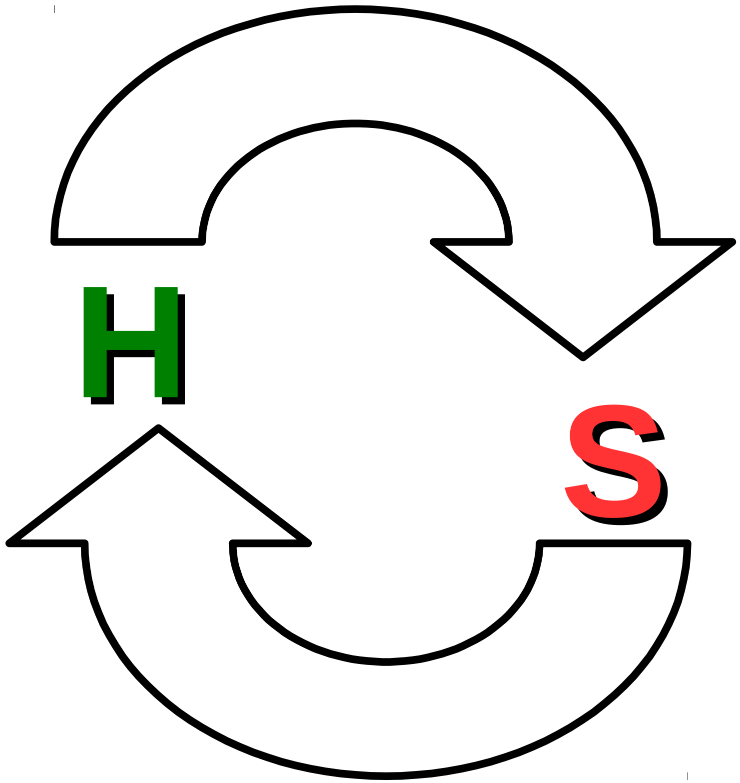
Stimulated Transitions and Self Interactions



Jim Kneller
Yue Yang
Kennedy Perkins
NC State University

Stimulated transitions

Patton, Kneller & McLaughlin, PRD **91** 025001 (2015)

Patton, Kneller & McLaughlin, PRD **89** 073022 (2014)

Kneller, McLaughlin & Patton, JPG **40** 055002 (2013)

The evolution of a neutrino traveling through a fluctuating matter potential is well described by the Stimulated Transition model.

The ν state at r is related to the initial state through a matrix *S*.

S obeys a differential equation

$$i\frac{dS}{d\lambda} = H\,S$$

H is the Hamiltonian, λ is an affine parameter.

H is composed of two terms:

- the vacuum contribution,
- the underlying smooth matter potential,
- the perturbing potential δH.

$\}\ H_0$ (vacuum contribution and underlying smooth matter potential)

The perturbing potential has one non-zero element which we write as a Fourier series with wavenumbers {q} and amplitudes {C} .

$$\delta H_{ee} = \sum_a C_a \sin(q_a r + \eta_a)$$

It is possible to derive an analytic solution for the case of a constant background potential using the Rotating Wave Approximation.

In the basis of the solutions of H_0 - with eigenvalues $\{k_1, k_2, \ldots\}$ - we find a set of integers, one for each wavenumber.

The integers typically try to satisfy

$$\delta k_{ij} + n_1 q_1 + n_2 q_2 + \ldots \approx 0$$

The neutrino behaves like an illuminated polarized molecule.

- It picks out the Fourier modes in the turbulence with frequencies that match the eigenvalue splitting.

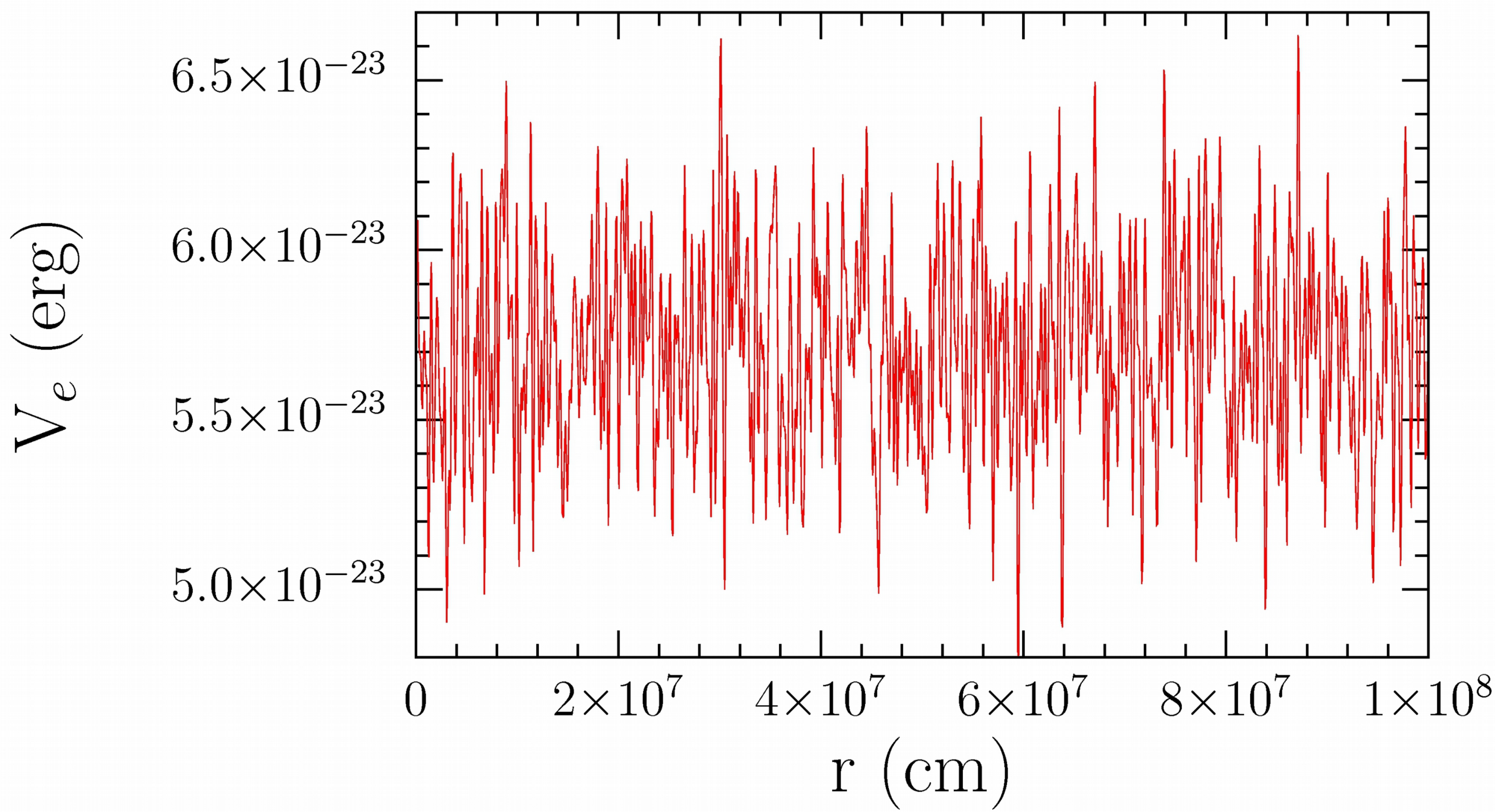
For two flavors the solution is particularly simple:

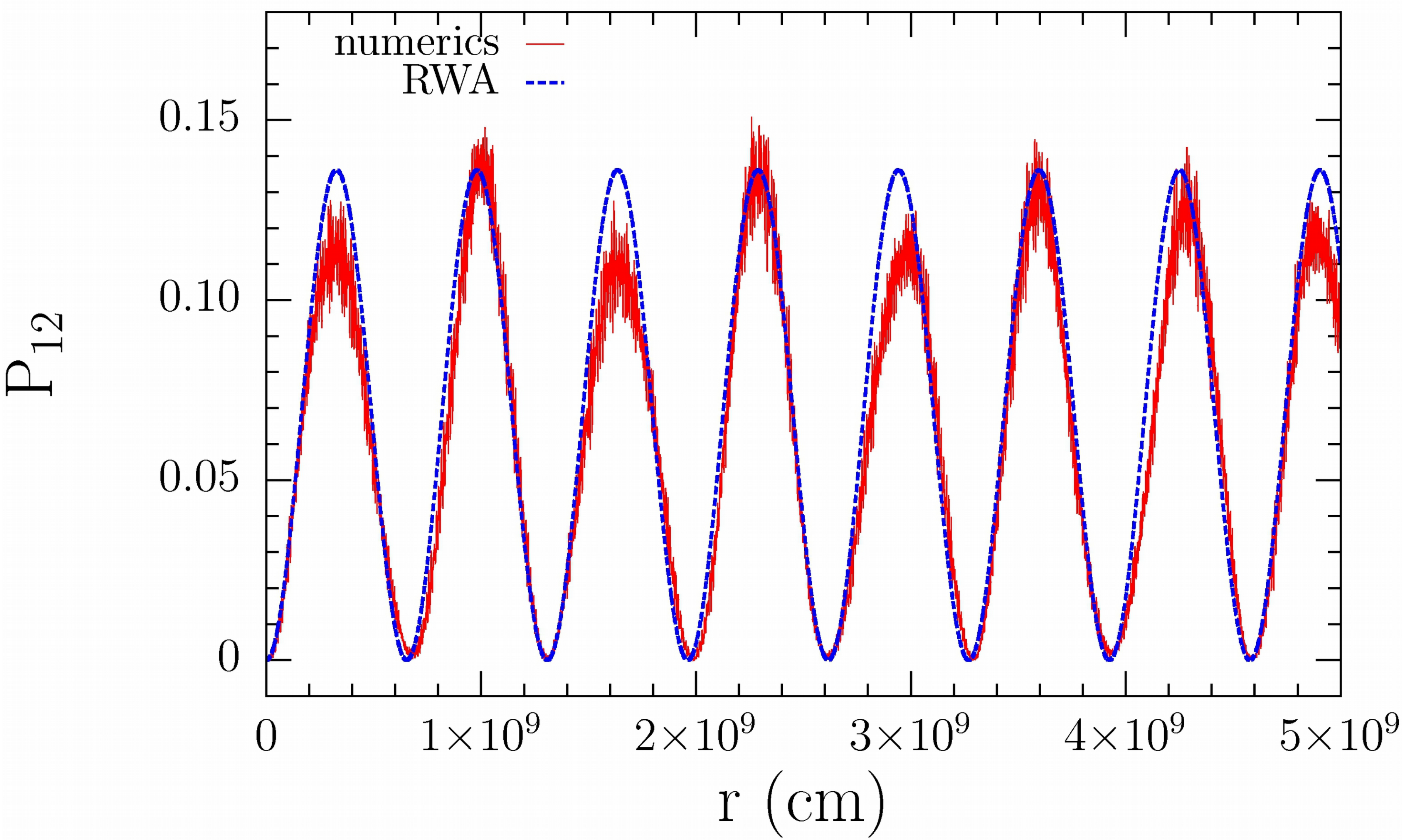
$$P_{12} = \frac{\kappa^2}{Q^2} \sin^2(Q r)$$

- The quantities κ and Q are functions of the amplitudes {C} and wavenumbers {q} of the Fourier modes.

A realization of turbulence created using 50 Fourier modes.

Patton, Kneller & McLaughlin, PRD **89** 073022 (2014)





numerics
RWA
0.15
0.10
0.05
0
P_{12}
0
1×10^9
2×10^9
3×10^9
4×10^9
5×10^9
r (cm)

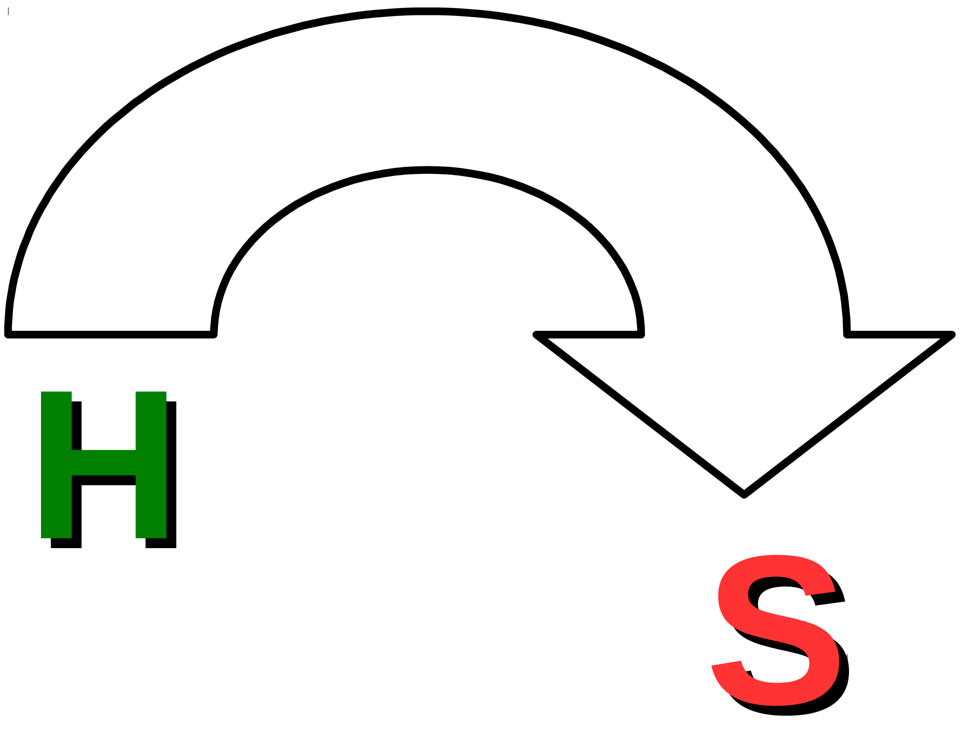
Generalizing

Can the Stimulated Transition description be used for other (neutrino flavor) evolution problems?

How does S evolve for an arbitrary Hamiltonian?

What if H is a function of S?

From H to S



We generalize to an arbitrary perturbing Hamiltonian.

In some basis (f) we write:

$$\delta H^{(f)} = \sum_{a=1}^{N_q} C_a e^{-i q_a r} + C_a^{\dagger} e^{i q_a r}$$

We transform to the eigenbasis of H_0 using a matrix U and solve for the evolution matrix S_0 in that basis.

If H_0 is a constant then $S_0 = \exp(-i\, K\, r)$ where K is the diagonal matrix of eigenvalues of H_0, K = diag(k_1,k_2,...).

In the eigenbasis of H_0, we write $S = S_0 A$ and find A evolves according to

$$i \frac{dA}{dr} = \sum_a \left\{ e^{i K r} U^{\dagger} \left[C_a e^{i q_a r} + C_a^{\dagger} e^{-i q_a r} \right] U e^{-i K r} \right\} A = H^{(A)} A$$

In general $H^{(A)}$ has diagonal *and* offdiagonal elements.

We pull out the diagonal elements of C_a and write them as

$$diag\left(U^{\dagger} C_a U\right) = \frac{1}{2i} e^{i\Phi_a} F_a$$

Where $\Phi_a = diag(\varphi_{a,1}, \varphi_{a,2}, \ldots)$ and $F_a = diag(F_{a,1}, F_{a,2}, \ldots)$.

We now write A as A = W B where the diagonal matrix W given by

$$W = \exp\left(-i \sum_a \Xi_a\right)$$

$$\Xi_a = \frac{F_a}{q_a}\left[\cos\Phi_a - \cos\left(\Phi_a + q_a r\right)\right]$$

Ξ_a is also a diagonal matrix: $\Xi_a = diag(\xi_{a,1}, \xi_{a,2}, \ldots)$.

The purpose of W is to remove the diagonal elements of $H^{(A)}$.

We also define the matrix G_a by

$$offdiag\left(U^{\dagger} C_a U\right) = G_a$$

The matrix B evolves according to

$$i\frac{dB}{dr} = e^{iKr + i\sum_b \Xi_b}\left(\sum_a \left\{G_a e^{iq_a r} + G_a^{\dagger} e^{-iq_a r}\right\}\right)e^{-iKr - i\sum_b \Xi_b} B = H^{(B)} B$$

The element ij of $H^{(B)}$ is

$$H_{ij}^{(B)} = \sum_a \left\{G_{a,ij} e^{i\left(\left[q_a + \left(k_i - k_j\right)\right]r + \sum_b \left[\xi_{b,i} - \xi_{b,j}\right]\right)} + c.c\right\}$$

The term $\exp(i[\xi_{b,i} - \xi_{b,j}])$ needs attention.

In full this term is

$$\xi_{b,i}-\xi_{b,j}=\frac{\left[F_{b,i}\cos\varphi_{b,i}-F_{b,j}\cos\varphi_{b,j}\right]}{q_b}\left(1-\cos\left(q_b r\right)\right)$$
$$+\frac{\left[F_{b,i}\sin\varphi_{b,i}-F_{b,j}\sin\varphi_{b,j}\right]}{q_b}\sin\left(q_b r\right)$$

which can be simplified by introducing $x_{b,ij}$ and $y_{b,ij}$, and then rewriting it using $(z_{b,ij})^2 = (x_{b,ij})^2+(y_{b,ij})^2$ and $\tan \Psi_{b,ij} = y_{b,ij} / x_{b,ij}$

$$\xi_{b,i}-\xi_{b,j}=x_{b,ij}-z_{b,ij}\cos\left(q_b r+\psi_{b,ij}\right)$$

The term $\exp(i[\xi_{b,i} - \xi_{b,j}])$ can be expanded using Jacobi-Anger

$$e^{i[\xi_{b,i}-\xi_{b,j}]}=e^{i x_{b,ij}}\sum_{m_b=-\infty}^{+\infty}\left(-i\right)^{m_b}J_{m_b}\left(z_{b,ij}\right)e^{i m_b\left[q_b r+\psi_{b,ij}\right]}$$

And the element ij of $H^{(B)}$ is

$$H_{ij}^{(B)} = -i\, e^{i[k_i - k_j]r} \sum_a \left(\sum_{m_a} \kappa_{a m_a, ij}\, e^{i m_a q_a r} \left\{ \prod_{b \neq a} \sum_{m_b} \lambda_{b m_b, ij}\, e^{i m_b q_b r} \right\} \right)$$

$$\kappa_{a m_a, ij} = (-i)^{m_a} e^{i[x_{a,ij} + m_a \psi_{a,ij}]} \left[G_{a,ij}\, e^{-i \psi_{a,ij}} J_{m_a - 1} - G^*_{a,ij}\, e^{i \psi_{a,ij}} J_{m_a + 1} \right]$$

$$\lambda_{b m_b, ij} = (-i)^{m_b} e^{i[x_{b,ij} + m_b \psi_{b,ij}]} J_{m_b}$$

The Hamiltonian for B looks simple ☺ but, again,we cannot obtain a solution for B without making the Rotating Wave Approximaiton.

We assume that for each Fourier mode there is only one* important contribution to the series – n_a.

$$H_{ij}^{(B)} = -i\, e^{i[k_i - k_j]r} \sum_a \kappa_{a n_a, ij}\, e^{i n_a q_a r} \prod_{b \neq a} \lambda_{b, n_b, ij}\, e^{i n_b q_b r}$$

Other than a generalization of various terms, the Hamiltonian has exactly the same form as the fluctuating matter problem!

For the case of two flavors:

$$B=\begin{pmatrix} e^{ipr}\left[\cos Qr - i\frac{p}{Q}\sin Qr\right] & -ie^{ipr}\frac{\kappa}{Q}\sin Qr \\ -ie^{-ipr}\frac{\kappa^*}{Q}\sin Qr & e^{-ipr}\left[\cos Qr + i\frac{p}{Q}\sin Qr\right] \end{pmatrix}$$

where

$$\kappa=\sum_a \kappa_{a,n_a}\prod_{b\neq a}\lambda_{b,n_b}$$

$$2\mathrm{p}=k_1-k_2+\sum_a n_a q_a$$

$$Q^2=p^2+\kappa^2$$

The transition probability in the eigenbasis of H_0 is

$$P_{12}=\frac{\kappa^2}{Q^2}\sin^2(Qr)$$

A simple self-interaction problem

Consider a simple self-interaction problem for monoenergetic neutrinos and antineutrinos for two flavors

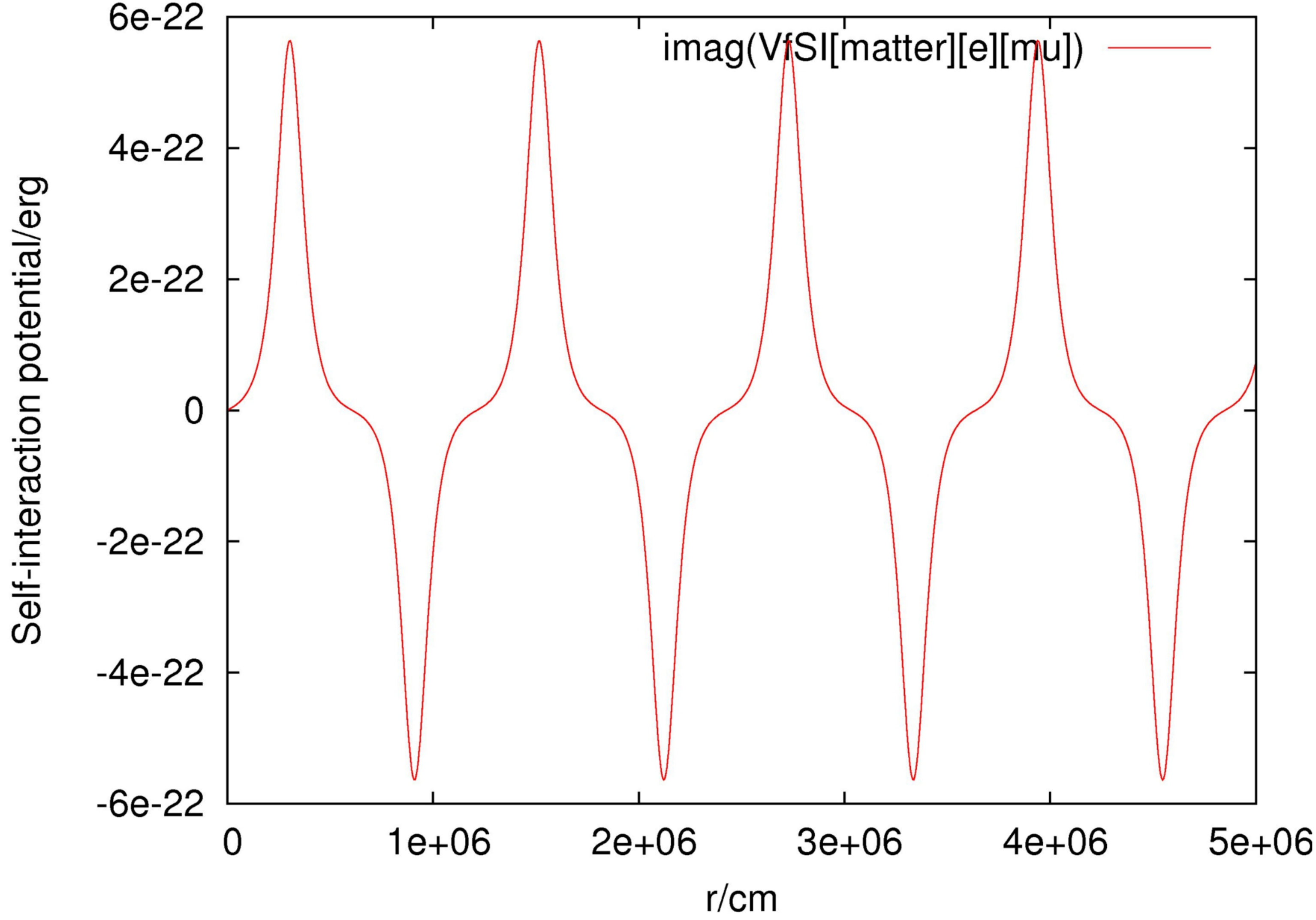
The self-interaction Hamiltonian is

$$H_{SI}=\mu\left(\rho-\alpha\bar{\rho}^{*}\right)=\mu\left(S\rho(0)S^{\dagger}-\alpha\left(\bar{S}\bar{\rho}(0)\bar{S}^{\dagger}\right)^{*}\right)$$

α is the asymmetry, μ is the strength of the self-interaction.

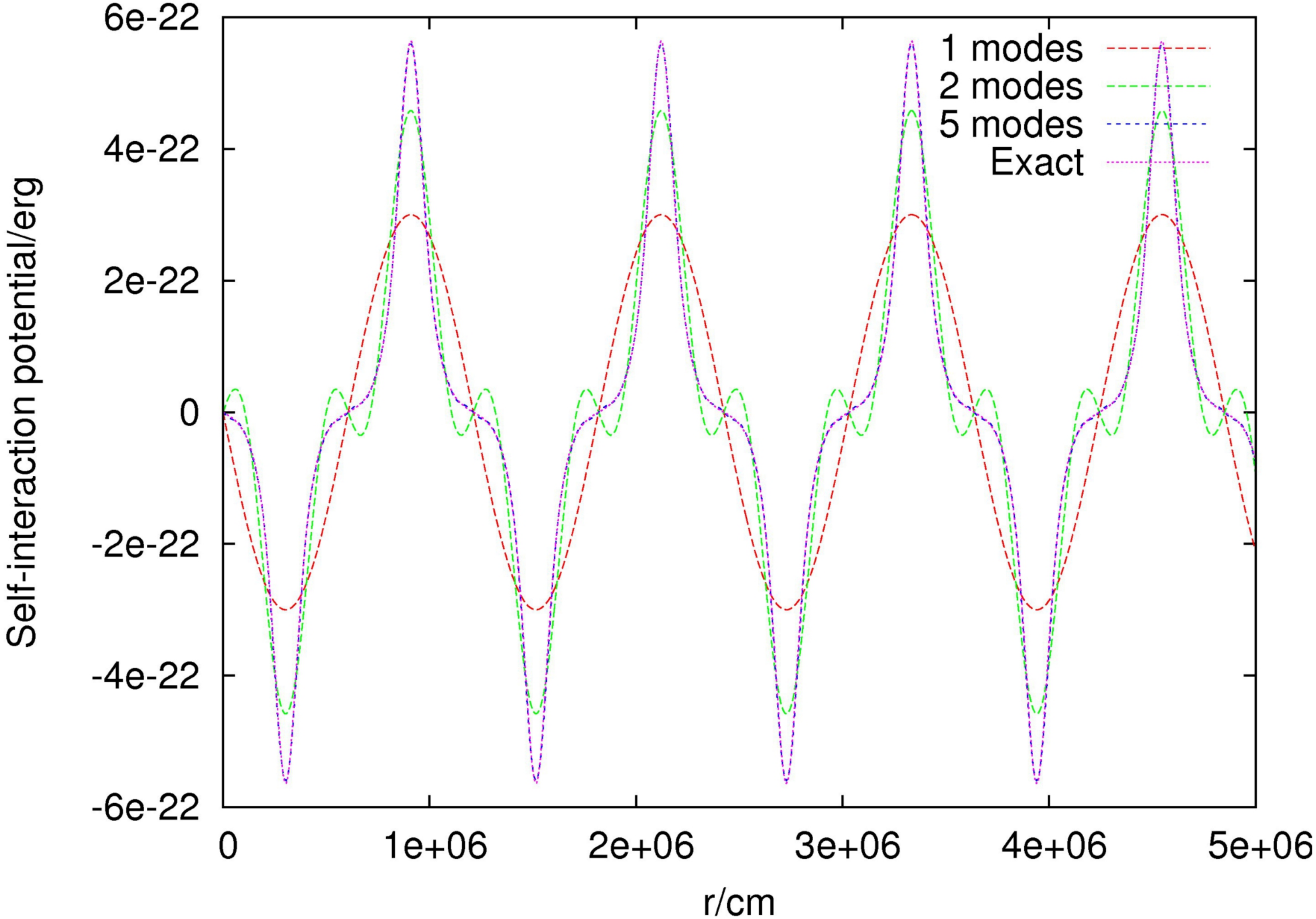
We consider first the case $\alpha=1$.

- This is the first problem found in Hannestad et al PRD **74** 105010 (2006).

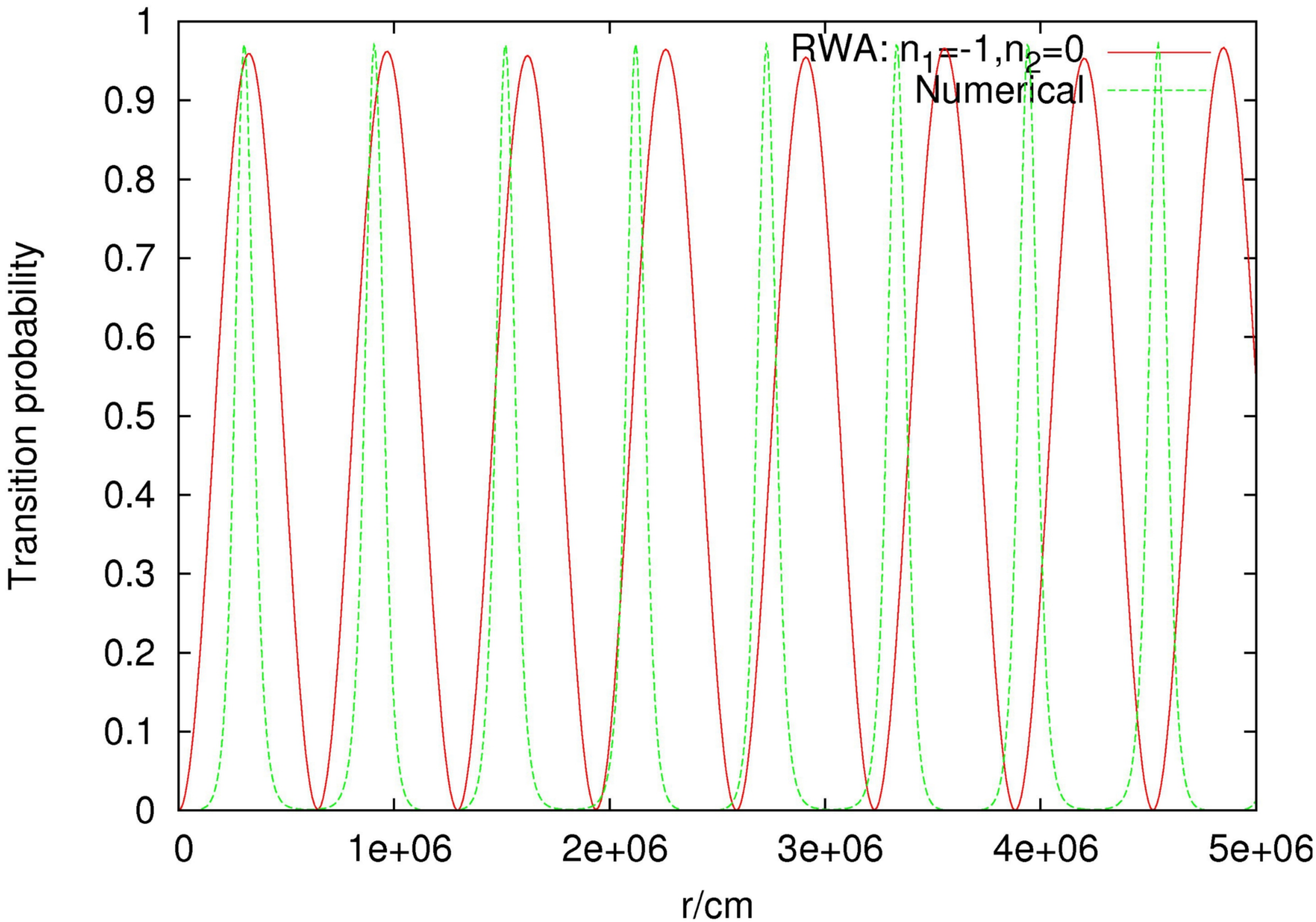


There is only one non-zero element in H_{SI} for this case.

We decompose H_{SI} into its Fourier modes.



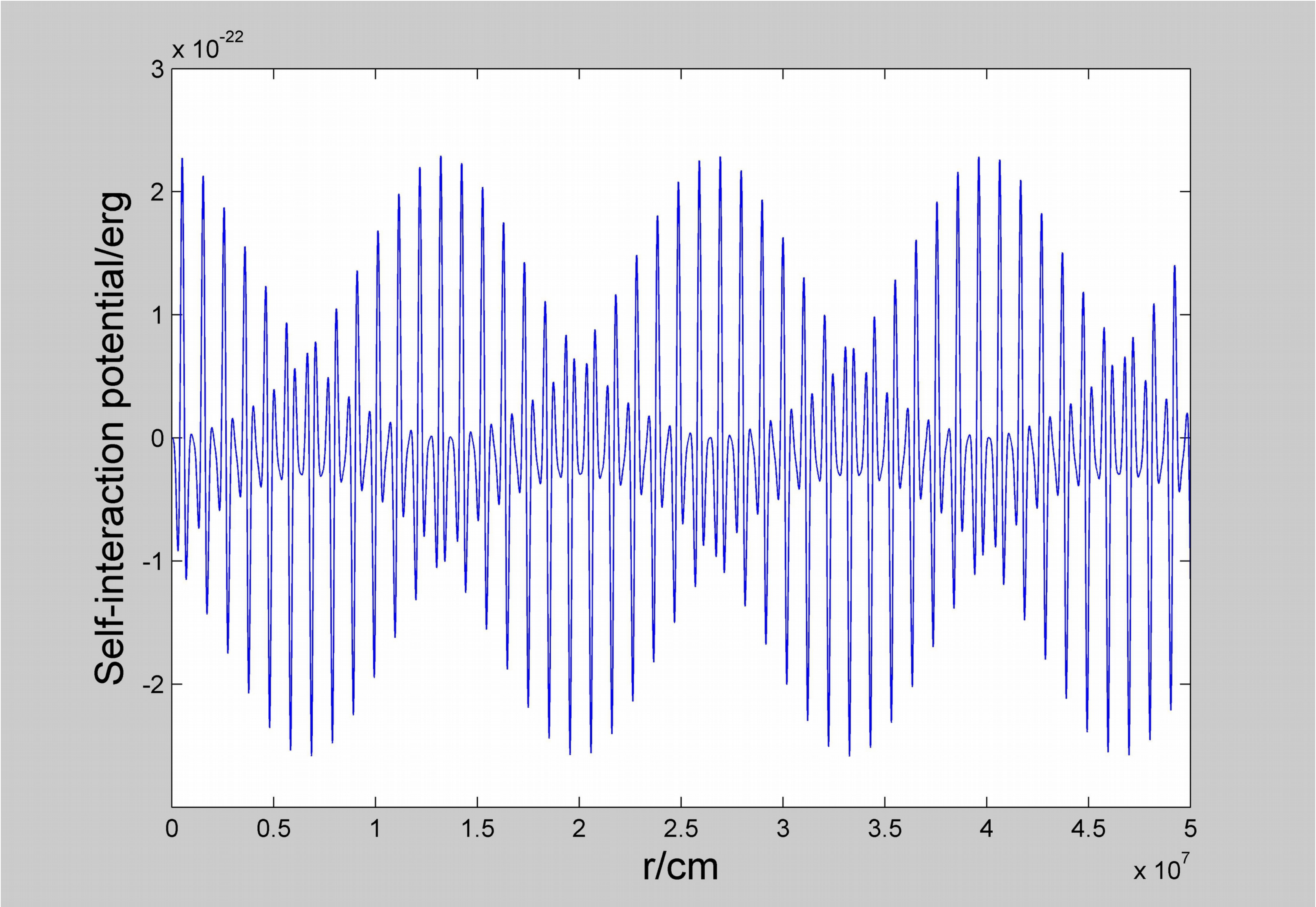
Only the odd harmonics of the fundamental q_1 contribute.



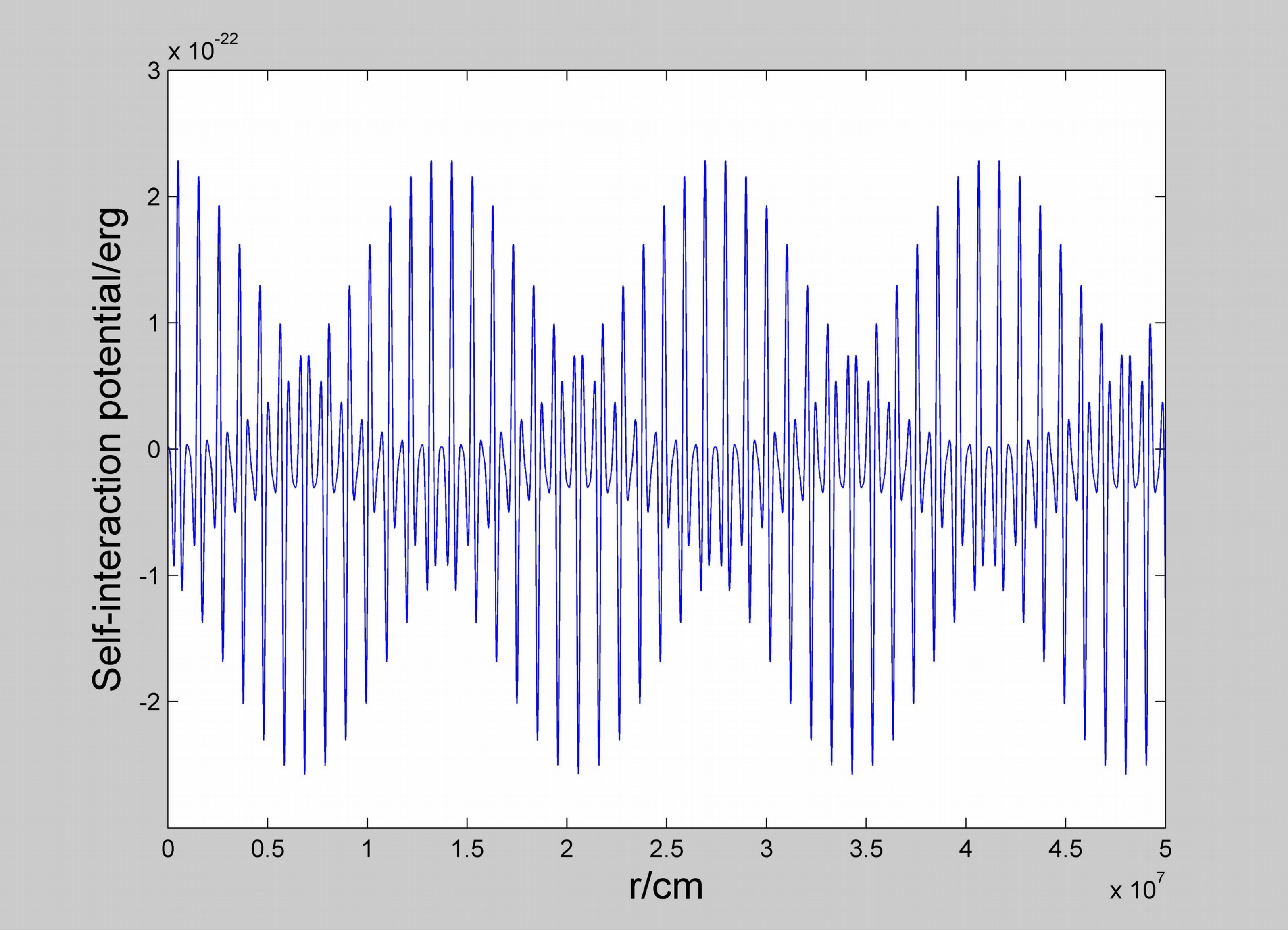
The RWA does pretty well*.

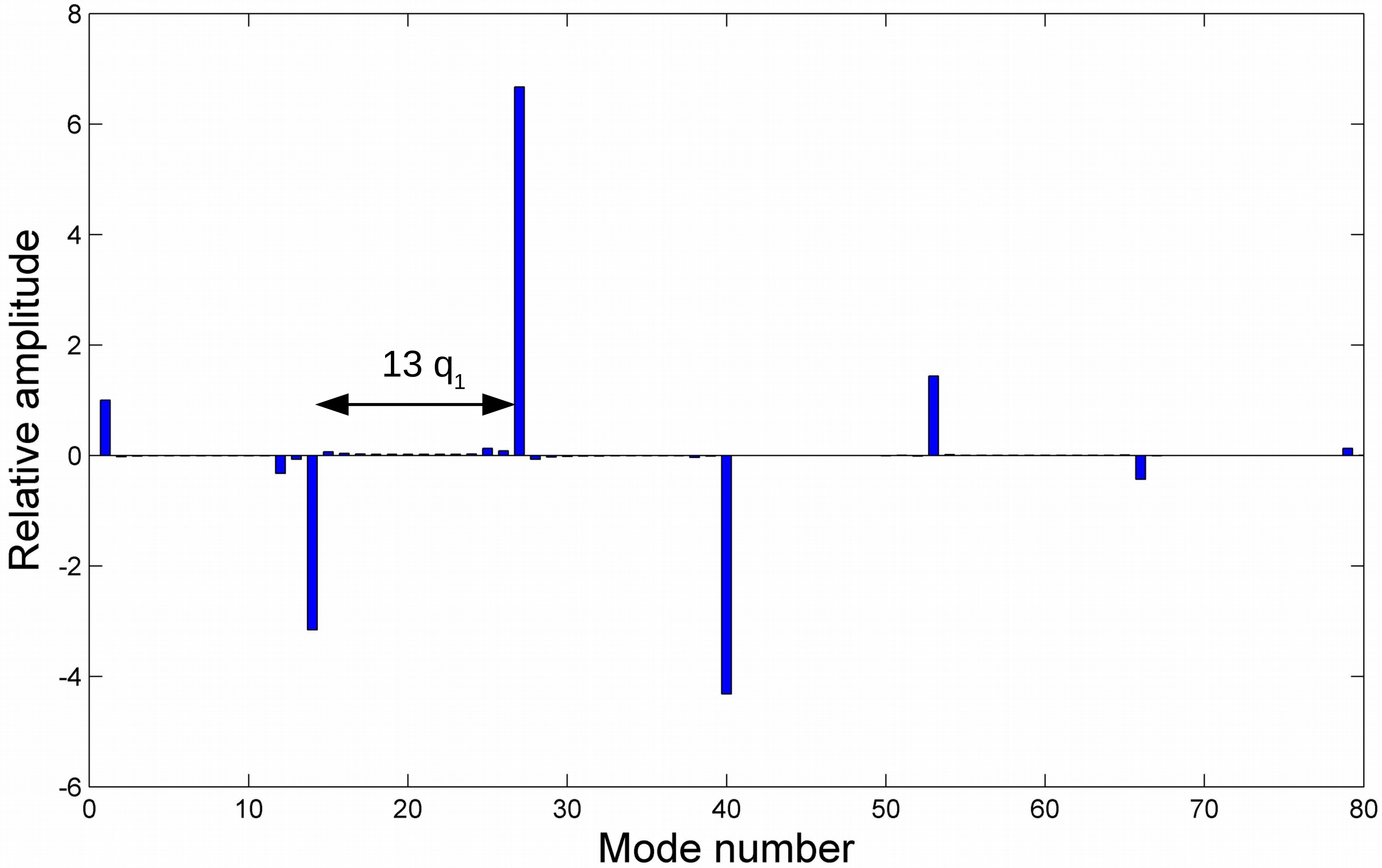
*From past experience with the MSW problem, matching the frequency is hard, the amplitude is easier.

Consider the asymmetric case $\alpha = 0.5$.

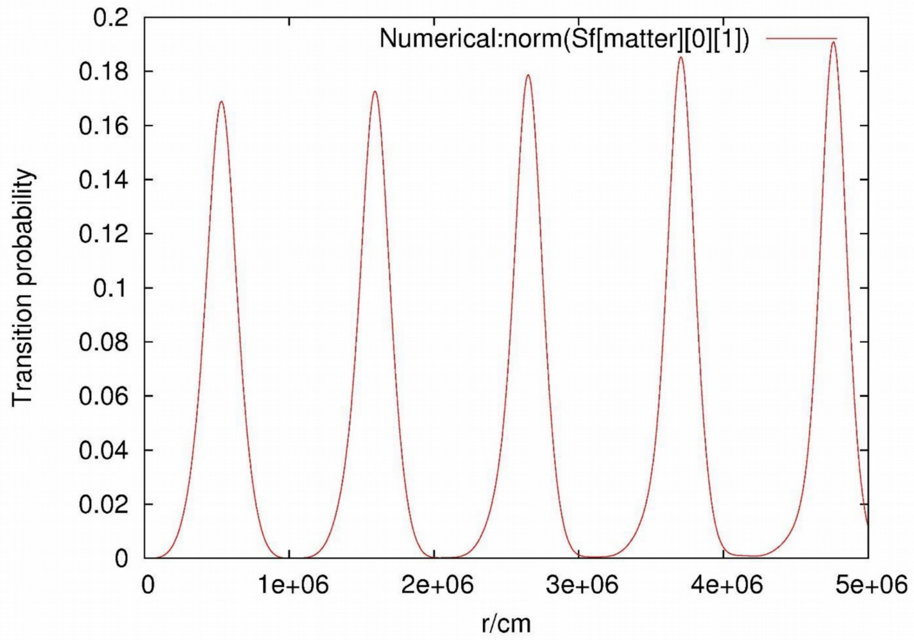


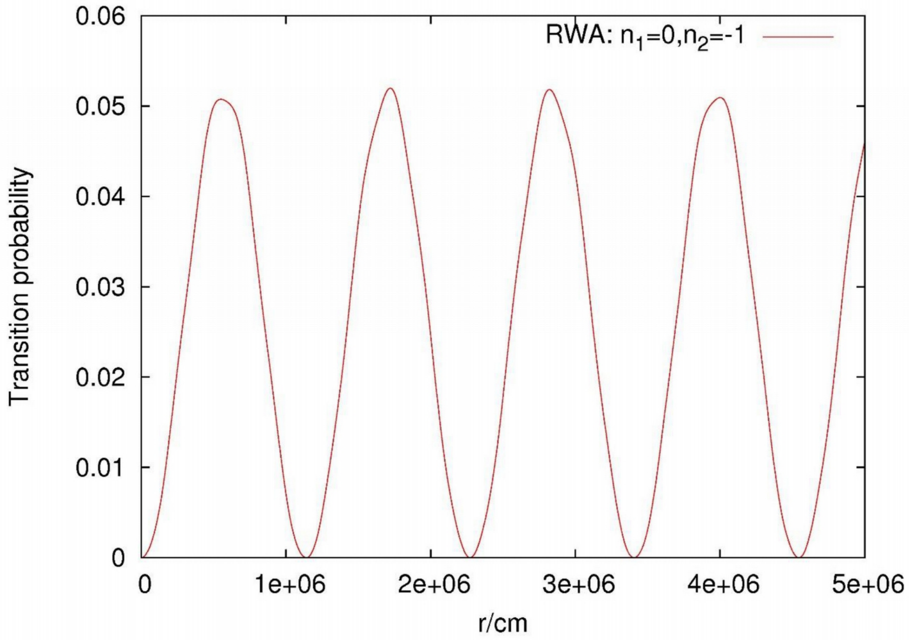
We can again find a Fourier decomposition which matches the potential well using ~7 modes.





The spacing between the harmonics is 13 q_1 (?!)



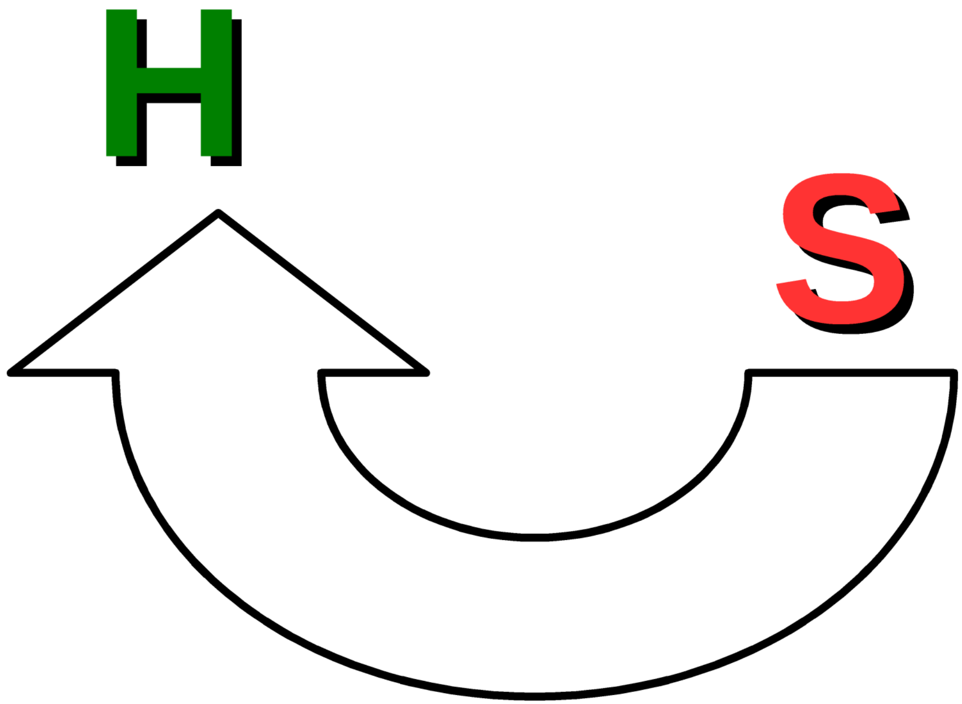


The RWA solution with mode 27(n1=0) plus mode 53(n2=-1).

Using just two modes, the frequency is almost right, the amplitude is too small.

- We probably need all 7 modes to get the amplitude right and we need to include all combinations of {n} with the same detuning frequency.

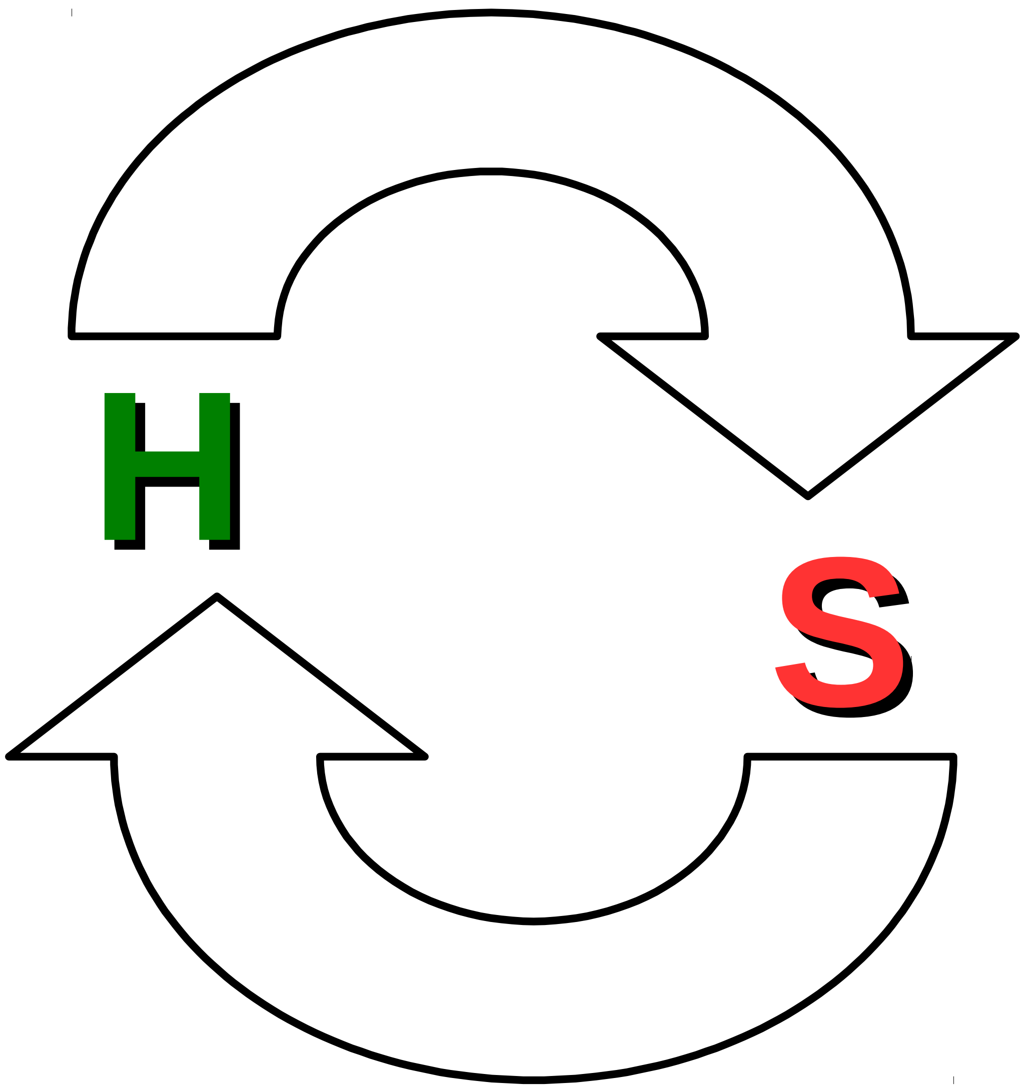
From S to H



Knowing the general form for S, it is possible to construct a self-interaction Hamiltonian H_{SI}' in the original basis.

- The general form for H_{SI}' is very messy involving products of infinite series i.e. $\Pi(\Sigma\ldots)$ just as in the derivation of the solution for B.

Self Consistency



We are working on the self-consistency question $H_{SI} = H_{SI}'$.

- In general, we do not have equal numbers of Fourier modes in H_{SI} and H_{SI}'.

$$\sum_{a=1}^{N_q} \left(\# e^{i q_a r} \right) = \prod_{b=1}^{N_q} \left(\sum_{m_b=-\infty}^{+\infty} \# e^{i m_b q_b r} \right)$$

N_q must be infinite and the wavenumbers cannot be independent.

- The wavenumbers must form a harmonic series.

Questions we're working on:

- Can we find the fundamental wavenumber q_1?
- Why don't all harmonics appear?
- How do we compute the Fourier coefficient matrices?

Summary

- Using the RWA, it is possible to solve for the evolution with the general perturbing Hamiltonian

$$\delta H = \sum_a C_a e^{-iq_a r} + C_a^{\dagger} e^{iq_a r}$$

- The RWA predicts the amplitude and frequency of the solution to the symmetric self-interaction problem.

- Given the general form of the solution we can construct a self-interaction Hamiltonian from it.

- How do we find a self-consistent solution?