Resumming large radiative corrections in the high-energy evolution of the Color Glass Condensate

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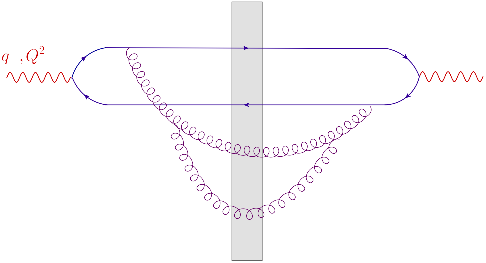
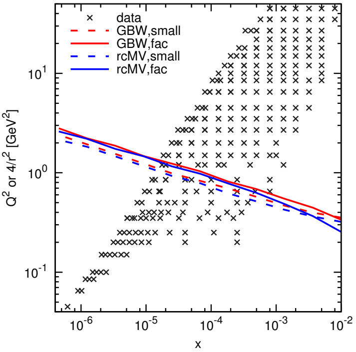




Introduction

- The Color Glass Condensate effective theory is the pQCD description of the wavefunction of a very energetic hadron
  - non-linear effects associated with the high gluon density
- Via appropriate factorization schemes, it controls various high-energy processes: DIS, proton-nucleus, nucleus-nucleus (initial conditions)
- "Effective theory" : high-energy evolution described by pQCD
  - Balitisy-JIMWLK hierarchy ≈ BK equation (at large N_c)
  - non-linear generalizations of the BFKL equation
- The non-linear evolution has recently been promoted to NLO
  - running coupling corrections to BK *(Kovchegov, Weigert; Balitsky, 07)*
  - full NLO version of the BK equation *(Balitsky, Chirilli, 08)*
  - JIMWLK evolution at NLO *(Kovner, Lublinsky, Mulian, 2013)*
- Large corrections enhanced by double or single transverse logarithms

Introduction

- Similar corrections were previously identified in the context of NLO BFKL *(Fadin, Kotsky, Lipatov, Camici, Ciafaloni ... 95-98)*
- Associated with high transverse momenta (weak scattering/low gluon density) $\Longrightarrow$ cannot be cured by non-linear effects
- For NLO BFKL, one has devised powerful resummation schemes ...

  (Salam, Ciafaloni, Colferai, Stasto, 98-03; Altarelli, Ball, Forte, 00-03)

  ... which however were formulated in Mellin space
- The non-linear evolution equation (JIMWLK, BK) are naturally formulated in the transverse coordinate space (eikonal approximation)
- A new strategy for resumming the large transverse logarithms, better suited for non-linear evolution *(arXiv:1502.05642)*
- An all-order resummed ('collinearly-improved') version of the BK equation, as a powerful tool for phenomenology *(arXiv:1507.03651)*

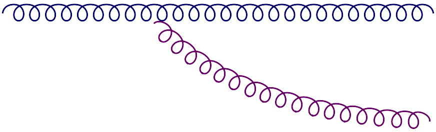
Outline

- The prototype process at high energy: dipole-hadron scattering
- BK equation at leading order (LO)
- A glimpse at the NLO corrections
- Double transverse logarithms & Time ordering
- Adding single transverse logarithms (DGLAP, running coupling)

  $\Longrightarrow$ a “collinearly improved” version of BK equation
- Fits to the HERA data

QCD evolution via Bremsstrahlung

- Bremsstrahlung favors soft and collinear emissions:

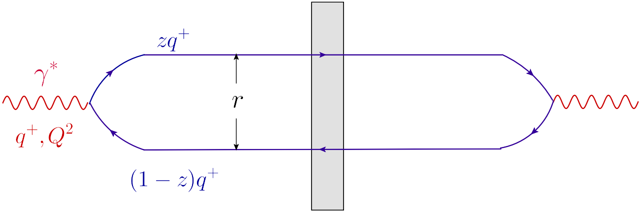


$$\mathrm{d}\mathcal{P} = \frac{\alpha_s N_c}{\pi}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}k_\perp^2}{k_\perp^2} \equiv \bar{\alpha}_s\,\mathrm{d}Y\,\mathrm{d}\rho$$

  - double logarithmic enhancement: $Y \equiv \ln(1/x)$, $\rho \equiv \ln(k_\perp^2/Q_0^2)$
  - evolution with increasing Y (decreasing x) : BFKL, BK, JIMWLK
  - evolution with increasing ρ (or $k_\perp$) : DGLAP
  - common limit: 'double logarithmic approximation' (DLA): $(\bar{\alpha}_s Y\rho)^n$
- Beyond leading-order (LO) : 'cross-terms' between BFKL and DGLAP

Dipole–hadron scattering

- A high-energy process interesting for both DIS and pA collisions



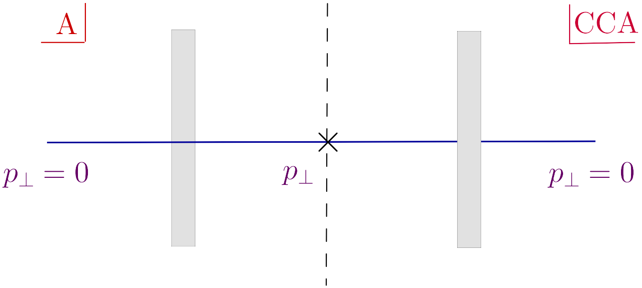
$$\sigma_{\gamma^* p}(Q^2, x) = 2\pi R_p^2 \sum_f \int \mathrm{d}^2 r \int_0^1 \mathrm{d}z \left|\Psi_f(r, z; Q^2)\right|^2 T(r, x)$$

$$x = \frac{Q^2}{2P \cdot q} \simeq \frac{Q^2}{s} \ll 1 \quad \text{(Bjoerken' } x\text{)}$$

- $T(r, x)$: scattering amplitude for a $q\bar{q}$ color dipole with transverse size r
  - $r^2 \sim 1/Q^2$: the resolution of the dipole in the transverse plane
  - x : longitudinal fraction of a gluon from the target that scatters

Dipole–hadron scattering

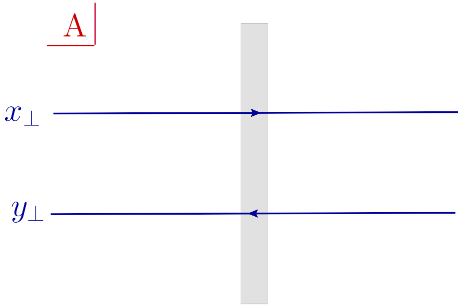
- A high-energy process interesting for both DIS and pA collisions



- Forward quark production in pA collisions
  - a quark from the proton undergoes transverse momentum broadening via rescattering off the nucleus
  - cross-section $=$ amplitude $\times$ complex conjugate amplitude

Dipole–hadron scattering

- A high-energy process interesting for both DIS and pA collisions

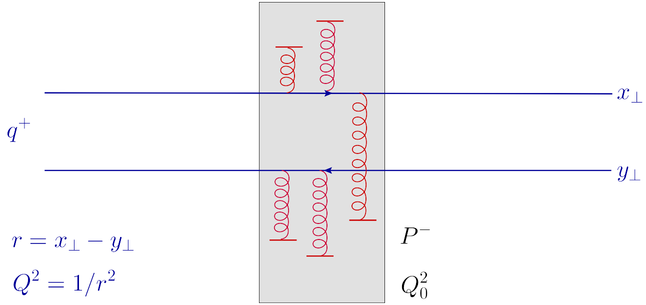


- mathematically equivalent (to the accuracy of interest) to the elastic scattering of a dipole (amplitude only)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\eta \mathrm{d}^2\boldsymbol{p}} = xq(x)\,\frac{1}{(2\pi)^2}\int \mathrm{d}^2\boldsymbol{x}\mathrm{d}^2\boldsymbol{y}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\,S_{\boldsymbol{xy}}$$

- $S_{\boldsymbol{xy}} = 1 - T_{\boldsymbol{xy}}$: the dipole S-matrix (survival of the singlet state)

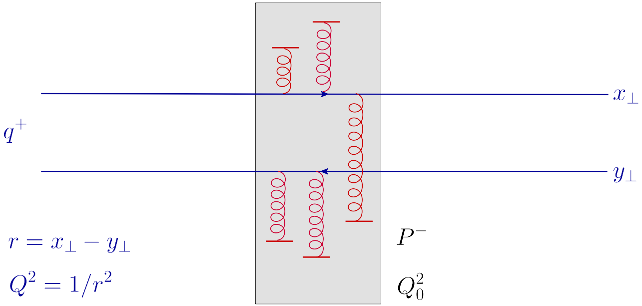
Dipole–hadron scattering ($\gamma^* p$, $\gamma^* A$, pA, ...)



- Dipole ('projectile'): large q^+, transverse resolution $Q^2 = 1/r^2$
- Hadron ('target'): large P^-, saturation momentum Q_0^2
- Wilson lines : multiple scattering in the eikonal approximation

$$S_{\boldsymbol{xy}} = \frac{1}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}}^\dagger V_{\boldsymbol{y}}\big)\,, \qquad V^\dagger(\boldsymbol{x}) = \mathrm{P}\exp\left\{ig\int \mathrm{d}x^+\, A_a^-(x^+,\boldsymbol{x})t^a\right\}$$

The target average: CGC



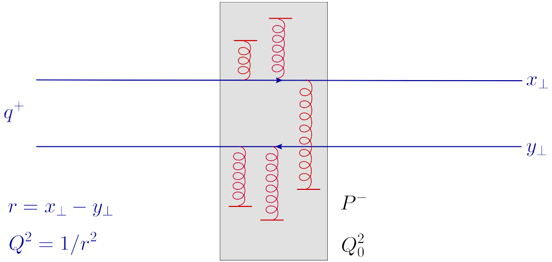
- Average over the color fields (or charges) in the target :

$$\langle S_{\boldsymbol{xy}} \rangle = \int [DA^-]\, W[A^-]\, \frac{1}{N_c}\, \mathrm{tr}\big(V_{\boldsymbol{x}}^\dagger V_{\boldsymbol{y}}\big)[A^-]$$

- The CGC weight function $W[A^-]$: kind of 'functional pdf'

  ▷ semi-classical treatment of the gluons in the target: high density

An instructive model ('low energy')



- MV model *(McLerran, Venugopalan, '93)* : a Gaussian weight function

$$\langle S_{\boldsymbol{xy}} \rangle \,=\, \mathrm{e}^{-\langle T_0(r) \rangle} \,\simeq\, \exp\left\{-\frac{1}{4} r^2 Q_0^2 \,\ln\frac{1}{r^2\Lambda^2}\right\}$$

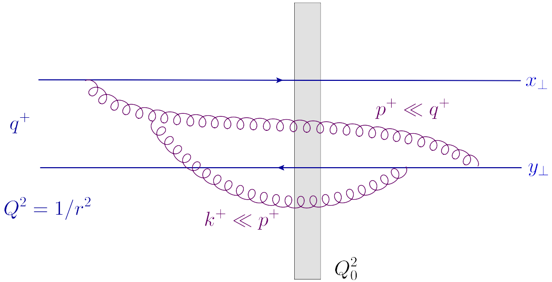
▷ $Q_0^2 \propto$ color charged squared/unit $\perp$ area; $\Lambda =$ IR cutoff ('confinement')

▷ $\langle T_{\boldsymbol{xy}} \rangle = 1 - \langle S_{\boldsymbol{xy}} \rangle \,\propto\, r^2$ as $r \to 0$: 'color transparency'

▷ $\langle T_{\boldsymbol{xy}} \rangle \,\simeq\, 1$ as $r \gtrsim 1/Q_{s0}$: 'unitarity'

High energy evolution

- Probability $\sim \alpha_s \ln(1/x)$ to have additional, soft $(x \ll 1)$, gluons
  - $x \equiv k^+/q^+$: longitudinal momentum fraction of the emitted gluon



$$x_2 = \frac{k^+}{q^+} \ll x_1 = \frac{p^+}{q^+} \ll 1\,, \qquad Y = \ln\frac{1}{x_{\min}} = \ln\frac{s}{Q_0^2}$$

- Leading logarithmic approx: resum $(\bar{\alpha}_s Y)^n$ with $n \geq 1$
  - strong ordering in x, no special ordering in $k_\perp$

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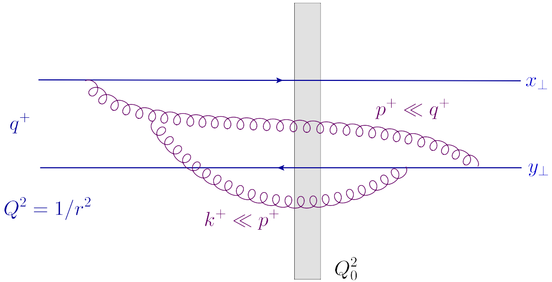
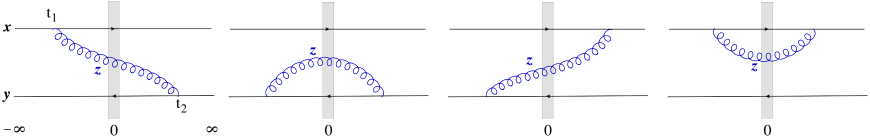


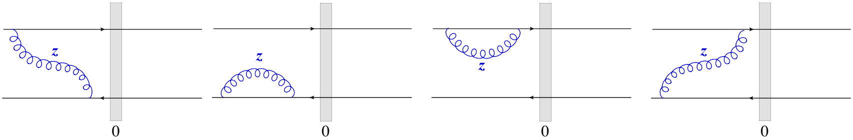
- Non-linear evolution, due to high gluon density in the target
  - projectile (dipole) evolution: Balitsky hierarchy, BK equation
  - target (CGC) evolution: JIMWLK equation (functional)
  - linear approximation (weak scattering) : BFKL

One step in the high energy evolution

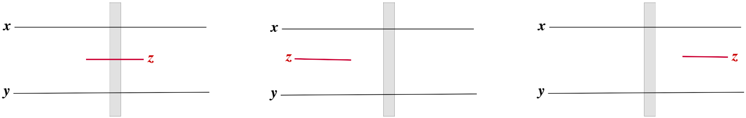
- 'Real corrections' : the soft gluon crosses the shockwave



- 'Virtual corrections' : evolution in the initial/final state



- Large N_c : the original dipole splits into two new dipoles



The BK equation

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\mathcal{M}_{\boldsymbol{xyz}}\big[S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\big]$$

- Dipole kernel: BFKL kernel in the dipole picture *(Al Mueller, 1990)*

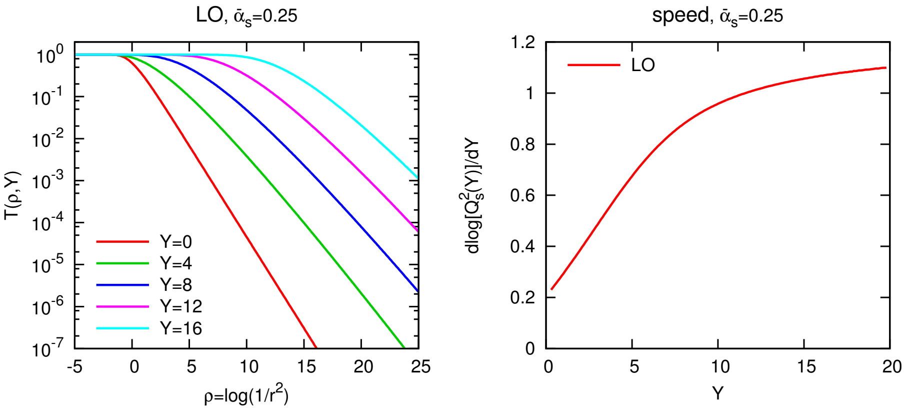
$$\mathcal{M}_{\boldsymbol{xyz}} = \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} = \left[\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2} - \frac{z^i - y^i}{(\boldsymbol{z}-\boldsymbol{y})^2}\right]^2$$

  - color transparency : $\mathcal{M}_{\boldsymbol{xyz}} \propto r^2$, hence $T_{\boldsymbol{xy}} \propto r^2$ as $r \to 0$
  - good 'infrared' (large $z_\perp$, small $k_\perp$) behavior : dipole

$$\mathcal{M}_{\boldsymbol{xyz}} \simeq \frac{r^2}{(\boldsymbol{z}-\boldsymbol{x})^4} \quad \text{when } |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$$

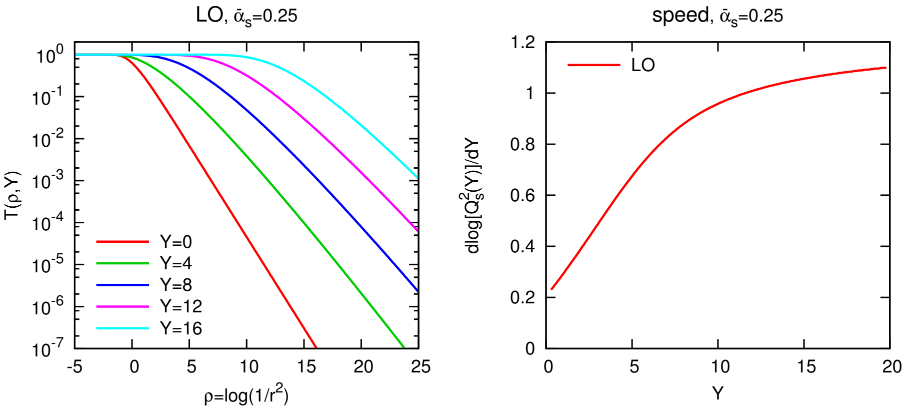
  - 'ultraviolet' poles ($\boldsymbol{z}=\boldsymbol{x}$ or $\boldsymbol{z}=\boldsymbol{y}$) cancel between 'real' and 'virtual'
  - non-linear effects $\Longrightarrow$ unitarity bound : $T_{\boldsymbol{xy}} \leq 1$

LO BK : numerical solutions



- $T(\rho, Y)$ as a function of $\rho = \ln(1/r^2 Q_0^2)$ with increasing Y
  - color transparency at large ρ (small r) : $T(r, Y) \propto r^2 Q_0^2 = \mathrm{e}^{-\rho}$
  - unitarity ('black disk limit') at small ρ (large r) : $T(r, Y) \simeq 1$

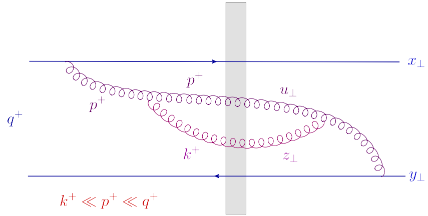
LO BK : numerical solutions



- $T(\rho, Y)$ as a function of $\rho = \ln(1/r^2 Q_0^2)$ with increasing Y
  - saturation momentum $Q_s(Y)$: $T(r, Y) = 0.5$ when $r = 2/Q_s(Y)$
  - saturation exponent: $\lambda_s \equiv \frac{\mathrm{d}\ln Q_s^2(Y)}{\mathrm{d}Y} \simeq 4.88\bar{\alpha}_s \simeq 1$ for $Y \gtrsim 10$

Next-to-leading order

- Two successive emissions strongly ordered in p^+ (or x) : $\sim (\bar{\alpha}_s Y)^2$
  - two iterations of BK : part of the LO evolution



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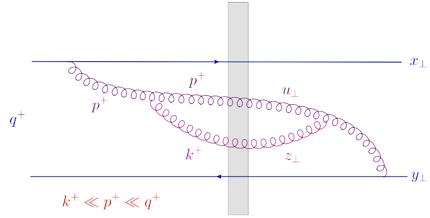
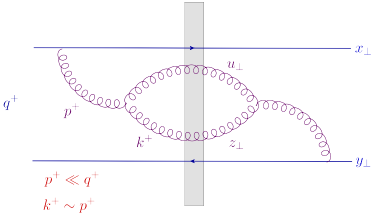
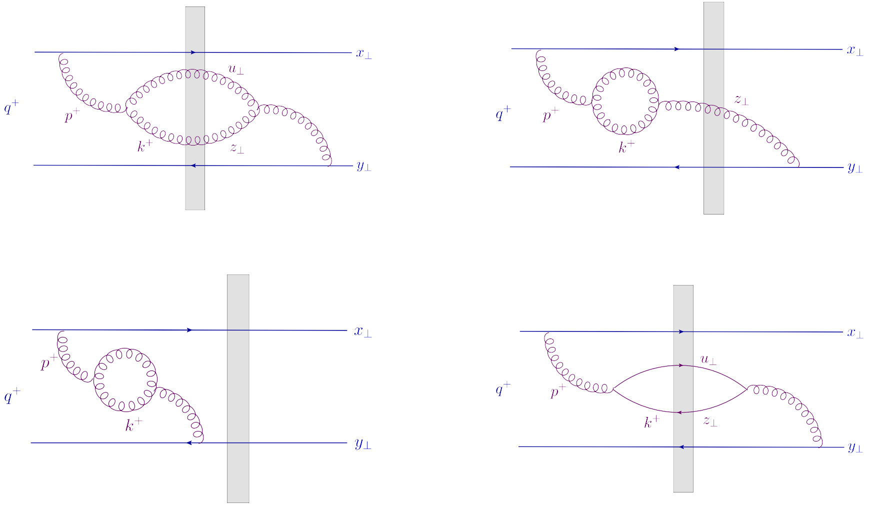




- 'NLO' : any effect of $\mathcal{O}(\bar{\alpha}_s^2 Y)$
  - the prototype: two successive emissions, one soft and one non-soft
- Caution: two strongly-ordered emissions contribute to NLO as well
  - in fact, they give the largest NLO corrections (see below)

Some NLO graphs



- NLO graphs too can be 'real' or 'virtual'
- They can involve quark loops as well

BK equation at NLO

Balitsky, Chirilli (arXiv:0710.4330 [hep-ph]) : $N_{\rm f} = 0$, *large* $N_{\rm c}$

$$\frac{\mathrm{d}S_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}_s}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\,\left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)\Bigg\{1 + \bar{\alpha}_s\left[\bar{b}\,\ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\,\frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2} + \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right]\Bigg\}$$

$$+\frac{\bar{\alpha}_s^2}{8\pi^2}\int\frac{\mathrm{d}^2\boldsymbol{u}\,\mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)\Bigg\{-2 + \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 + (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2 - 4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} + \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1 + \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

- blue : *leading-order (LO) terms*
- red : *NLO terms enhanced by (double or single) transverse logarithms*
- black : *pure* $\bar{\alpha}_s^2$ *effects (no logarithms)*

Large transverse logarithms

- Running coupling corrections: proportional to $\bar{b} = \frac{11}{12} - \frac{1}{6}\frac{N_{\rm f}}{N_{\rm c}}$
  - taken care off via 'standard' prescriptions; see below
- Collinear logarithms: ratios of widely separated dipole sizes
  - the double-logarithmic correction is already manifest

$$-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \simeq -\frac{1}{2}\ln^2\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2} \quad \text{if} \quad |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$$

  - the single logs are still hidden: needs to perform the integral over $\boldsymbol{u}$

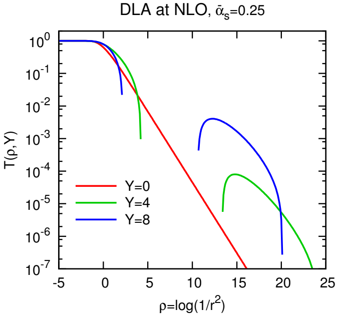
$$1/Q_s \gg |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \simeq |\boldsymbol{z}-\boldsymbol{u}| \gg |\boldsymbol{u}-\boldsymbol{x}| \simeq |\boldsymbol{u}-\boldsymbol{y}| \gg r$$

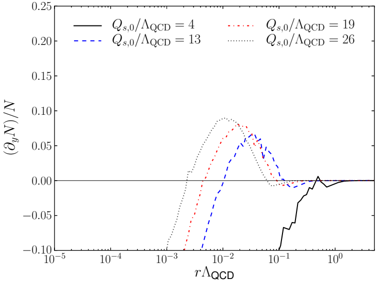
    - all dipoles are relatively small ($\ll 1/Q_s$): weak scattering
    - ... but such that their sizes are strongly increasing:
      $\Longrightarrow$ logarithmic phase-space for the intermediate gluon at $\boldsymbol{u}$
- N.B. Collinear logs are important at weak scattering (dilute target)

Unstable numerical solution

- Keeping just the collinear logarithms $\Longrightarrow$

$$\frac{\mathrm{d}T(r)}{\mathrm{d}Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_s^2} \mathrm{d}z^2 \, \frac{r^2}{z^4} \left(1 - \frac{1}{2}\,\bar{\alpha}_s \ln^2 \frac{z^2}{r^2} - \frac{11}{12}\,\bar{\alpha}_s \ln \frac{z^2}{r^2}\right) T(z)$$



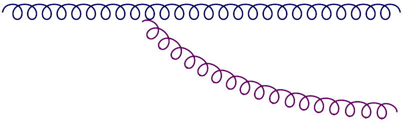


  - Left: LO BK + the double collinear logarithm
  - Right: full NLO BK *(Lappi, Mäntysaari, arXiv:1502.02400)*

- The main source of instability: the double collinear logarithm

Double logarithms in the QCD evolution

- Where do the double collinear logs come from ?



- Bremsstrahlung naturally introduces double-logarithmic corrections ...

$$\mathrm{d}\mathcal{P} = \frac{\alpha_s N_c}{\pi} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}k_\perp^2}{k_\perp^2} = \bar{\alpha}_s \, \mathrm{d}Y \, \mathrm{d}\rho$$

... but an energy logarithm times a collinear one !

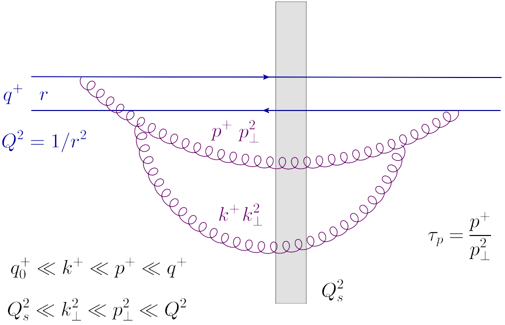
- Hint: From a constraint on the energy phase-space ...

$$Y > \rho \Longrightarrow \bar{\alpha}_s Y \rho \longrightarrow \bar{\alpha}_s (Y - \rho)\rho = \bar{\alpha}_s Y \rho - \bar{\alpha}_s \rho^2$$

... which is introduced by time ordering

Time ordering

- To get double logs, successive emissions must be strongly ordered in ...



  - in longitudinal momentum : $q^+ \gg p^+ \gg k^+ \cdots \gg q_0^+$
  - in transverse momentum/size: $Q^2 \gg p_\perp^2 \gg k_\perp^2 \cdots \gg Q_s^2$
  - in lifetime: $\tau_p = p^+/p_\perp^2 \gg \tau_k = k^+/k_\perp^2$ (factorization)

- Both p^+ and $p_\perp^2$ are decreasing $\Rightarrow$ potential conflict with time ordering

Time ordering

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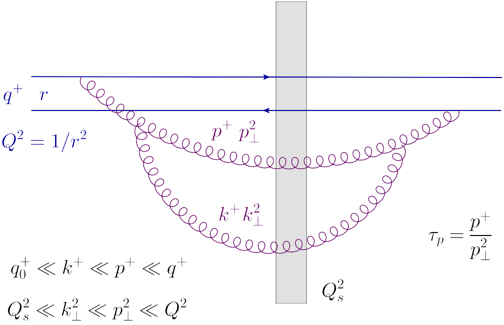


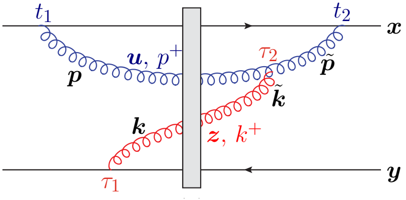
$$\frac{p^+}{p_\perp^2} > \frac{k^+}{k_\perp^2} \Longrightarrow \frac{p^+}{k^+} > \frac{p_\perp^2}{k_\perp^2} \Longrightarrow \Delta Y \equiv \ln\frac{p^+}{k^+} > \Delta\rho \equiv \ln\frac{p_\perp^2}{k_\perp^2}$$

- This condition enters perturbation theory via energy denominators

Two successive emissions: $p^+ \gg k^+$

- Light-cone (time-ordered) perturbation theory: physics is transparent
  - mixed F representation $(p^+, t \equiv x^+)$, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$
  - the time integrals yield energy denominators

$$\frac{1}{\sum_{i \in interm} k_i^- - P_0^-} \simeq \frac{1}{p^- + k^-}$$



$$\frac{1}{p^+u^2 + k^+z^2} = \frac{1}{p^+u^2}\frac{p^+u^2}{p^+u^2 + k^+z^2} \simeq \frac{1}{p^+u^2}\,\Theta(p^+u^2 - k^+z^2)$$

Two successive emissions: $p^+ \gg k^+$

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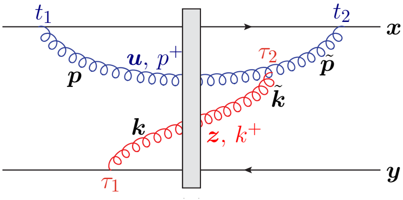

- Integrate out the harder gluon $(p^+, \boldsymbol{u})$ to double-log accuracy:

$$\bar{\alpha}_s \int_{r^2}^{z^2} \frac{\mathrm{d}u^2}{u^2} \int_{k^+}^{q^+} \frac{\mathrm{d}p^+}{p^+}\, \Theta(p^+u^2 - k^+z^2) = \bar{\alpha}_s Y \rho - \frac{\bar{\alpha}_s \rho^2}{2}$$

  - the expected LO contribution $\bar{\alpha}_s Y \rho$
  - NLO contribution $\bar{\alpha}_s \rho^2$ to the kernel for emitting a softer gluon $(k^+, \boldsymbol{z})$

The double logarithmic approximation (DLA)

- Enforce time-ordering in the 'naïve' DLA limit of BFKL $\Longrightarrow$ DLA 2.0

$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \mathcal{M}_{\boldsymbol{xyz}} \big[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}}\big]$$

  - large daughter dipoles: $1/Q_0 \gg |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg r$

$$\Longrightarrow \ \mathcal{M}_{\boldsymbol{xyz}} \simeq \frac{r^2}{(\boldsymbol{z}-\boldsymbol{x})^4}$$

  - $T(r) \propto r^2 \Longrightarrow T_{\boldsymbol{xz}} \simeq T_{\boldsymbol{zy}} \gg T_{\boldsymbol{xy}}$: only 'real' terms matter

$$\frac{\partial T(Y,r^2)}{\partial Y} = \bar{\alpha}_s \int_{r^2}^{1/Q_0^2} \frac{\mathrm{d}z^2}{z^2}\, \frac{r^2}{z^2}\, T(Y,z^2)$$

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  - introduce logarithmic variable $\rho \equiv \ln(1/r^2 Q_0^2)$

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$$\frac{\partial T(Y,\rho)}{\partial Y} = \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1\, \mathrm{e}^{-(\rho-\rho_1)}\, T(Y-\rho+\rho_1,\rho_1)$$

  - introduce time-ordering $\Longrightarrow$ non-local in Y

DLA 2.0

- The importance of time-ordering has since long been recognized
  - coherence effects, kinematical constraint, choice of rapidity scale ...
    Ciafaloni (88), CCFM (90), Lund group (Andersson et al, 96), Kwiecinski et al (96), Salam (98) ... Motyka, Stasto (09), G. Beuf (14)
- This is more than a prescription: it is a systematic approximation
  - resums powers of $\bar{\alpha}_s Y \rho$ and $\bar{\alpha}_s \rho^2$ to all orders
  - more precisely: all terms of order $\bar{\alpha}_s^n Y^k \rho^{2n-k}$, $n \geq 1$ and $0 \leq k \leq n$
- However, in order to be useful, this should also include
  - the terms of order $(\bar{\alpha}_s Y)^n$, $n \geq 1$ ('BFKL')
  - the non-linear effects expressing saturation ('BK')
- To that aim, one would need a 'genuine' evolution equation, local in Y
- Does it exist ? Not *a priori* clear !

Getting local

$$T(Y,\rho) = T(0,\rho) + \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1 \, \mathrm{e}^{-(\rho-\rho_1)} \int_0^{Y-\rho+\rho_1} \mathrm{d}Y_1 \, T(Y_1,\rho_1)$$

- For $Y \geq \rho$, the solution $T(Y,\rho)$ to the above equation coincides with the solution $\tilde{T}(Y,\rho)$ to the following problem:

$$\tilde{T}(Y,\rho) = \tilde{T}(0,\rho) + \bar{\alpha}_s \int_0^{Y} \mathrm{d}Y_1 \int_0^{\rho} \mathrm{d}\rho_1 \, \mathrm{e}^{-(\rho-\rho_1)} \, \mathcal{K}_{\mathrm{DLA}}(\rho-\rho_1) \tilde{T}(Y_1,\rho_1)$$

with the following, all-orders, kernel:

$$\mathcal{K}_{\mathrm{DLA}}(\rho) = \frac{\mathrm{J}_1\big(2\sqrt{\bar{\alpha}_s \rho^2}\big)}{\sqrt{\bar{\alpha}_s \rho^2}} = 1 - \frac{\bar{\alpha}_s \rho^2}{2} + \frac{(\bar{\alpha}_s \rho^2)^2}{12} + \cdots$$

... and the following, all-orders, initial condition:

$$\tilde{T}(0,\rho) = T(0,\rho) - \sqrt{\bar{\alpha}_s} \int_0^{\rho} \mathrm{d}\rho_1 \, \mathrm{e}^{-(\rho-\rho_1)} \, \mathrm{J}_1\big(2\sqrt{\bar{\alpha}_s (\rho-\rho_1)^2}\big) T(0,\rho_1)$$

Adding the single transverse logarithms

- Recall the NLO equation with all the single logs

$$\frac{\mathrm{d}T(r)}{\mathrm{d}Y} = \bar{\alpha}_s \int \mathrm{d}z^2 \, \frac{r^2}{z^4} \left\{ 1 - \bar{\alpha}_s \left(\frac{1}{2} \ln^2 \frac{z^2}{r^2} + \frac{11}{12} \ln \frac{z^2}{r^2} - \bar{b} \, \ln r^2 \mu^2 \right) \right\} T(z)$$

- The double-logarithm is already included within $\mathcal{K}_{\rm DLA}(\rho)$ ✓
- The collinear single-log comes from the DGLAP regime:
  - one soft ($x \ll 1$) emission + one non-soft ($x \sim 1$) one
  - coefficient $A_1 = 11/12$ related to the DGLAP anomalous dimension:

$$\gamma(\omega) = \int_0^1 \mathrm{d}z \, z^\omega \left[P_{\rm gg}(z) + \frac{C_{\rm F}}{N_{\rm c}} P_{\rm qg}(z) \right] = \frac{1}{\omega} - A_1 + \mathcal{O}\left(\omega, \frac{N_{\rm f}}{N_{\rm c}^3} \right)$$

  - can be resummed by including the A_1 piece of $\gamma(\omega)$ into $\mathcal{K}_{\rm DLA}(\rho)$ ✓
- The running coupling log is resummed by replacing $\bar{\alpha}_s \to \bar{\alpha}_s(r^2)$ ✓

- In the DLA regime (strong ordering in $k_\perp$) we have so far obtained

$$\frac{\partial \tilde{T}(Y,\rho)}{\partial Y} = \bar{\alpha}_s(\rho) \int_0^\rho \mathrm{d}\rho_1\, \mathrm{e}^{-(1+\bar{\alpha}_s A_1)(\rho-\rho_1)}\, \mathcal{K}_{\rm DLA}(\rho-\rho_1)\, \tilde{T}(Y,\rho_1)$$

- LLA: daughter dipoles can also be comparable/smaller than the parent
  - match onto the known result at NLLA:

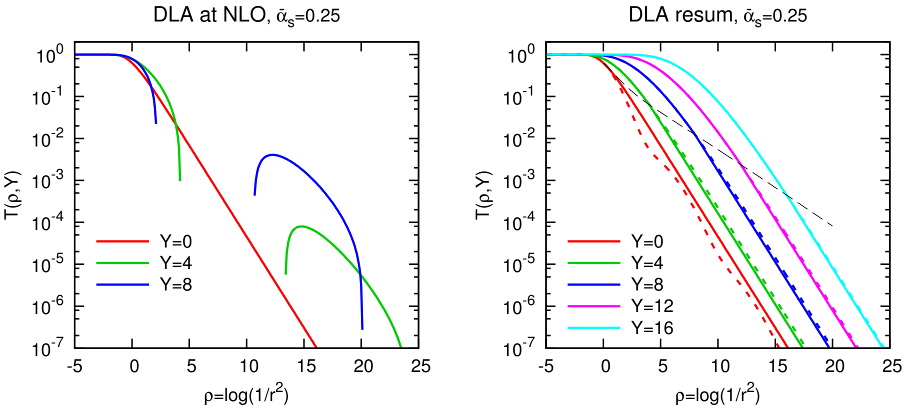
$$\mathcal{K}_{\rm DLA} \simeq 1 - \frac{\bar{\alpha}_s}{2} \ln^2 \frac{z^2}{r^2} \;\to\; 1 - \frac{\bar{\alpha}_s}{2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}$$

  - restore the virtual and the non-linear terms
  - extend the prescription for RC using guidance from DGLAP

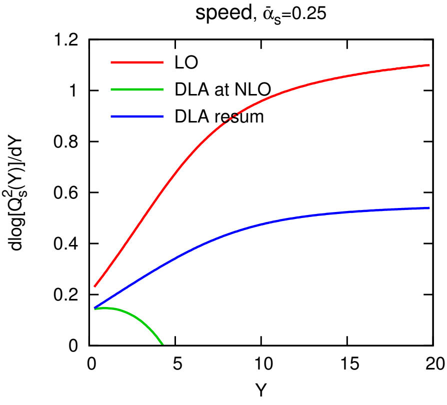
$$\frac{\mathrm{d}\tilde{T}_{\boldsymbol{xy}}}{\mathrm{d}Y} = \int \frac{\mathrm{d}^2\boldsymbol{z}}{2\pi}\, \bar{\alpha}_s(r_{\min})\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left(\tilde{T}_{\boldsymbol{xz}} + \tilde{T}_{\boldsymbol{zy}} - \tilde{T}_{\boldsymbol{xy}} - \tilde{T}_{\boldsymbol{xz}}\tilde{T}_{\boldsymbol{zy}}\right)$$

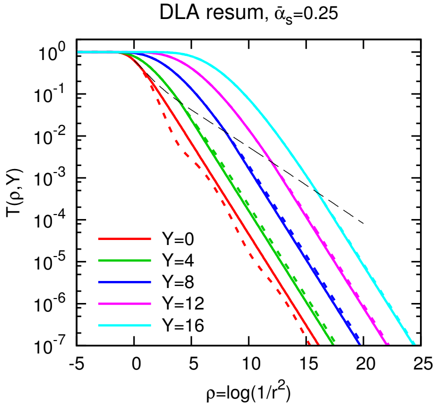
$$\times \left[\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{\min\{(\boldsymbol{x}-\boldsymbol{z})^2, (\boldsymbol{y}-\boldsymbol{z})^2\}}\right]^{\pm\bar{\alpha}_s A_1} \mathcal{K}_{\rm DLA}\big(\bar{\rho}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\big)$$

Numerical solutions

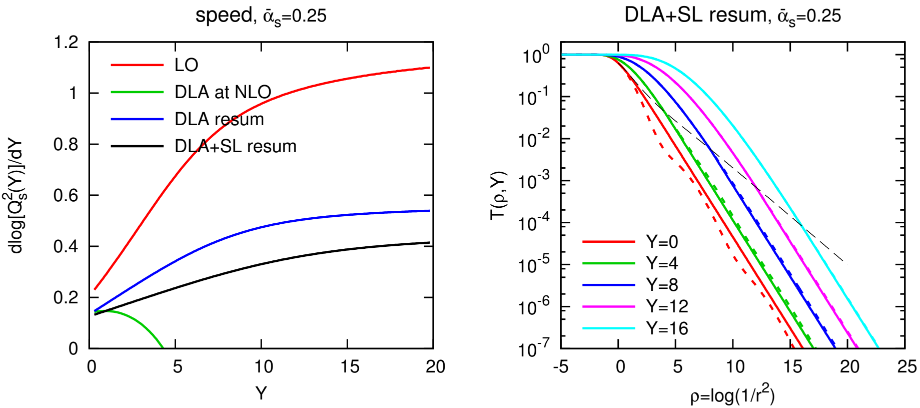


- Fixed coupling $\bar{\alpha}_s = 0.25$, double collinear logs alone
  - left: expanded to NLO
  - right: resummed to all orders
- The resummation stabilizes & slows down the evolution



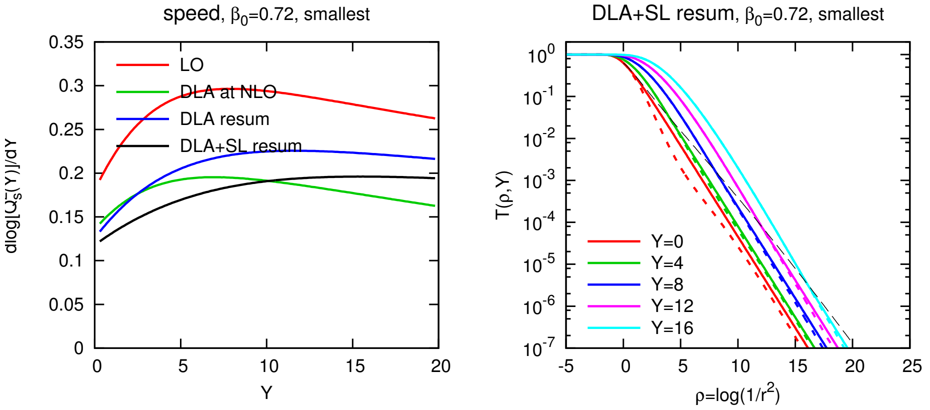


- Fixed coupling $\bar{\alpha}_s = 0.25$, double collinear logs alone
  - left: saturation exponent $\lambda_s \equiv \mathrm{d}\ln Q_s^2(Y)/\mathrm{d}Y$
  - LO: $\lambda_s \simeq 4.88\bar{\alpha}_s \simeq 1$ (for $Y \gtrsim 10$)
  - DLA resummed: $\lambda_s \simeq 0.5$



- Further slowing down when including the single collinear logs

Numerical solutions



- ... and even more so after also using a running coupling

Fitting the HERA data (1)

- Use numerical solutions to collinearly-improved running-coupling BK equation using initial conditions which involve free parameters
  - a similar strategy as for the DGLAP fits
- Various choices for the initial condition at $Y = Y_0$:

$$\text{GBW}: \qquad T(Y_0, r) = \left\{1 - \exp\left[-\left(\frac{r^2 Q_0^2}{4}\right)^p\right]\right\}^{1/p}$$

$$\text{rcMV}: \quad T(Y_0, r) = \left\{1 - \exp\left[-\left(\frac{r^2 Q_0^2}{4}\,\bar{\alpha}_s(C_{\rm MV} r)\left[1 + \ln\left(\frac{\bar{\alpha}_{\rm sat}}{\bar{\alpha}_s(C_{\rm MV} r)}\right)\right]\right)^p\right]\right\}^{1/p}$$

- One loop running coupling with scale $\mu = 2C_\alpha / r$:

$$\alpha_s(r) = \frac{1}{b_0 \ln\left[4C_\alpha^2/(r^2\Lambda^2)\right]}, \quad \text{with} \quad r = \min\{|\boldsymbol{x}-\boldsymbol{y}|, |\boldsymbol{x}-\boldsymbol{z}|, |\boldsymbol{y}-\boldsymbol{z}|\}$$

- Up to 5 free parameters: R_p (proton radius), Q_0, p, C_α, ($C_{\rm MV}$)

Fitting the HERA data (2)

- 3 light quarks + charm quark, all treated on the same footing
  - quark masses are not fit parameters, but they are varied to test the sensitivity of the fit
  - good quality fits for $m_{u,d,s} = 50 \div 140$ MeV and $m_c = 1.3$ or 1.4 GeV
- The most recent HERA data for the reduced photon-proton cross-section (combined analysis by ZEUS and H1)
  - small Bjoerken x: $x \leq 0.01$
  - not very high Q^2 : $Q^2 < Q^2_{\max}$ with $Q^2_{\max} = 50 \div 400$ GeV^2
- Good quality fits: χ^2 per point around 1.1-1.2
- Very discriminatory: the fits favor
  - rcMV initial condition (pQCD + saturation)
  - physical prescriptions for RC: smallest-dipole, FAC
  - physical values for the free parameters

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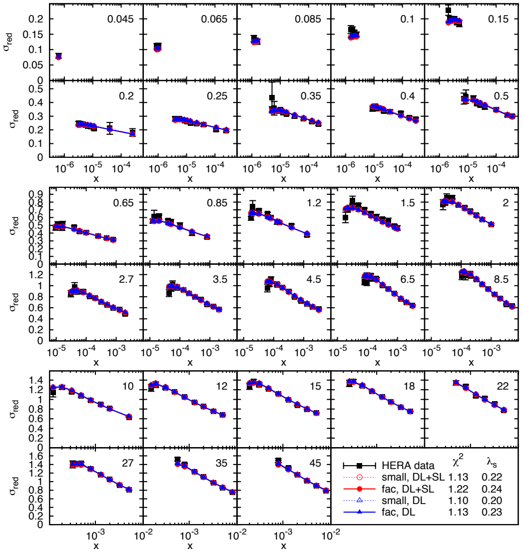
- Good quality fits: χ^2 per point around 1.1-1.2
- Very discriminatory: the fits disfavor
  - fixed coupling MV, GBW at high Q^2
  - Balitsky prescriptions for RC
  - 'anomalous dimension' $\gamma > 1$ in the initial condition

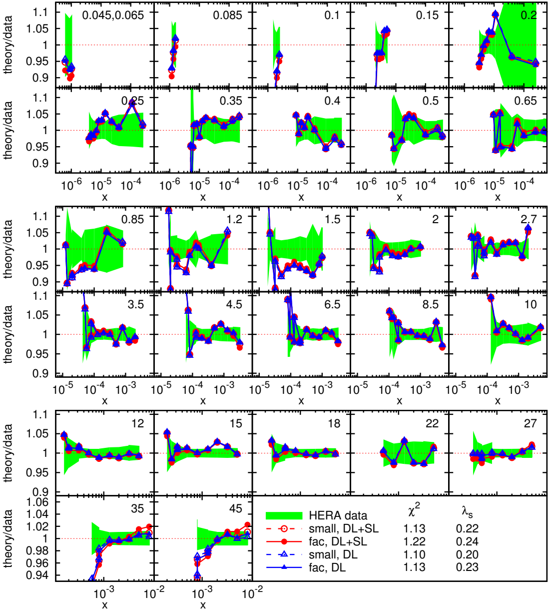
The Fit in tables

init cdt.	RC schm	sing. logs	χ^2 per data point			parameters					
			$\sigma_{\rm red}$	$\sigma_{\rm red}^{cc}$	F_L	R_p[fm]	Q_0[GeV]	C_α	p	$C_{\rm MV}$
GBW	small	yes	1.135	0.552	0.596	0.699	0.428	2.358	2.802	-
GBW	fac	yes	1.262	0.626	0.602	0.671	0.460	0.479	1.148	-
rcMV	small	yes	1.126	0.578	0.592	0.711	0.530	2.714	0.456	0.896
rcMV	fac	yes	1.222	0.658	0.595	0.681	0.566	0.517	0.535	1.550
GBW	small	no	1.121	0.597	0.597	0.716	0.414	6.428	4.000	-
GBW	fac	no	1.164	0.609	0.594	0.697	0.429	1.195	4.000	-
rcMV	small	no	1.097	0.557	0.593	0.723	0.497	7.393	0.477	0.816
rcMV	fac	no	1.128	0.573	0.591	0.703	0.526	1.386	0.502	1.015

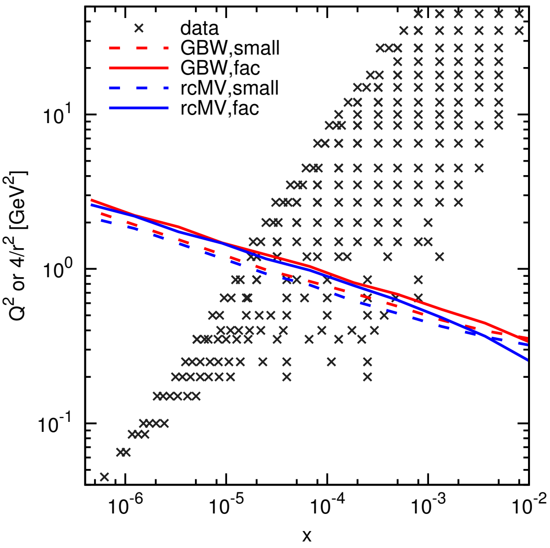
init cdt.	RC schm	sing. logs	χ^2/npts for $Q^2_{\rm max}$			
			50	100	200	400
GBW	small	yes	1.135	1.172	1.355	1.537
GBW	fac	yes	1.262	1.360	1.654	1.899
rcMV	small	yes	1.126	1.172	1.167	1.158
rcMV	fac	yes	1.222	1.299	1.321	1.317
GBW	small	no	1.121	1.131	1.317	1.487
GBW	fac	no	1.164	1.203	1.421	1.622
rcMV	small	no	1.097	1.128	1.095	1.078
rcMV	fac	no	1.128	1.177	1.150	1.131

The Fit in plots




0.045,0.065
0.085
0.1
0.15
0.2
0.25
0.35
0.4
0.5
0.65
0.85
1.2
1.5
2
2.7
3.5
4.5
6.5
8.5
10
12
15
18
22
27
35
45
theory/data
x
HERA data
small, DL+SL
fac, DL+SL
small, DL
fac, DL
χ^2
λ_s
1.13 0.22
1.22 0.24
1.10 0.20
1.13 0.23

The Fit in plots



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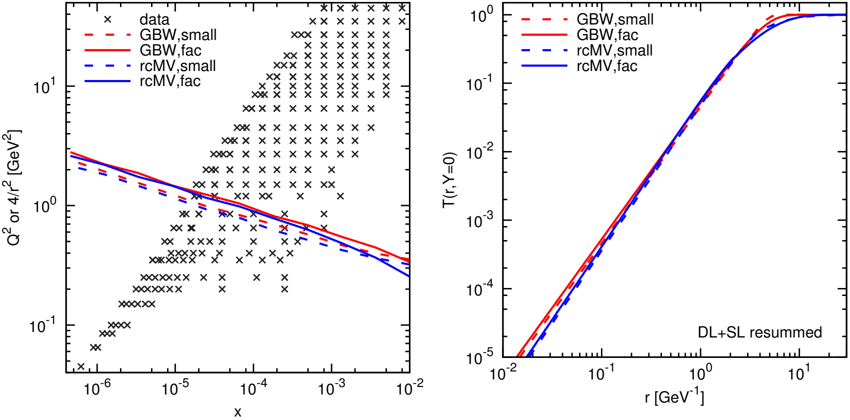

- Rather stable predictions for the saturation line and the shape of the initial amplitude