Exact solution to the RTA Boltzmann equation subject to Gubser flow

Michael Strickland
Kent State University
Kent, OH USA

Institute for Nuclear Theory, July 28 2015

Collaborators: D. Bazow, G. Denicol, W. Florkowski, U. Heinz, A. Jaiswal, E. Maksymiuk, M. Martinez, M. Nopoush, J. Noronha, and R. Ryblewski

References: NPA 916, 249 (2013), PRC 88, 024903 (2013), PRC 89, 054908 (2014), PRC 90, 044905 (2014), PRL 113, 202301 (2014), PRD 90, 125026 (2014), PRD 91, 045007 (2015)

Motivation I

- Want to have a quantitatively reliable understanding of soft-dynamics of the QGP that can be applied far from equilibrium
- Different causal formulations on the market:
    - Israel-Stewart
    - Grad-14
    - Chapman-Enskog

      } assume isotropic background
    - Anisotropic hydrodynamics } assume anisotropic background
    - ... which is best?
- Leads to systematic uncertainties in extraction of transport coefficients, etc.
- Could use data as the arbitrar, but it would be better to have exact solutions that could be used to test frameworks/codes


It’s important to calibrate your tools


The curvature of the earth must be much larger than we thought!?
It’s important to calibrate your tools

Motivation II

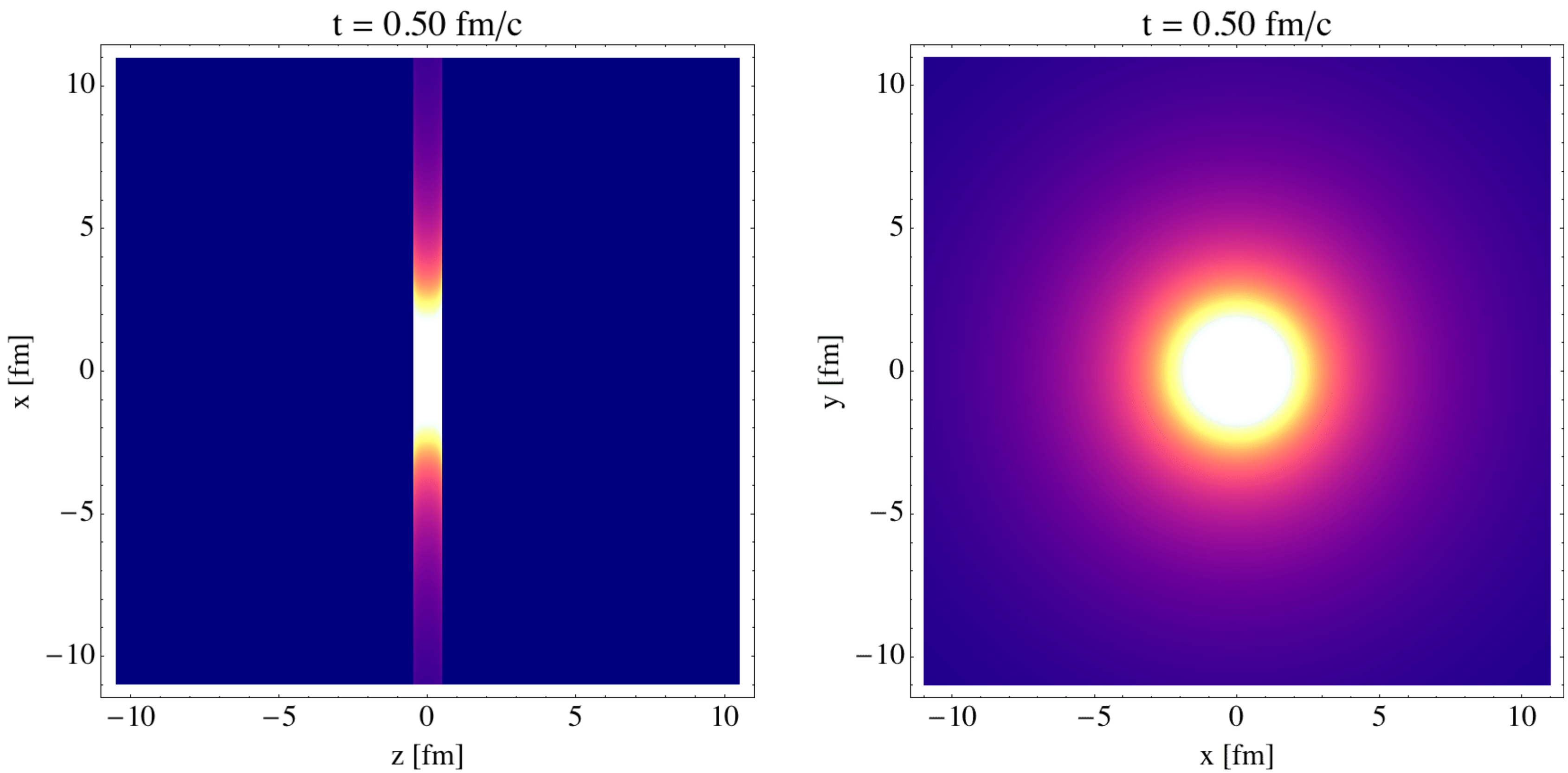
- In the past, approximate analytic “shock-tube” solutions in the non-expanding case have been used to test codes
- Would be nice to have exact solutions that include both strong transverse and longitudinal expansion
- Test case(s) should be close in spirit to the conditions generated dynamically in heavy ion collisions
- Minimally, we can target a system that is boost invariant and cylindrically symmetric → solutions will be functions of longitudinal proper time τ and radial coordinate r
- Of course, we would like to go beyond this, but for now even this would be a good starting point

Motivation III

- For this purpose we will exactly solve the Boltzmann equation with a simple collisional kernel (relaxation time approximation = RTA) subject to a class of non-trivial flows
- <u>The resulting exact solution holds for arbitrary shear viscosity to entropy density ratio</u>
- Allows us to span physical situations ranging from ideal hydrodynamics all the way to free streaming
- Can be applied to large and small systems
- Using the same starting point (RTA Boltzmann), we then derive the corresponding hydrodynamical equations of motion using hydro frameworks on the market

Results of the 1+1d kinetic solution

[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]



Visualization of the effective temperature

0+1d Solution

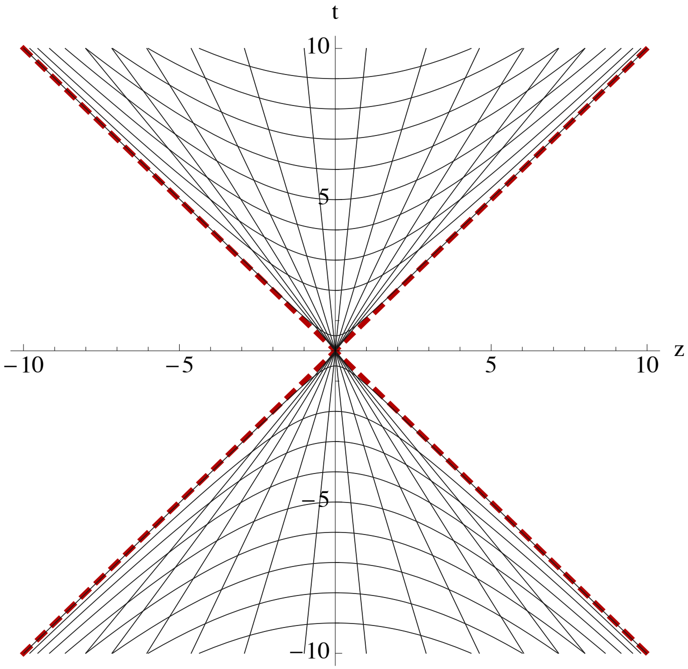
Bjorken Flow

Boost-invariant and transversally homogeneous flow → Bjorken flow

$$t = \tau \cosh \varsigma\,, \qquad z = \tau \sinh \varsigma\,,$$

Milne coordinates

$$\tau^2 = t^2 - z^2$$
$$\varsigma = \text{arctanh}(z/t)$$



Cartesian components

$$u^\mu = (\cosh \varsigma, 0, 0, \sinh \varsigma) = \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau}\right)$$

Milne components

$$(t, x, y, z) \to (\tau, x, y, \varsigma)$$

$$u_\tau = 1 \qquad u_\varsigma = 0$$

In Milne coordinates Bjorken flow maps to a static flow, i.e. all spacelike components of the four-velocity are vanishing

Simple kinetic equation

Relativistic Boltzmann equation

$$p^\mu \partial_\mu f(x,p) = C[f(x,p)]$$

Use RTA for the collisional kernel

$$C[f] = \frac{p \cdot u}{\tau_{\rm eq}} \left(f^{\rm eq} - f \right)$$

$\tau_{\rm eq}$ = relaxation time

For conformal RTA

$$\tau_{\rm eq}(\tau) = \frac{5\eta}{T\mathcal{S}}$$

Assume (for now) classical statistics

$$f^{\rm eq} = \exp\left(- \frac{p \cdot u}{T} \right)$$

Conformal (massless) thermodynamic vars

$$n_{\rm eq} = \frac{2 g_0 T^3}{\pi^2} \qquad \mathcal{S}_{\rm eq} = \frac{8 g_0 T^3}{\pi^2}$$

$$\mathcal{E}_{\rm eq} = \frac{6 g_0 T^4}{\pi^2} \qquad \mathcal{P}_{\rm eq} = \frac{2 g_0 T^4}{\pi^2}$$

In addition, assume (for now) that all particles are massless

See W. Florkowski and E. Maksymiuk, arXiv:1411.3666 for quantum statistics

Bjorken-flow symmetries

$$ISO(2) \times SO(1,1) \times \mathbf{Z}_2$$

transverse homogeneity	boost invariance	reflection symmetry around collision plane

- The one-particle distribution function is a Lorentz scalar
- It can only depend on longitudinal-boost-invariant variables $\tau,\, w \text{ and } \vec{p}_T$ with

$$w = tp_L - zE$$

- RTA Boltzmann equation simplifies dramatically

$$\boxed{\frac{\partial f}{\partial \tau} = \frac{f^{\rm eq} - f}{\tau_{\rm eq}}}$$

$$f^{\rm eq}(\tau, w, p_T) = \exp\left[-\frac{\sqrt{w^2 + p_T^2\tau^2}}{T(\tau)\tau}\right]$$

Solution for the distribution function

[W. Florkowski, R. Ryblewski, and MS, 1304.0665 and 1305.7234]

Solution for the distribution function is now straightforward to obtain

$$
\begin{aligned}
f(\tau, w, p_T) &= D(\tau, \tau_0) f_0(w, p_T) \\
&+ \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')} D(\tau, \tau')\, f^{\rm eq}(\tau', w, p_T)
\end{aligned}
$$

with the damping function D given by

$$
D(\tau_2, \tau_1) = \exp\left[- \int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{\rm eq}(\tau'')} \right] \qquad \text{with} \qquad \tau_{\rm eq}(\tau) = \frac{5\eta}{T\mathcal{S}}
$$

One could solve this 3d integral equation for f, but the problem can be reduced to solving a 1D integral equation by integrating the left and right to obtain the energy density

$$
\mathcal{E}(\tau) = \frac{g_0}{\tau^2} \int dP\, v^2\, f(\tau, w, p_T)
$$

$$
dP = 2\, d^4p\, \delta\left(p^2 - m^2\right) \theta(p^0)/(2\pi)^3 = \frac{dw}{v} d^2p_T
$$

$$
v \equiv Et - p_L z = \sqrt{w^2 + p_T^2 \tau^2}
$$

Energy density integral equation

[W. Florkowski, R. Ryblewski, and MS, 1304.0665 and 1305.7234]

$$\bar{\mathcal{E}}(\tau) = D(\tau,\tau_0)\,\mathcal{R}\big(\xi_{\rm FS}(\tau)\big)/\mathcal{R}\left(\xi_0\right) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{\rm eq}(\tau')}\, D(\tau,\tau')\,\bar{\mathcal{E}}(\tau')\,\mathcal{R}\left(\left(\frac{\tau}{\tau'}\right)^2 - 1\right)$$

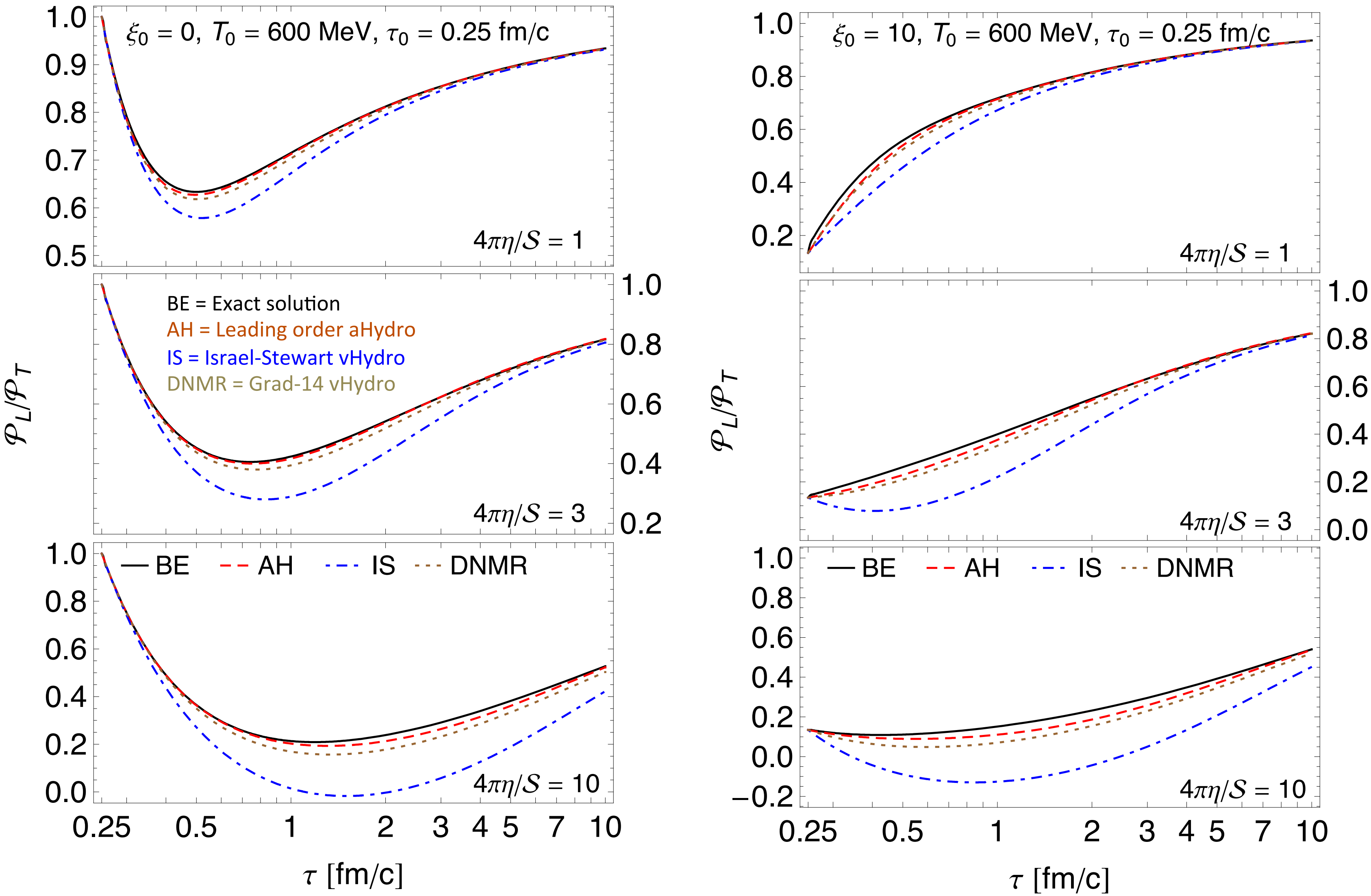
$$\xi_{\rm FS}(\tau) = (1+\xi_0)(\tau/\tau_0)^2 - 1 \qquad \bar{\mathcal{E}} = \mathcal{E}/\mathcal{E}_0$$

$$\mathcal{R}(z) = \frac{1}{2}\left[(1+z)^{-1} + \arctan\big(\sqrt{z}\big)/\sqrt{z}\right]$$

- Allows for arbitrarily momentum-space anisotropic initial condition
- Can be solved by guessing proper-time dependence of the energy density and then iterating until convergence is achieved
- Once the energy density profile is obtained, this can be used to determine the effective temperature, all other components of the energy-momentum tensor, number density, and the full one-particle distribution function

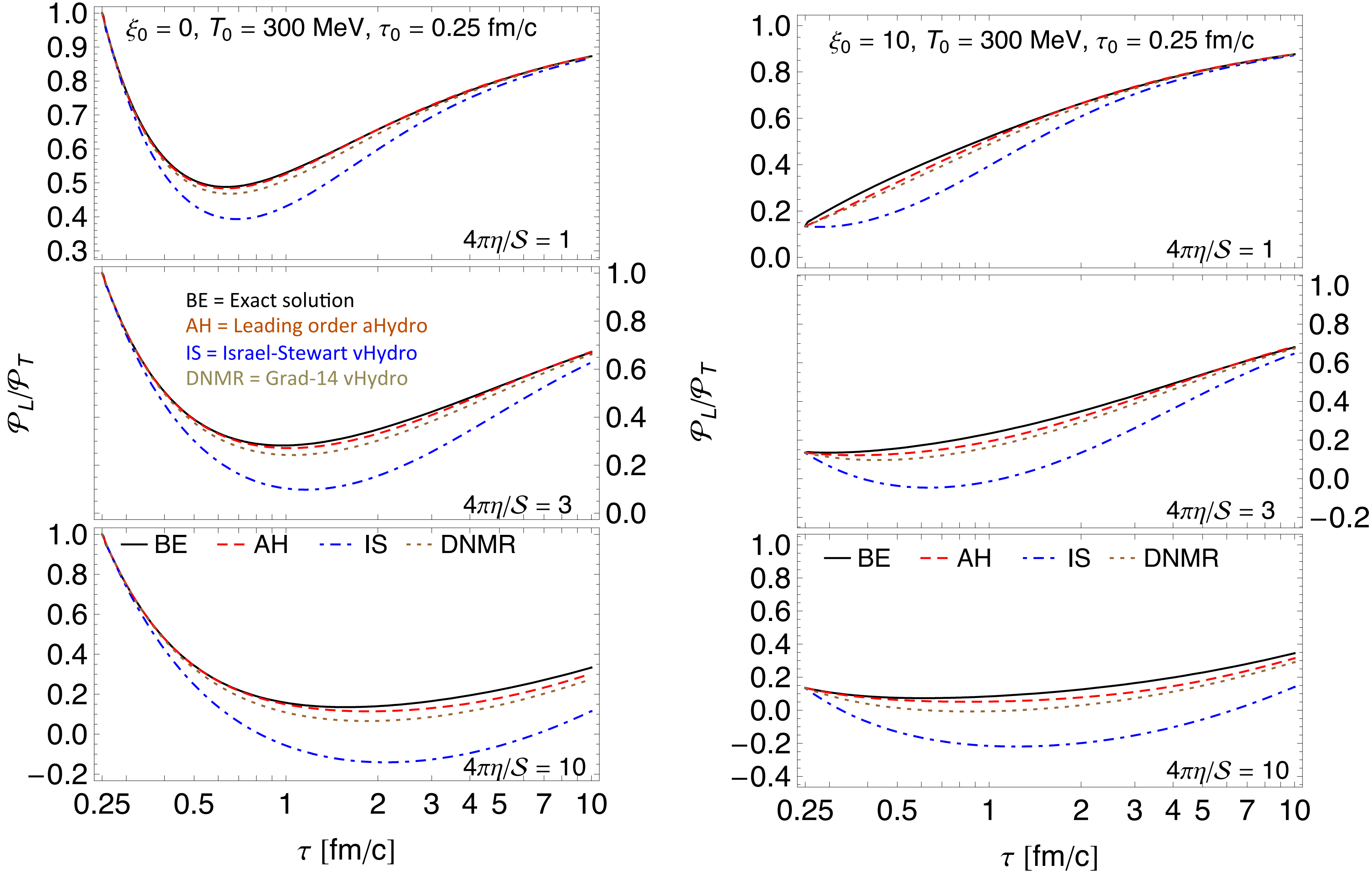
Conformal 0+1d results

[W. Florkowski, R. Ryblewski, and MS, 1304.0665 and 1305.7234]



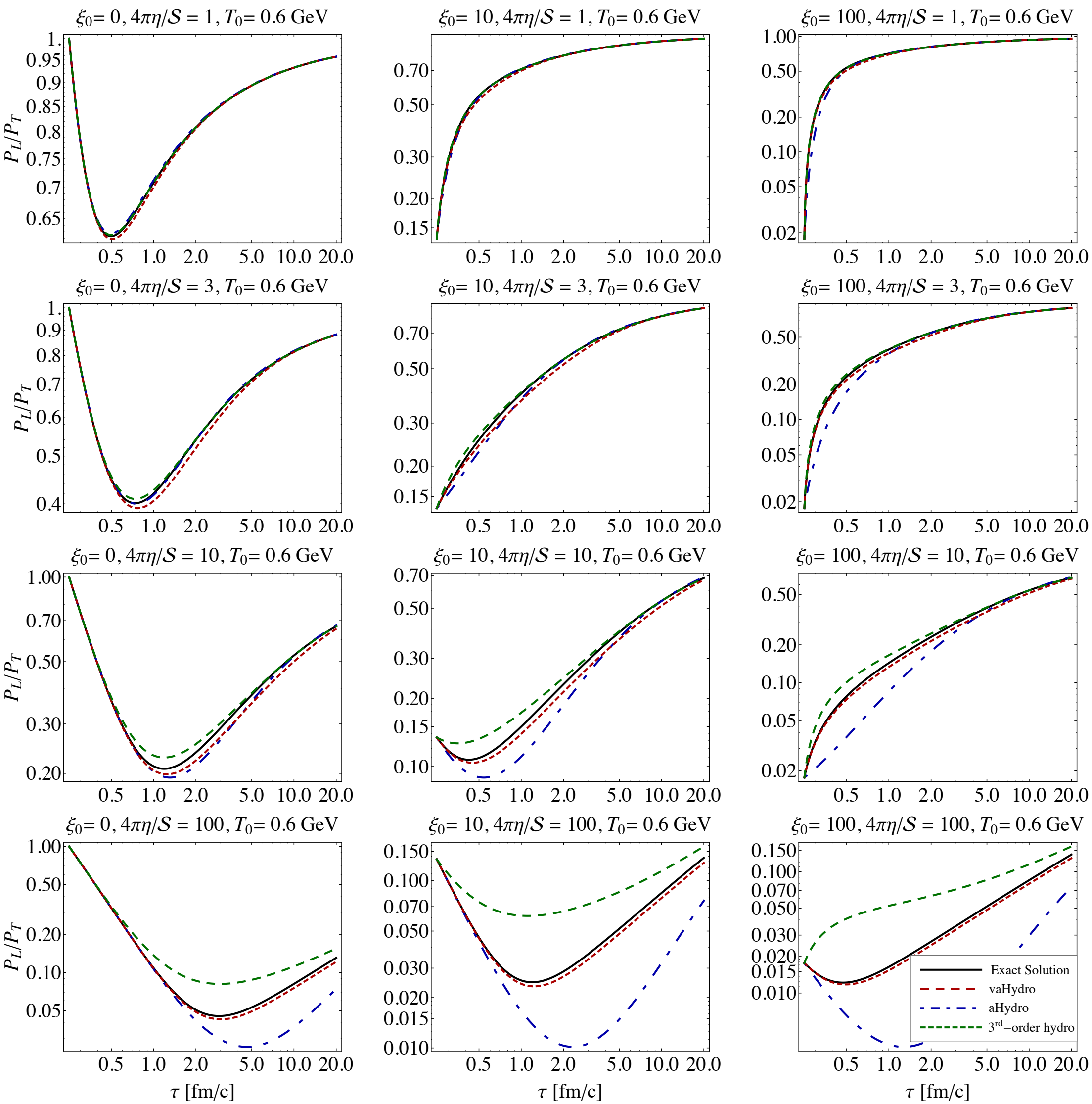
Conformal 0+1d results

[W. Florkowski, R. Ryblewski, and MS, 1304.0665 and 1305.7234]



Conformal 0+1d results - vaHydro

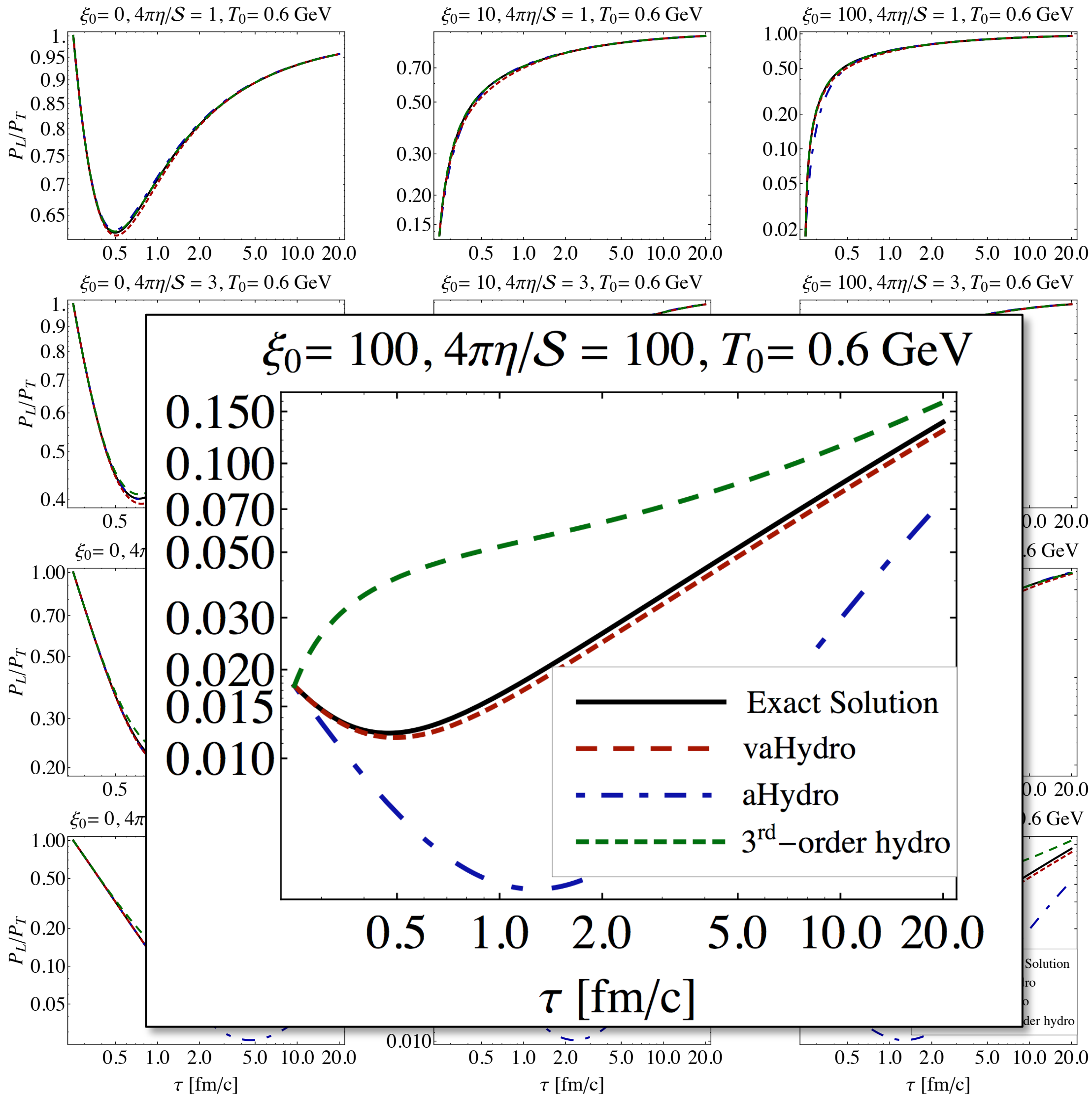
[D. Bazow, U. Heinz, and MS, 1311.6720]



- Second-order aHydro ("vaHydro") further improves the agreement with the exact solutions
- Shown on the left is the pressure anisotropy is quite sensitive to the framework used
- Same conclusion obtained for all observables studied
- Note that I don't even show Israel-Stewart any longer, only 3rd order viscous hydro of Jaiswal

Conformal 0+1d results

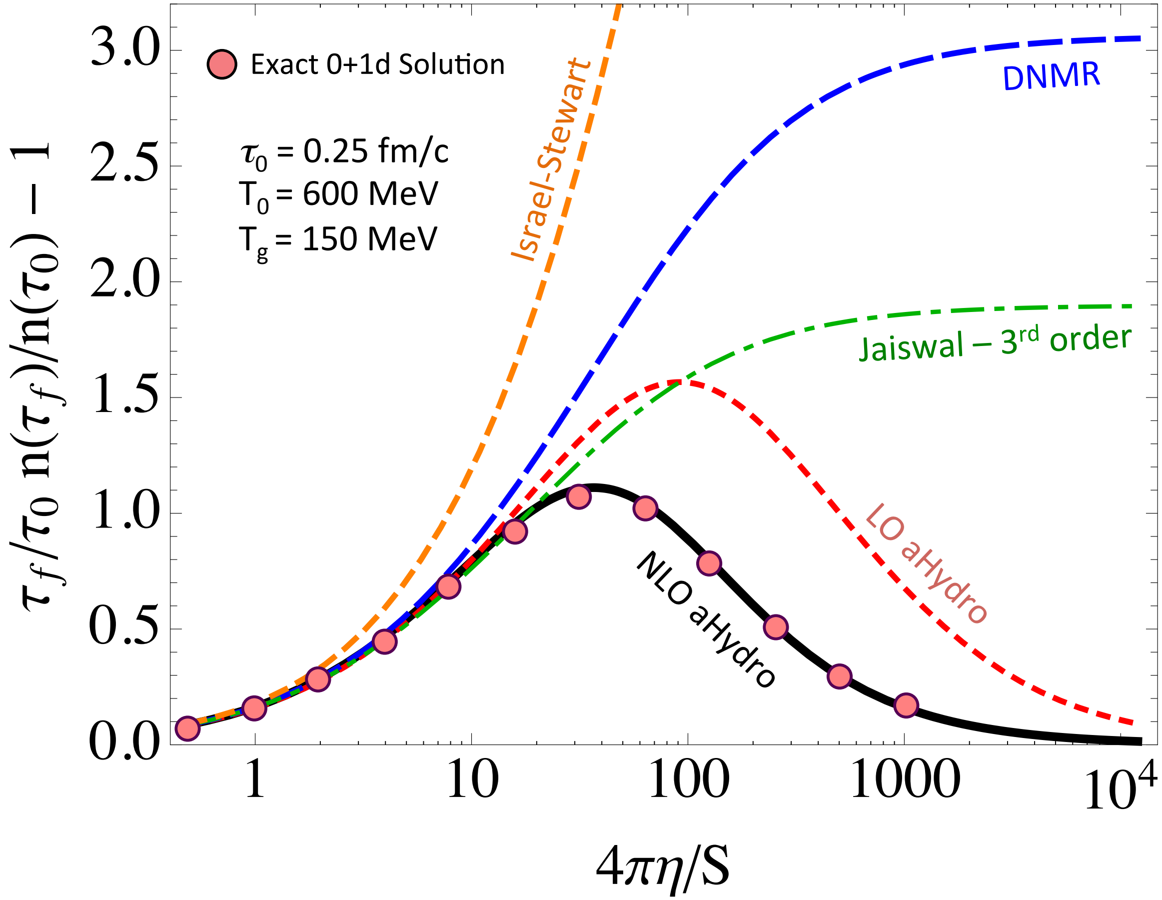
[D. Bazow, U. Heinz, and MS, 1311.6720]



- Second-order aHydro (“vaHydro”) further improves the agreement with the exact solutions
- Shown on the left is the pressure anisotropy is quite sensitive to the framework used
- Same conclusion obtained for all observables studied
- Note that I don’t even show Israel-Stewart any longer, only 3rd order viscous hydro of Jaiswal

Conformal 0+1d results

[D. Bazow, U. Heinz, and MS, 1311.6720]

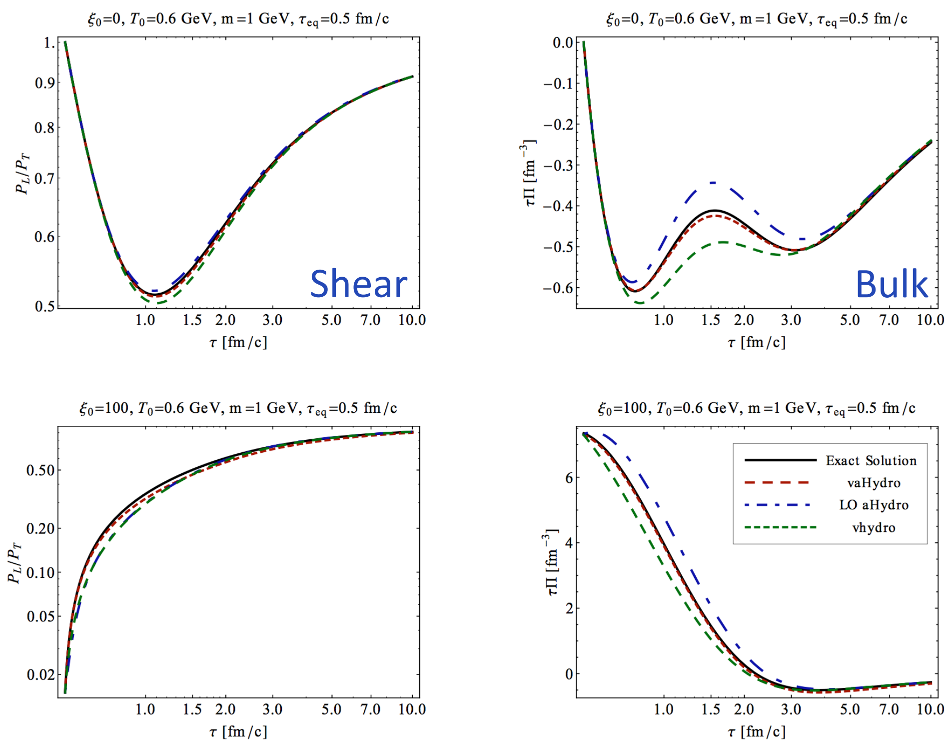


- Number (entropy) production vanishes in two limits: ideal hydrodynamic and free streaming limits
- In the conformal model which we are testing with, number density is proportional to entropy density

Non-conformal (massive) gas

[W. Florkowski, E. Maksymiuk, R. Ryblewski, and MS, 1402.7348]

$$2m^2T(\tau)\left[3T(\tau)K_2\left(\frac{m}{T(\tau)}\right)+mK_1\left(\frac{m}{T(\tau)}\right)\right]$$
$$=D(\tau,\tau_0)\Lambda_0^4\tilde{\mathcal{H}}_2\left[\frac{\tau_0}{\tau\sqrt{1+\xi_0}},\frac{m}{\Lambda_0}\right]+\int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{\rm eq}}D(\tau,\tau')T^4(\tau')\tilde{\mathcal{H}}_2\left[\frac{\tau'}{\tau},\frac{m}{T(\tau')}\right]$$



- Can use a same method to obtain the exact solution for a massive gas
- Allows one to assess different methods for inclusion of bulk viscous effects
- The overarching conclusion is that it is important to include shear-bulk couplings (Israel-Stewart fails completely).

M. Nopoush, R. Ryblewski, and MS, 1405.1355
G.S. Denicol, W. Florkowski, R. Ryblewski, and MS, 1407.4767
A. Jaiswal, R. Ryblewski, and MS, 1407.7231

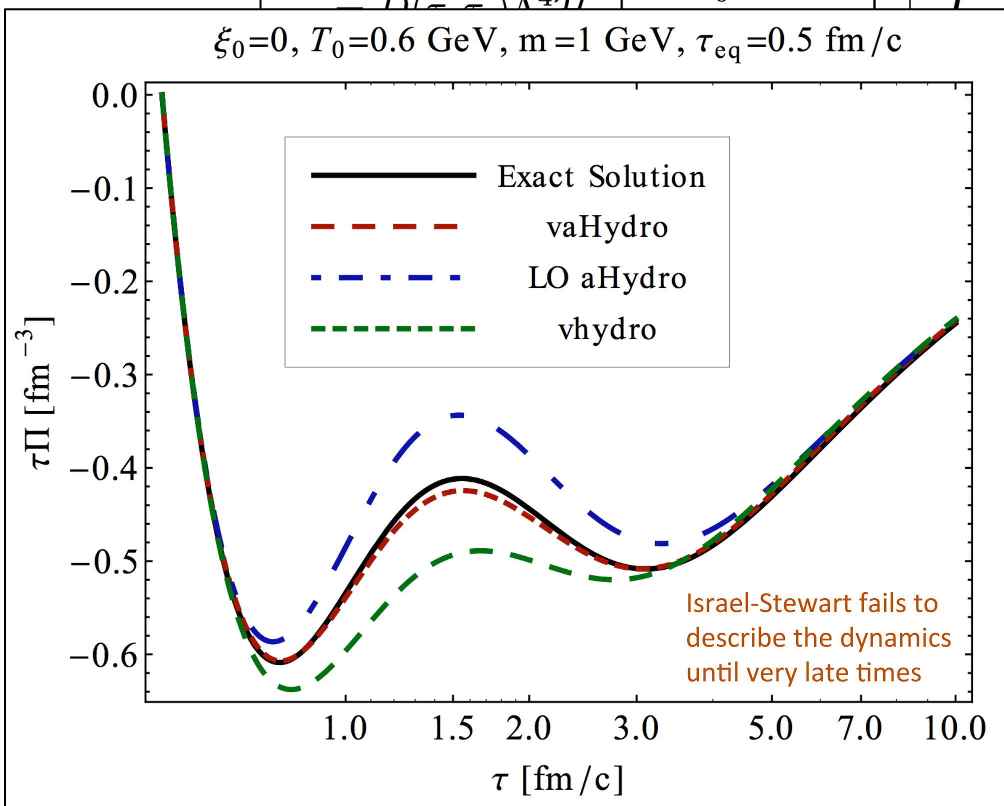
- Plots on the left show recent results of Bazow, Heinz, and Martinez 1503.07443

Non-conformal (massive) gas

[W. Florkowski, E. Maksymiuk, R. Ryblewski, and MS, 1402.7348]

$$2m^2T(\tau)\left[3T(\tau)K_2\left(\frac{m}{T(\tau)}\right)+mK_1\left(\frac{m}{T(\tau)}\right)\right]$$

$$\int^{\tau}\frac{d\tau'}{\tau_{eq}}D(\tau,\tau')T^4(\tau')\tilde{\mathcal{H}}_2\left[\frac{\tau'}{\tau},\frac{m}{T(\tau')}\right]$$



aHydro = M. Nopoush, R. Ryblewski, and MS, 1405.1355
vaHydro = D. Bazow, U. Heinz, and M. Martinez, 1503.07443
vHydro = G.S. Denicol, W. Florkowski, R. Ryblewski, and MS, 1407.4767

- Can use a same method to obtain the exact solution for a massive gas
- Allows one to assess different methods for inclusion of bulk viscous effects
- The overarching conclusion is that it is <u>important to include shear-bulk couplings</u> (Israel-Stewart fails completely).

M. Nopoush, R. Ryblewski, and MS, 1405.1355
G.S. Denicol, W. Florkowski, R. Ryblewski, and MS, 1407.4767
A. Jaiswal, R. Ryblewski, and MS, 1407.7231

- Plots on the left show recent results of Bazow, Heinz, and Martinez 1503.07443

1+1d Conformal Solution

Gubser Flow

[S. Gubser, 1006.0006;
S. Gubser and Y.Yarom, 1012.1314]

Gubser flow is a cylindrically-symmetric and boost-invariant flow that possesses a high degree of symmetry when mapped to Weyl-rescaled deSitter space

$$SO(3)_q \times SO(1,1) \times Z_2$$

$SO(3)_q$	$SO(1,1)$	Z_2
Related to rotational symmetry around beam axis	boost invariance	reflection symmetry around collision plane

The parameter q above is an arbitrary energy scale that sets the radial extent of the system at a given proper time.

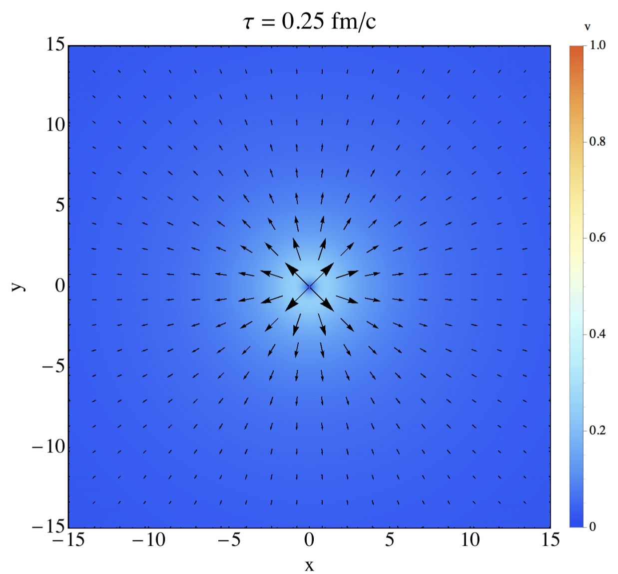
Polar Milne components

$$\tilde{u}^\tau = \cosh(\theta_\perp)$$
$$\tilde{u}^r = \sinh(\theta_\perp)$$
$$\tilde{u}^\phi = 0$$
$$\tilde{u}^\varsigma = 0$$

Transverse rapidity

$$\theta_\perp = \tanh^{-1}\left(\frac{2q^2\tau r}{1+q^2\tau^2+q^2r^2}\right)$$

This flow is quite strong:
The velocity gradients grow exponentially in time!



Weyl-rescaled de Sitter Coordinates

- Conformal theories are invariant under arbitrary Weyl transformations
- An (m,n) tensor with canonical dimension Δ transforms as

$$Q^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n}(x) \;\to\; \Omega^{\Delta+m-n} Q^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n}(x)$$

  where Ω is an arbitrary function of spacetime
- For the problem at hand we take $\Omega = \tau$ → "Weyl-rescaling"
- Next we transform to de Sitter coordinates defined via

$$\sinh\rho = -\frac{1 - q^2\tau^2 + q^2 r^2}{2q\tau}$$
$$\tan\theta = \frac{2qr}{1 + q^2\tau^2 - q^2 r^2}$$

- ρ = de Sitter "time"
- $SO(3)_q$ symmetry guarantees that, in the end, physical observables will not depend on θ
- Azimuthal angle and spatial rapidity are the same as in Minkowski coords

Weyl-rescaled de Sitter Coordinates

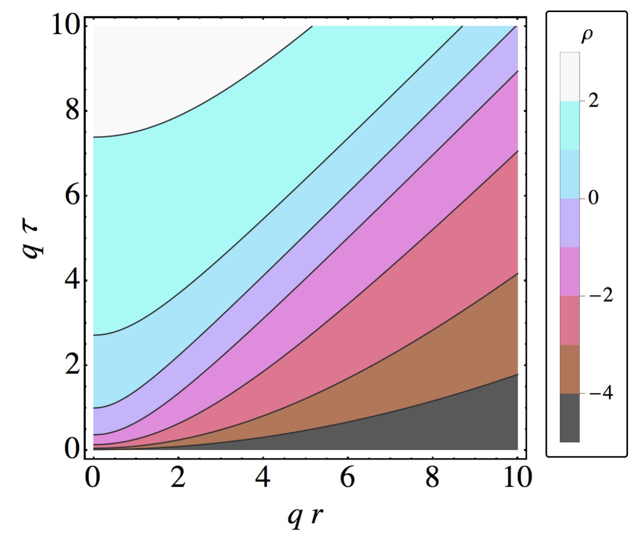
$$\boxed{dS_3 \times \mathbf{R}} \qquad \hat{g}_{\mu\nu} = \frac{1}{\tau^2}\frac{\partial x^\alpha}{\partial \hat{x}^\mu}\frac{\partial x^\beta}{\partial \hat{x}^\nu} g_{\alpha\beta} \qquad \boxed{\mathbf{R}^{3,1}}$$

$$\hat{g}_{\mu\nu} = \text{diag}(-1,\, \cosh^2\!\rho,\, \cosh^2\!\rho \sin^2\!\theta,\, 1)$$

$$d\hat{s}^2 = -d\rho^2 + \cosh^2\rho \underbrace{\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}_{SO(3)_q} + d\varsigma^2$$

$$\sinh\rho = -\frac{1 - q^2\tau^2 + q^2 r^2}{2q\tau}$$

$$\tan\theta = \frac{2qr}{1 + q^2\tau^2 - q^2 r^2}$$



Polar Milne components

$$\tilde{u}^\tau = \cosh(\theta_\perp)$$
$$\tilde{u}^r = \sinh(\theta_\perp)$$
$$\tilde{u}^\phi = 0$$
$$\tilde{u}^\varsigma = 0$$

After Weyl rescaling and coordinate transformation the Gubser flow four-velocity is static!

$$\longrightarrow \quad \hat{u}^\mu = \tau \frac{\partial \hat{x}^\mu}{\partial x^\nu} u^\nu \quad \longrightarrow \quad \hat{u}^\mu = (1,\, 0,\, 0,\, 0)$$

de Sitter space flow velocity

[S. Gubser, 1006.0006;
S. Gubser and Y.Yarom, 1012.1314]

Sketch of the 1+1d kinetic solution

[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]

- Start with the RTA Boltzmann equation subject to Gubser flow
- Make a Weyl-rescaling (homogeneous transformation of RTA Boltzmann eq.) + coord. transformation of the kinetic equation
- Use the fact that the distribution function can only depend on $SO(3)_q$ x SO(1,1) x Z_2 invariants

$SO(3)_q$ invariance $\longrightarrow$ $\hat{p}_\Omega^2 = \hat{p}_\theta^2 + \frac{\hat{p}_\phi^2}{\sin^2\theta}$

SO(1,1) invariance $\longrightarrow$ $\hat{p}_\varsigma$ (related to the w variable from 0+1d solution)

Z_2 $\longrightarrow$ $\varsigma \to -\varsigma$ Reflection symmetry

$$f(\hat{x}^\mu, \hat{p}_i) \longrightarrow f(\rho; \hat{p}_\Omega^2, \hat{p}^\varsigma)$$

Sketch of the 1+1d kinetic solution

[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]

- For a conformal system the relaxation time must be proportional to the inverse temperature (no other scale)

$$\tau_{\rm eq} = \frac{c}{T}$$

For RTA kernel $c = 5\eta/\mathcal{S}$

- This gives

$$\frac{\partial}{\partial\rho} f(\rho; \hat{p}_\Omega^2, \hat{p}_\varsigma) = -\frac{\hat{T}(\rho)}{c}\left[f(\rho; \hat{p}_\Omega^2, \hat{p}_\varsigma) - f_{\rm eq}\Big(\hat{p}^\rho/\hat{T}(\rho)\Big)\right]$$

with $\hat{p}^\rho = \sqrt{\frac{\hat{p}_\Omega^2}{\cosh^2\rho} + \hat{p}_\varsigma^2}$ (mass shell constraint)

- This looks exactly like the Bjorken-flow problem solved previously!

Sketch of the 1+1d kinetic solution

[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]
[M. Nopoush, R. Ryblewski, and MS, 1410.6790]

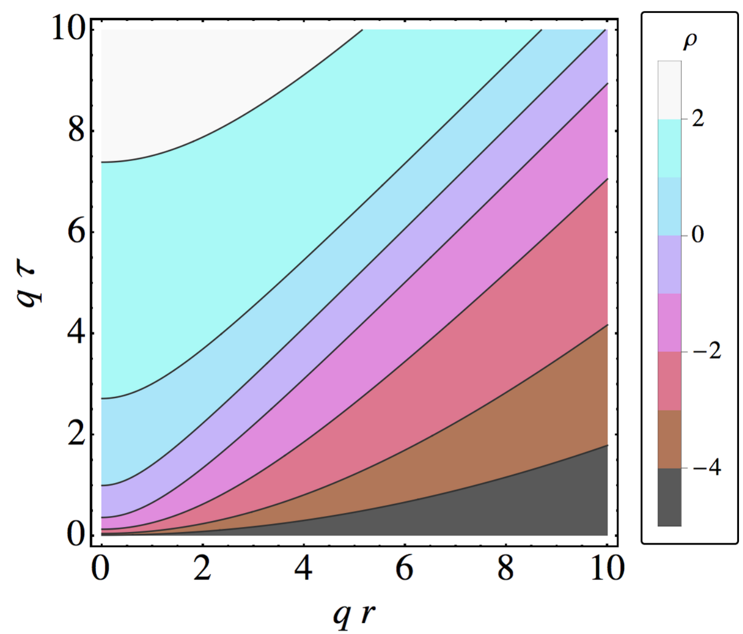
- As before, we can turn this into a 1d integral equation for the energy density and, once that it is solved, we can determine all components of the energy-momentum tensor and the full distribution function

$$\hat{\varepsilon}(\rho) = D(\rho,\rho_0)\hat{\varepsilon}_{\rm FS} + \frac{3}{\pi^2 c}\int_{\rho_0}^{\rho} d\rho' D(\rho,\rho')\mathcal{H}_\varepsilon\left(\frac{\cosh\rho'}{\cosh\rho}\right)\hat{T}^5(\rho')$$

$$\mathcal{H}_\varepsilon(x) \equiv \frac{x^2}{2} + \frac{x^4}{2}\frac{\tanh^{-1}\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

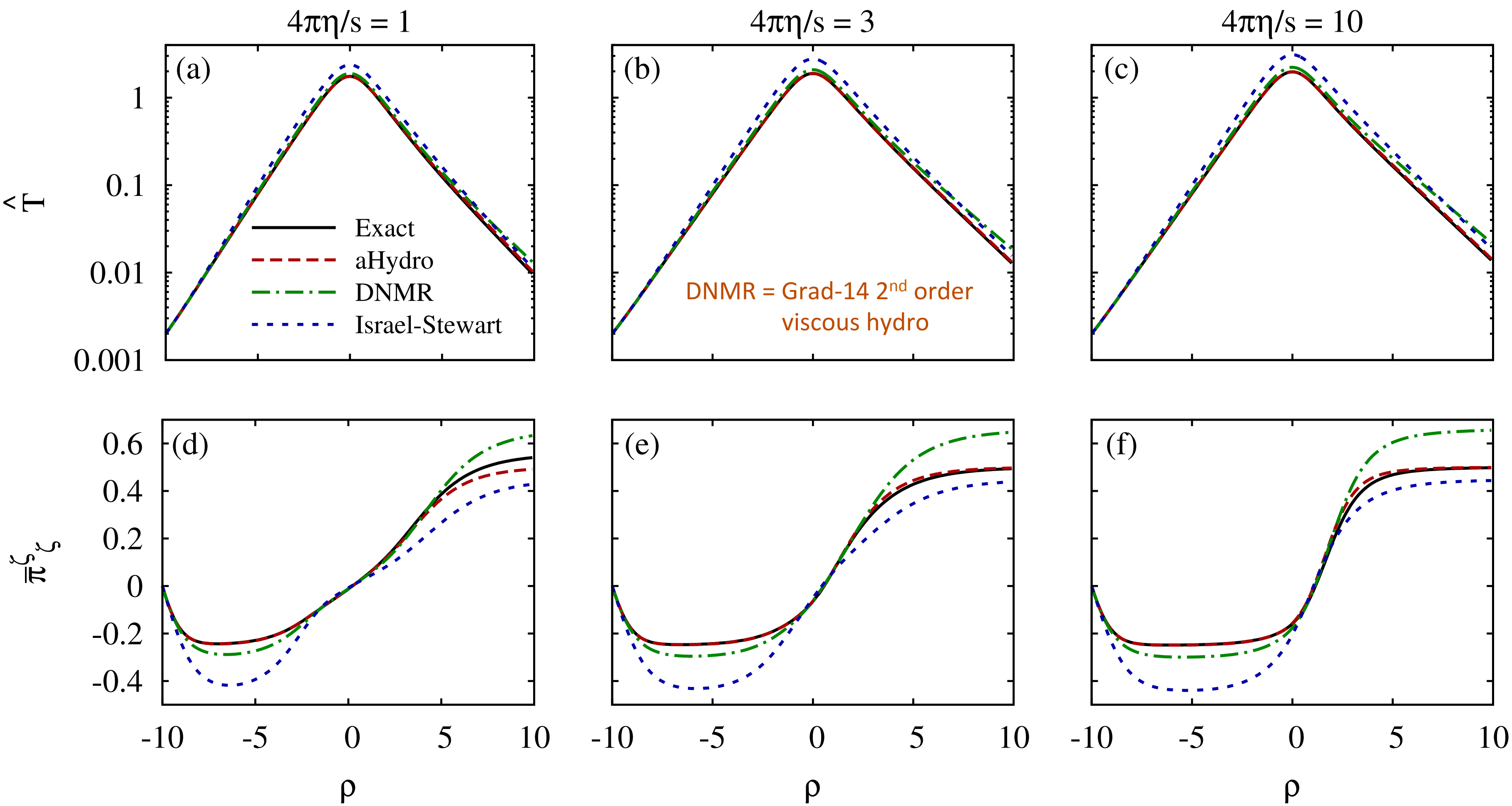
- For heavy ion application, the initial value for the de Sitter space energy density should be provided at $\rho_0 \rightarrow -\infty$ which maps to $\tau_0 \rightarrow 0^+$

- I will show results for $\rho_0 = -10$ which, for $q = 1$, maps to $\tau_0 < 5 \times 10^{-4}$ fm/c



Results of the 1+1d kinetic solution

[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]
[M. Nopoush, R. Ryblewski, and MS, 1410.6790]



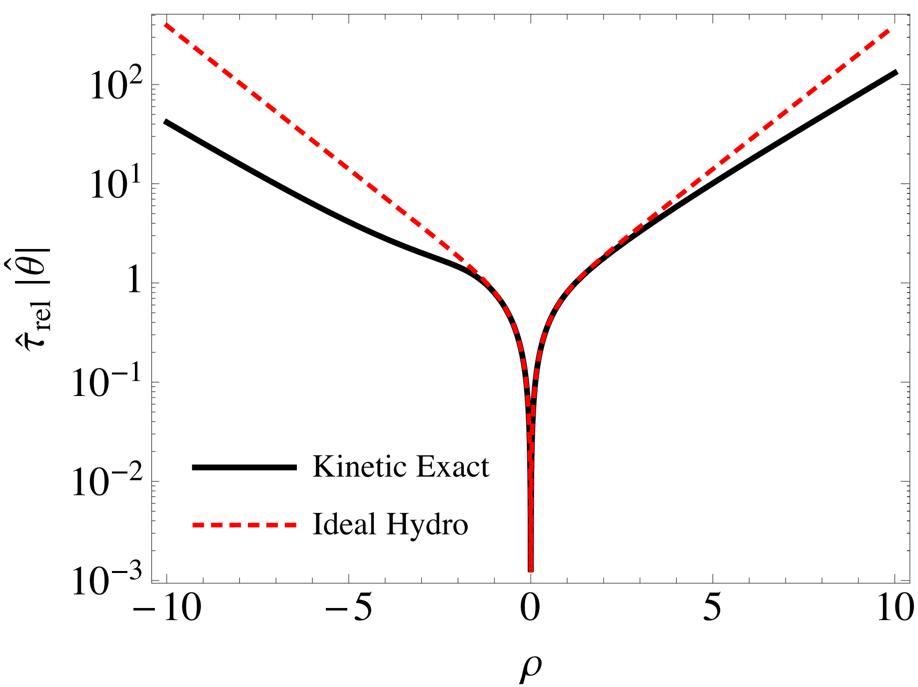
Isotropic initial conditions

Why is this nontrivial?

Knudsen number in de Sitter coordinates

$$\text{Kn} = \hat{\tau}_{\text{micro}}/\hat{\tau}_{\text{macro}} = \hat{\tau}_{\text{rel}}\,|\hat{\theta}| \equiv \underbrace{\hat{\tau}_{\text{rel}}}_{c/\hat{T}}\,\underbrace{|\hat{\nabla}\cdot\hat{u}|}_{2\tanh(\rho)}$$

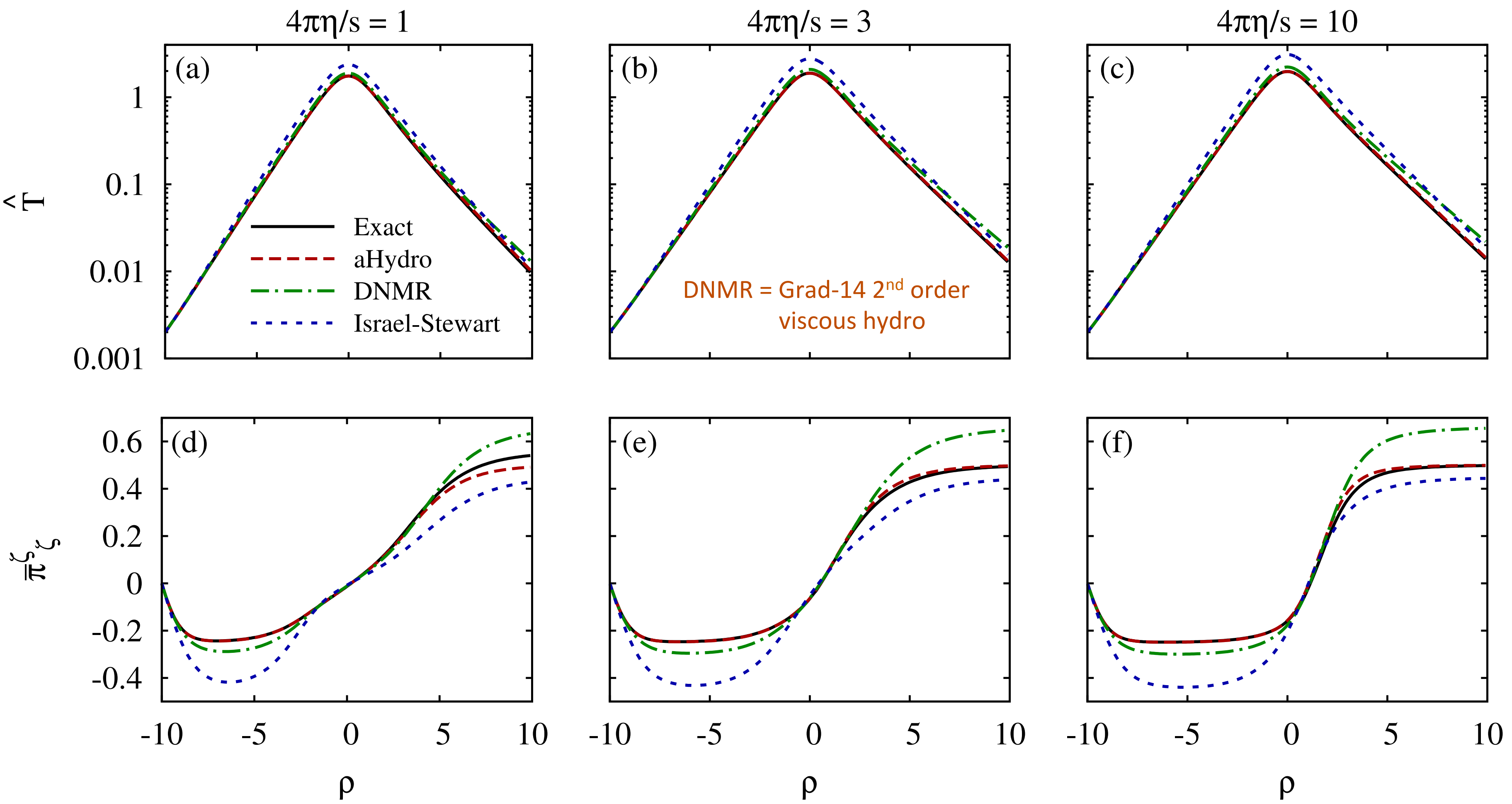
$4\pi\eta/s = 1 \quad \rho_0 = 0 \quad \hat{\mathcal{E}}(\rho_0) = 1$



Exponentially large gradients at early and late de Sitter times!

Results of the 1+1d kinetic solution

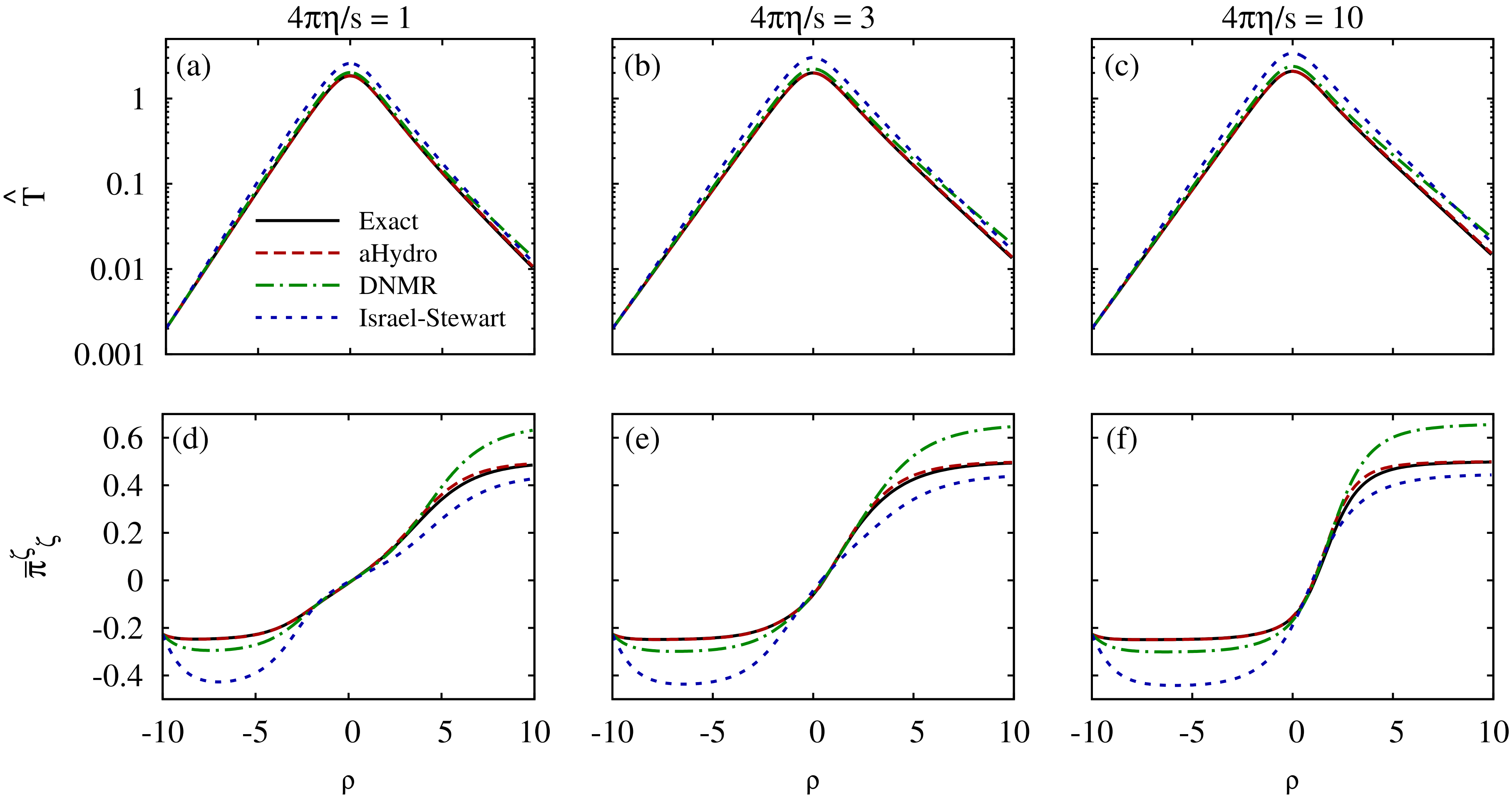
[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]
[M. Nopoush, R. Ryblewski, and MS, 1410.6790]



Isotropic initial conditions

Results of the 1+1d kinetic solution

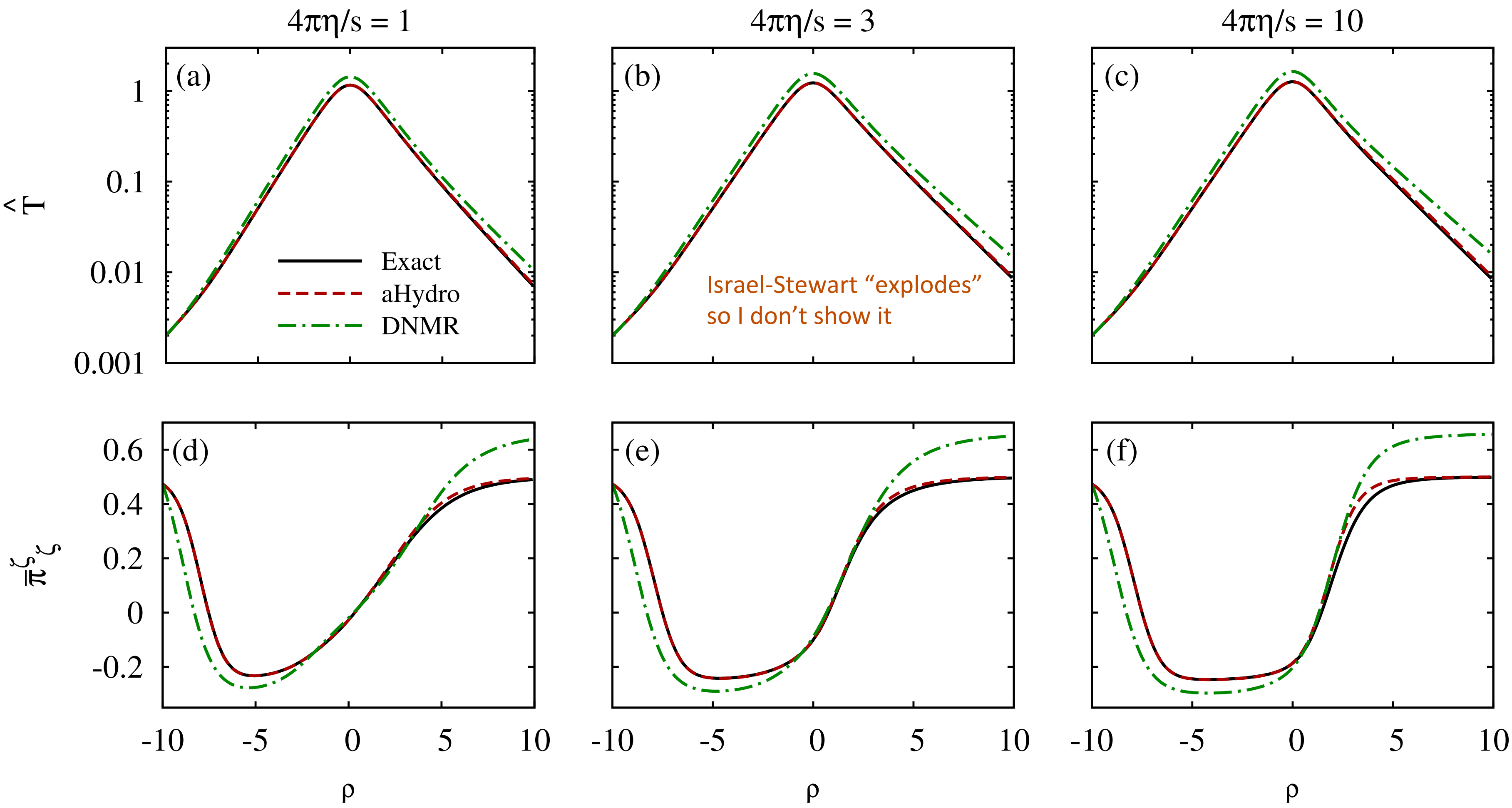
[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]
[M. Nopoush, R. Ryblewski, and MS, 1410.6790]



Oblate ($P_{L,0}$ / $P_{T,0}$ << 1) initial conditions

Results of the 1+1d kinetic solution

[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]
[M. Nopoush, R. Ryblewski, and MS, 1410.6790]

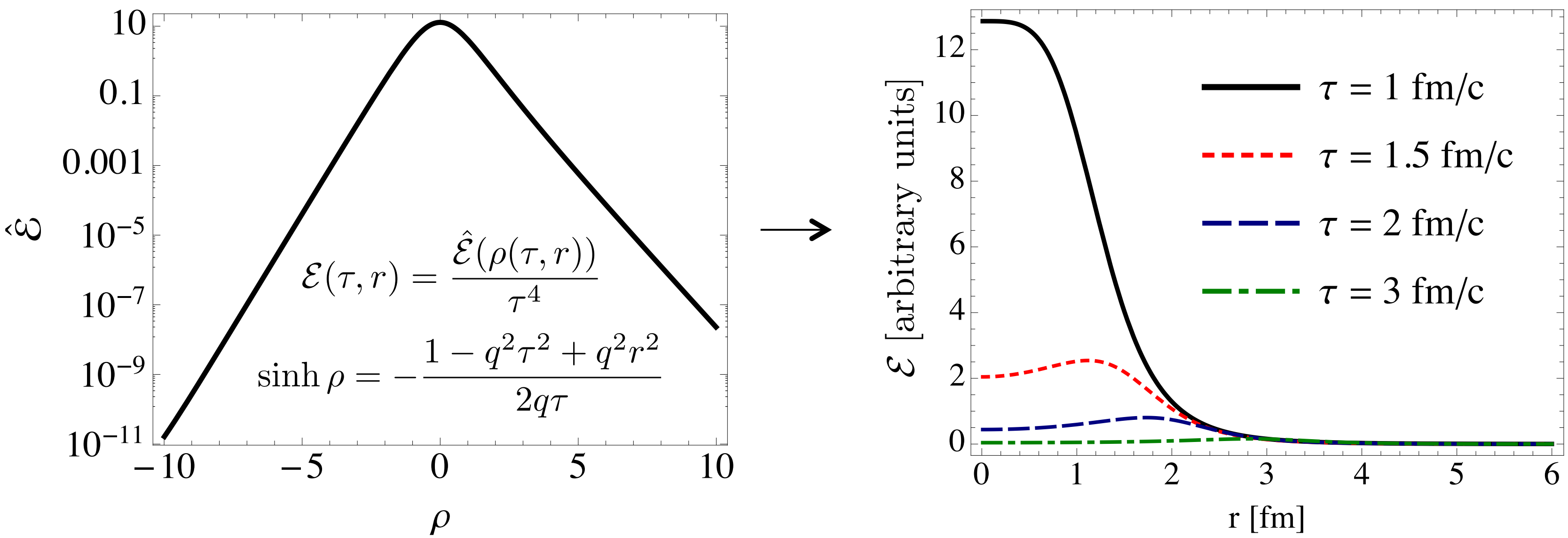


Prolate ($P_{L,0}$ / $P_{T,0}$ >> 1) initial conditions

Results of the 1+1d kinetic solution

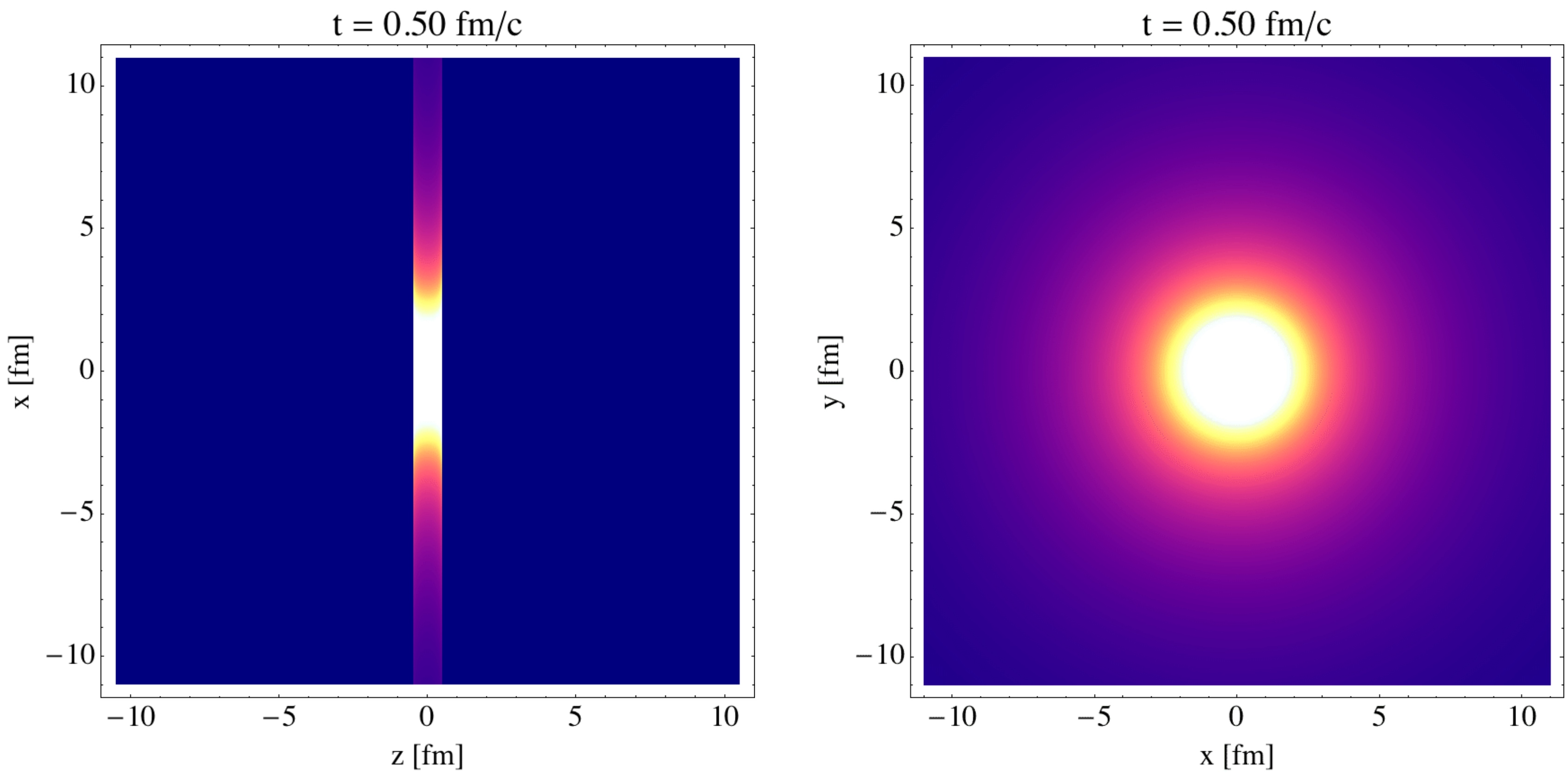
[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]

- Results are not very easy to interpret intuitively, so let's map back them back to Minkowski space by reversing the Weyl-rescaling and coordinate transformation, e.g.



Results of the 1+1d kinetic solution

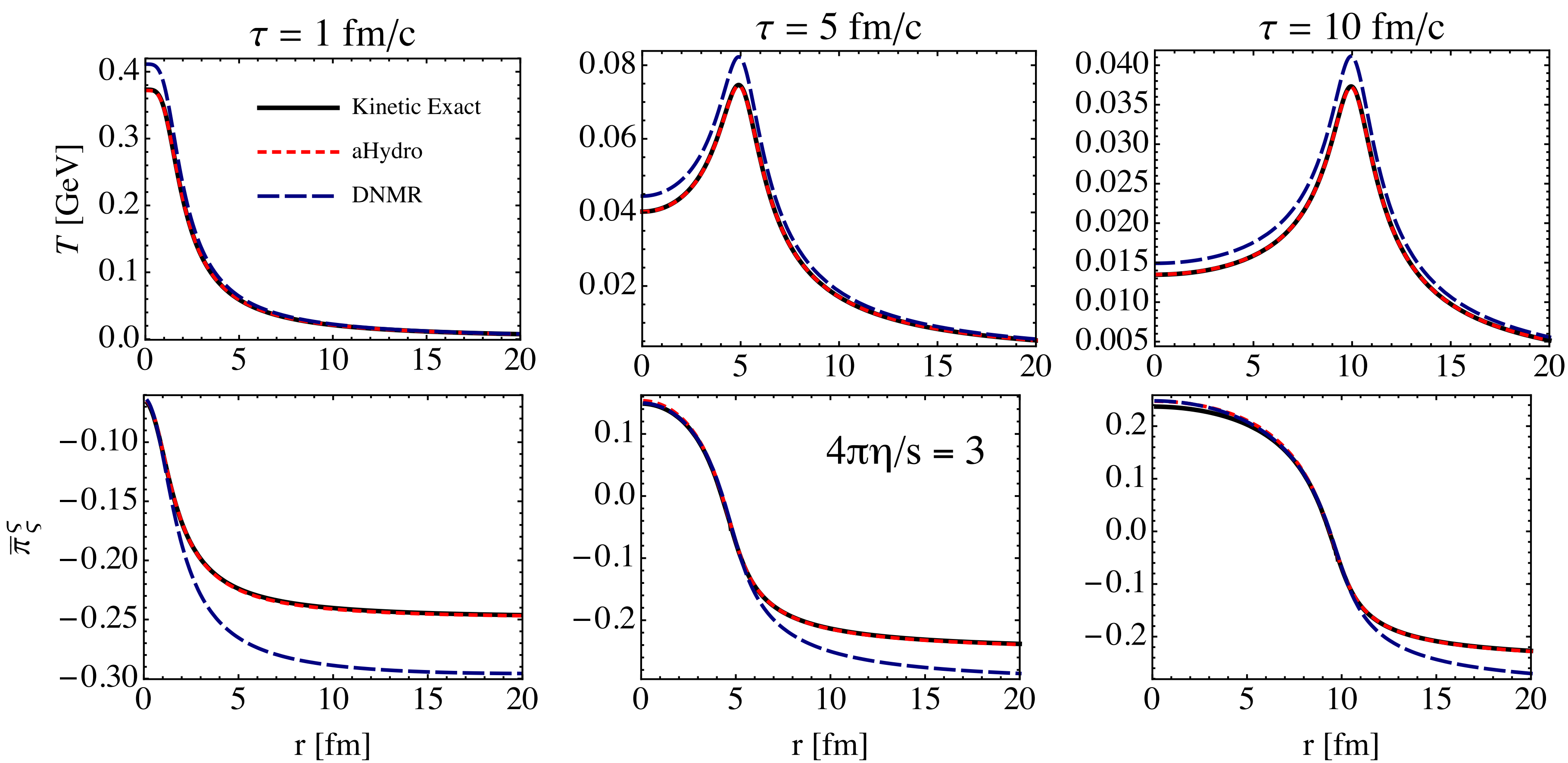
[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]



Visualization of the effective temperature

Results of the 1+1d kinetic solution

[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]
[M. Nopoush, R. Ryblewski, and MS, 1410.6790]



Conclusions

- All methods (exact solution and hydro formulations) agree qualitatively → momentum-space anisotropy, etc.
- Israel-Stewart (IS) equations are the worst approximation overall.
- Grad-14 and Jaiswal's Chapman-Enskog-like method work much better than IS for the conformal case and, if one includes the shear-bulk couplings, they also work reasonably well in the non-conformal case.
- Anisotropic hydrodynamics (aHydro and vaHydro) worked the best in all cases examined.
- aHydro can describe systems ranging from the ideal hydro limit to the free streaming limit and automatically includes higher-order couplings, e.g. shear-bulk couplings.

Results of the 1+1d kinetic solution

[G. Denicol, U.W. Heinz, M. Martinez, J. Noronha, and MS, 1408.5646 and 1408.7048]

Gives exact solution in the forward light cone.
Below I show the solution for the scaled shear correction.