within a few % with exact result, even for very large $\eta/\mathcal{S}$!

Bjorken flow (III)

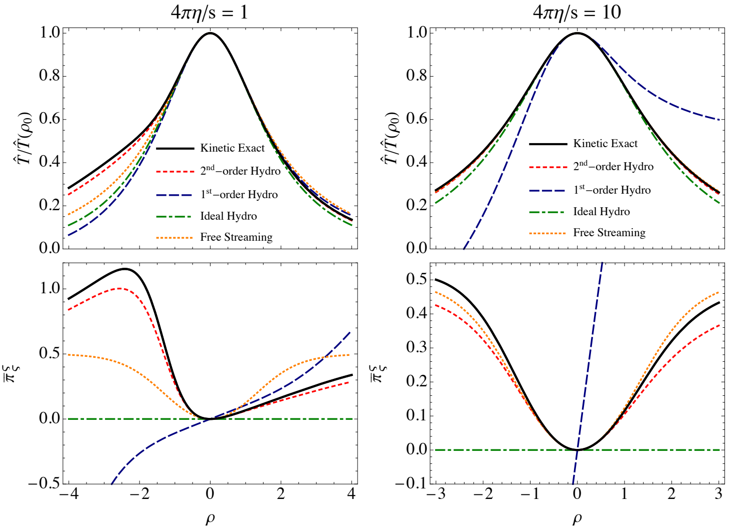
Total entropy (particle) production $\frac{n(\tau_f)\cdot\tau_f}{n(\tau_0)\cdot\tau_0} - 1$



Gubser flow I: temperature profile in (x, y) and (x, z)

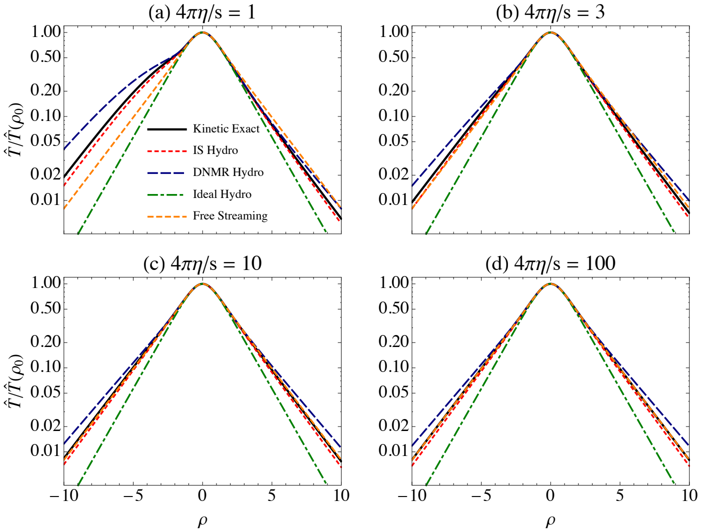


Gubser flow II: ρ-evolution of temperature and shear stress



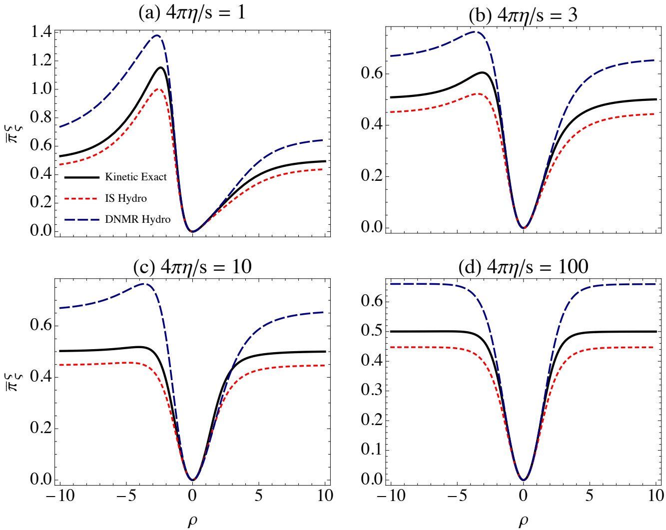
Note: $\bar{\pi}^\varsigma_\varsigma \equiv \bar{\pi}^\eta_\eta$! Thermal equil. initial conditions at $\rho_0 = 0$.

Gubser flow III: temperature evolution in de Sitter time ρ



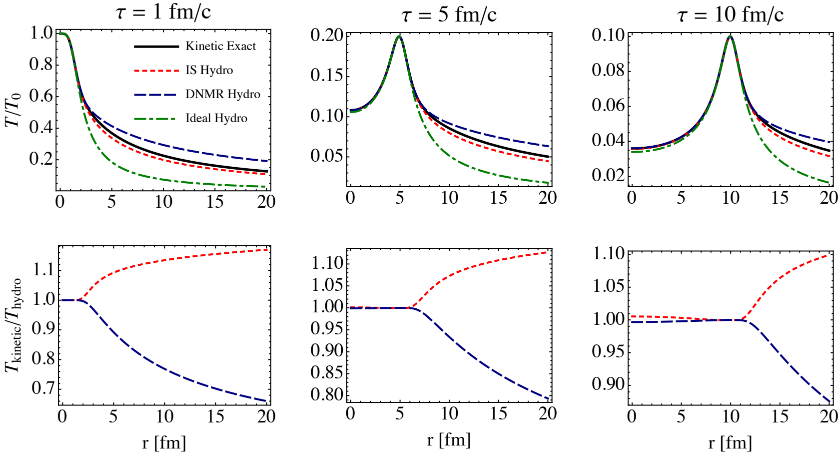
IS seems to work better than DNMR (!?)

Gubser flow IV: shear stress evolution in de Sitter time ρ



IS seems to work better than DNMR (!?)

Gubser flow V: temperature evolution in Minkowski space

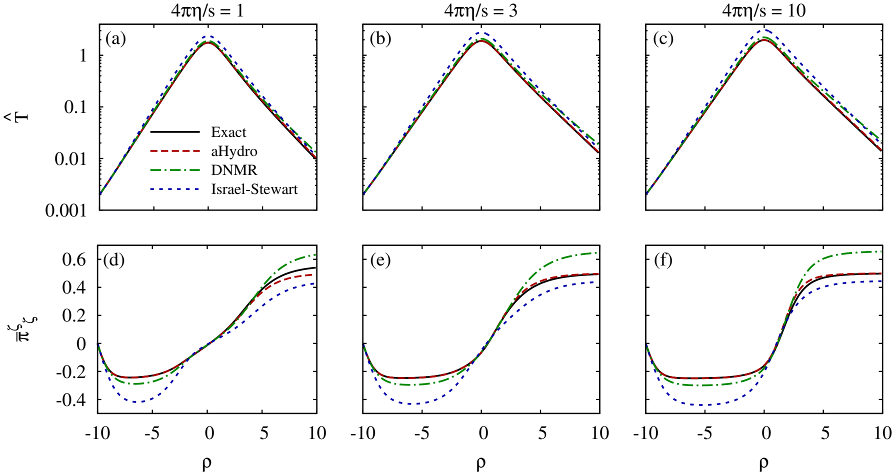


IS seems to work better than DNMR (!?)

Both seem to have problems at large $r \leftrightarrow$ large negative ρ

Gubser flow in aHydro: ρ-evolution of T and shear stress

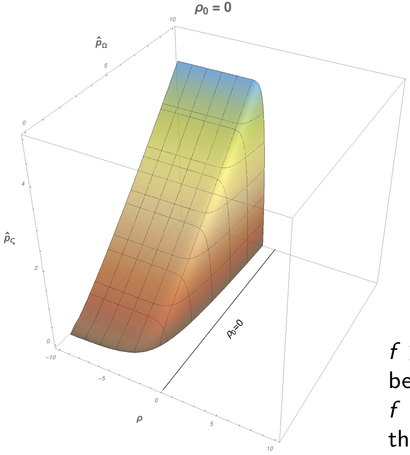
M. Nopoush, R. Ryblewski, M. Strickland, PRD 91 (2015) 045007

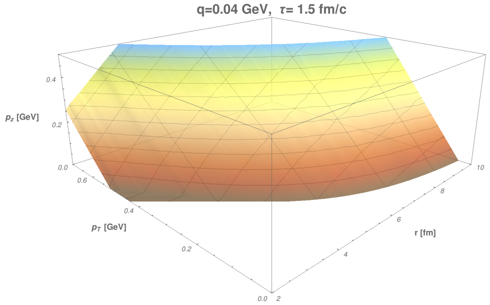


Thermal equil. initial conditions at $\rho_0 \to -\infty$. aHydro works better than IS & DNMR

Exact BE solution w/ Gubser flow: problems at $\rho - \rho_0 < 0$

At fixed $(\hat{p}_\Omega, w)$, $f(\rho; \hat{p}^2_\Omega, w)$ increases monotonically with ρ near $\rho_0 \Longrightarrow$ With thermal initial conditions at finite ρ_0, for some points in momentum space f eventually becomes negative for large enough negative $\rho - \rho_0$:





$f > 0$ (physical) above surface, $f < 0$ (unphysical) below surface.

f becomes unphysical at small $|p_z|$, large $p_\perp$, and the unphysical region grows with r.

Overview

Conclusions

- A new exact solution of the Boltzmann equation with a relaxation time collision term for systems undergoing Gubser flow enables **tests of hydrodynamic approximation schemes in situations that resemble heavy-ion collisions** where the created matter undergoes simultaneous longitudinal and transverse expansion.
- When compared with the exact solution, second-order viscous hydrodynamics (IS and DNMR) works better than NS theory, anisotropic hydrodynamics (aHydro) works better than hydrodynamic schemes based on an expansion around local mometum isotropy (IS and DNMR), and viscous anisotropic hydrodynamic (vaHydro) (which treats small viscous corrections as IS or DNMR but resums the largest viscous terms) outperforms aHydro.
  Performance hierarchy: vaHydro > aHydro > DNMR ∼ IS > NS > ideal fluid.
- When using the exact solution for such hydrodynamic tests, care must be taken to **avoid the region of large negative de Sitter times** (measured from the time of initialization) where the exact solution features negative distribution functions in part of momentum space.

The End