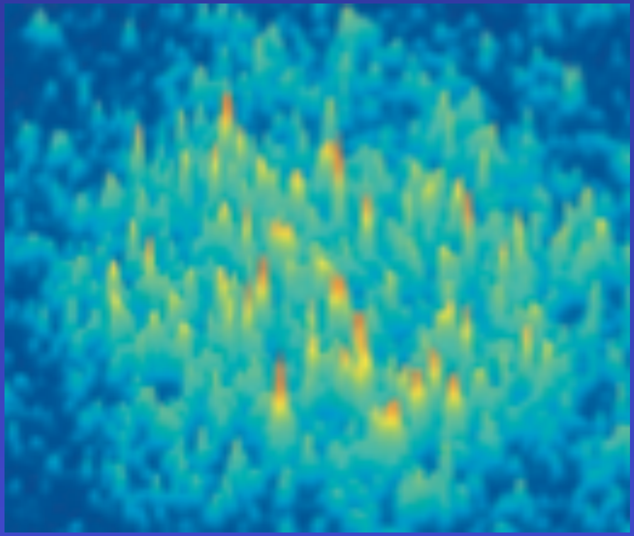
Probing the Optical Conductivity of Harmonically-confined Quantum Gases

Eugene Zaremba

Queen's University, Kingston, Ontario, Canada



Financial support from NSERC

Work done in collaboration with Ed Taylor and Zhigang Wu

The Art of Poking and Prodding

- parametric perturbations

$$V_{\text{trap}}(x,t) = \frac{1}{2} m\omega_0(t)^2 x^2$$

$$\omega_0(t) = \omega_0 + \Delta\omega_0 \theta(t), \quad \omega_0(t) = \omega_0 + \Delta\omega_0 \sin \Omega t$$

- modulation spectroscopy

$$V_{\text{trap}}(x) + V_{\text{opt}}(x,t)$$

$$V_{\text{opt}}(x,t) = V_0(t) \sin kx, \quad V_{\text{opt}}(x,t) = V_0 \sin k(x - x_0(t))$$

- effective dynamics

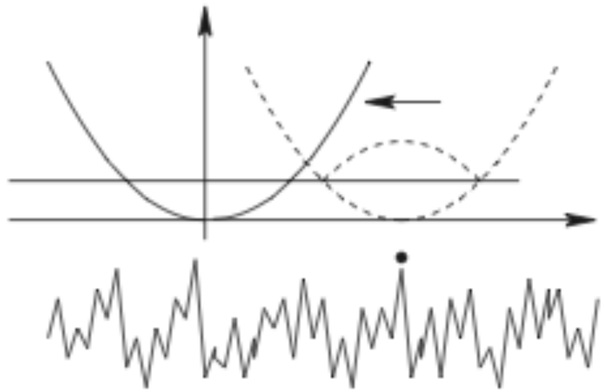
$$V_{\text{trap}}(x) + V_{\text{ext}}(x)$$

$$\Phi(x) \to \Phi(x - x_0) \Rightarrow V_{\text{ext}}(x - x_0 \cos \omega_0 t)$$

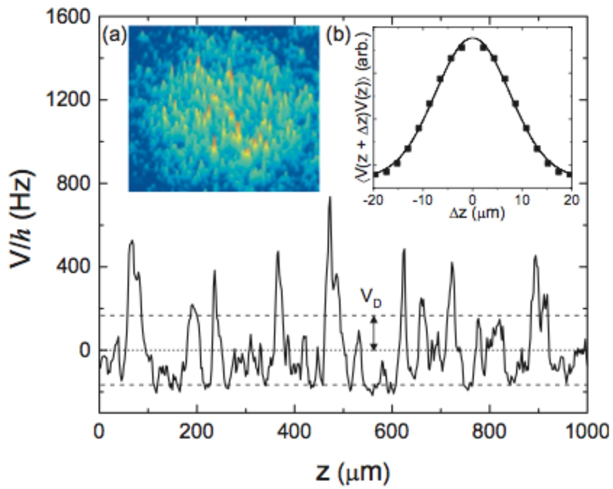
- oscillating harmonic potential – optical conductivity

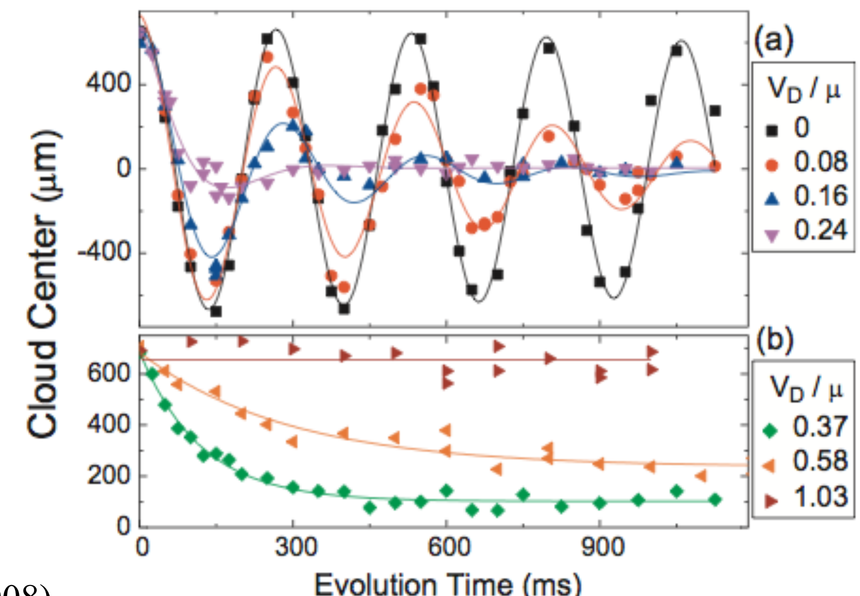
$$V_{\text{trap}}(x,t) = \frac{1}{2} m\omega_0^2 (x - x_0 \sin \omega t)$$

Experiment on Disorder-induced Damping



- oscillations of the centre of mass are induced by suddenly shifting the harmonic confining potential
- a disorder potential created by a laser speckle pattern leads to the decay of the centre of mass motion





Y.P. Chen *et al*., Phys. Rev. A**77**, 033632 (2008)

Equation of Motion in Presence of Disorder

- the Hamiltonian of the system is

$$\hat{H} = \sum_{i=1}^{N} \left[\frac{\hat{p}_i^2}{2m} + V_{\mathrm{trap}}(\hat{\mathbf{r}}_i) \right] + \hat{V}_{\mathrm{int}} + \sum_{i=1}^{N} V_{\mathrm{dis}}(\hat{\mathbf{r}}_i) \equiv \hat{H}_0 + \hat{V}_{\mathrm{dis}}$$

- for an arbitrary dynamical state, the c.m. coordinate $Z(t) = \langle \Psi(t) | \hat{R}_z | \Psi(t) \rangle$ satisfies

$$\frac{d^2 Z}{dt^2} + \omega_z^2 Z = \frac{F}{M}$$

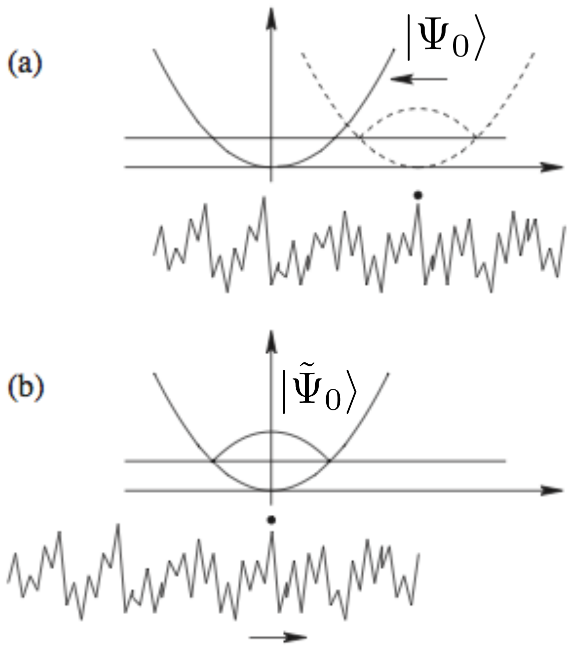
where

$$F(t) = \langle \Psi(t) | \widehat{F}_z | \Psi(t) \rangle$$

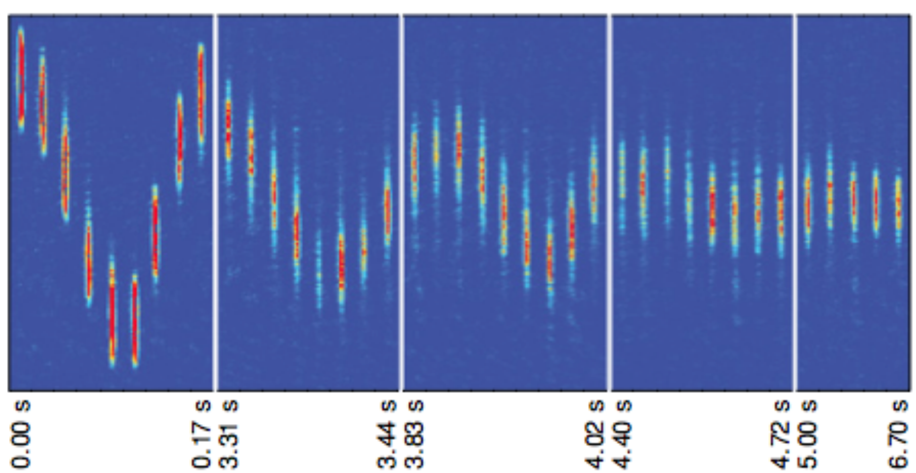
$$\widehat{F}_z = - \sum_{i=1}^{N} \frac{\partial V_{\mathrm{dis}}(\hat{\mathbf{r}}_i)}{\partial \hat{z}_i}$$

- evaluation of the force requires knowledge of the dynamical state

Extended Harmonic Potential Theorem



- in (a), the condensate is initially displaced relative to the centre of the harmonic trap and oscillates in the presence of a static disorder potential
- in (b), the condensate is initially at rest at the centre of the harmonic trap but the disorder potential is displaced and is made to oscillate at the trap frequency
- the force is *exactly the same* in these two physically distinct scenarios



D. Dries *et al.*, Phys. Rev. A**82**, 033603 (2010)

Linear Response

- the utility of this latter point of view is that the force can be calculated using conventional linear response theory if the perturbation is weak
- to second order in the disorder potential, we have

$$F(t) = -\frac{i}{\hbar}\int_0^t dt'\,\langle\Phi_0|[\widetilde{F}_{z,I}(t),\widetilde{V}_{\mathrm{dis},I}(\mathbf{x}(t'),t')]|\Phi_0\rangle$$
$$= \int_0^t dt' \int \frac{dk_z}{2\pi} R(k_z) k_z e^{ik_z[z(t)-z(t')]}\chi(k_z,k_z;t-t')$$

where

$$\chi(k_z,k_z;t-t') = \int d\mathbf{r}\int d\mathbf{r}' e^{ik_z(z-z')}\chi(\mathbf{r},\mathbf{r}';t-t')$$

$$\chi(\mathbf{r},\mathbf{r}';t-t') = \tfrac{i}{\hbar}\theta(t-t')\langle\Phi_0|[\hat{n}_I(\mathbf{r},t),\hat{n}_I(\mathbf{r}',t')]|\Phi_0\rangle$$

- the disorder-averaged speckle pattern gives

$$R(k_z) = \sqrt{\pi}\sigma\overline{V^2_{\mathrm{dis}}}e^{-\frac{1}{4}\sigma^2 k_z^2}$$

Density Response Function

- within the *Bogoliubov approximation*, the density response function is determined by the collective excitations of the system

$$\chi(\mathbf{r},\mathbf{r}';t-t') = \frac{i}{\hbar}\theta(t-t')\sum_i \left\{\delta n_i(\mathbf{r})\delta n_i^*(\mathbf{r}')e^{-i\omega_i(t-t')} - \text{c.c.}\right\}$$

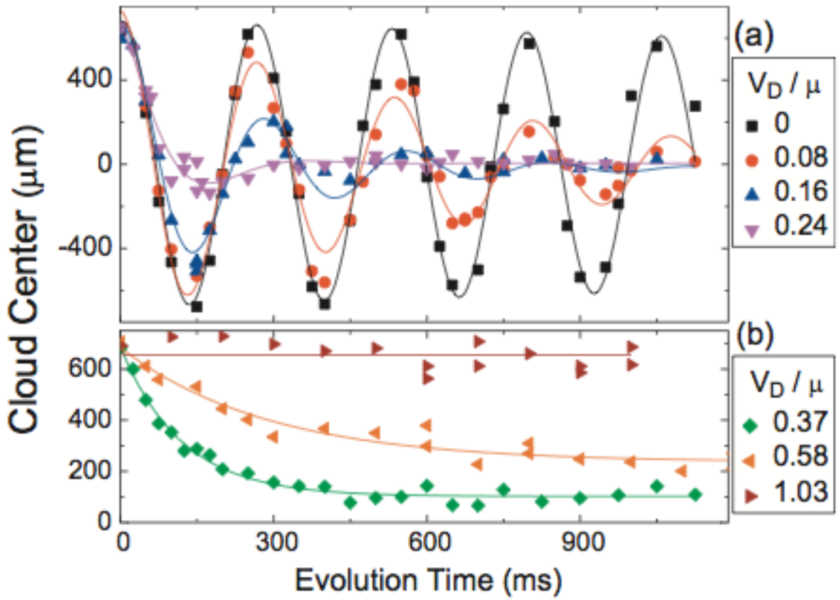
- the Bogoliubov excitations for an arbitrary anisotropic trap must be determined numerically

- a simpler approach is to make use of the *cylindrical local density approximation*, a variant of the more commonly used bulk local density approximation

$$\chi(k_z,k_z;\tau) \simeq \int dz\,\chi_{\text{cyl}}(k_z,\tau;\nu(z))$$

- in this approximation, the condensate responds to a perturbation as if it were locally part of an infinitely long cylindrical condensate with density per unit length $\nu(z)$

Damping of the Centre of Mass Oscillation



Y.P. Chen *et al.*, Phys. Rev. A**77**, 033632 (2008)

- the damping of the mode increases with increasing strength of the disorder potential

$$Z(t) = z_0 e^{-bt} \cos \omega_z t$$

- theoretical estimate of damping

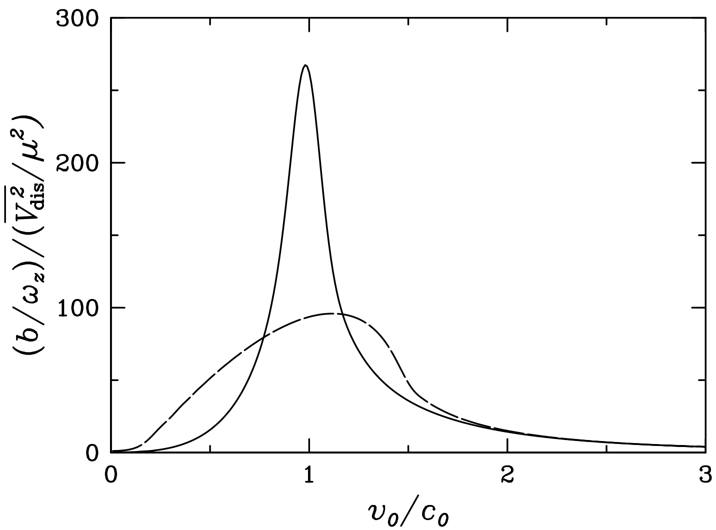
$$\frac{d^2 Z}{dt^2} + \omega_z^2 Z = \frac{F}{M}$$

$$\Delta Z_l \equiv Z(T_l) - Z(T_{l-1}) = -\frac{1}{M\omega_z} \int_{T_{l-1}}^{T_l} dt \sin \omega_z t F(t)$$

$$b \simeq -\frac{\Delta Z_1}{z_0 T} \simeq -\frac{\Delta Z_\infty}{z_0 T}$$

*Results**

$$\frac{b}{\omega_z} = \frac{2\pi}{M v_0^2 \hbar} \int_{-R_z}^{R_z} dz \int \frac{dk}{2\pi} R(k) \sum_j \psi_j^2(k) \sum_{n=1}^{\infty} n J_n^2(z_0 k) \delta(\omega_j(k) - n\omega_z)$$



Experimental Comparison

- initial displacement

$$z_0 \sim 700\,\mu\text{m} \quad v_0/c_0 \sim 2.9$$

- disorder strength

$$\overline{V_{\rm dis}^2}/\mu^2 \simeq 0.0064$$

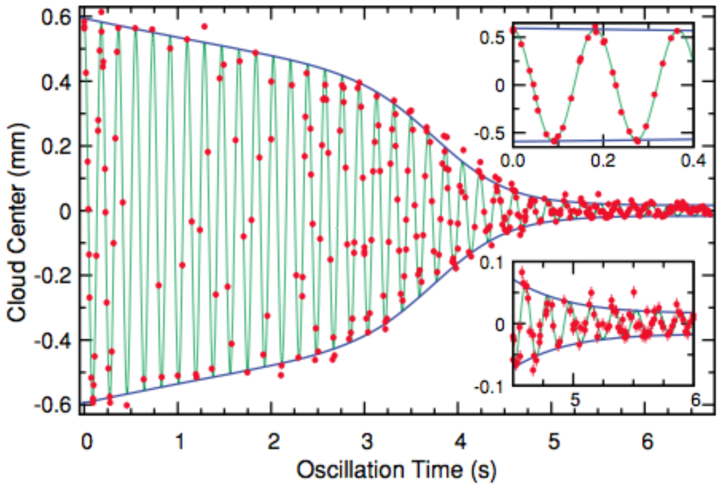
- damping

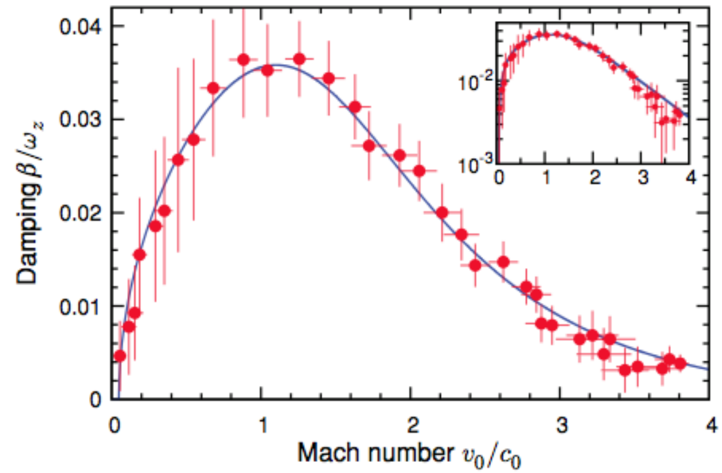
$$(b/\omega_z)_{\rm th} \simeq 0.028 \quad (b/\omega_z)_{\rm exp} \simeq 0.034$$

- the damping exhibits a resonant peak for a velocity equal to the sound speed at the centre of the trap
- the results for the bulk LDA differ quantitatively from the cylindrical LDA

*Z. Wu and E. Zaremba, Phys. Rev. Lett. **106**, 165301 (2011)

*Other Data**





- we do not find good agreement for this set of data – we have no explanation for this
- for comparable experimental parameters, the Chen *et al*. experiment seems to give different results from the Dries *et al.* experiment
- our theory can only account for the damping at early times immediately after excitation since the cloud heats up significantly during the course of the evolution

*D. Dries *et al.*, Phys. Rev. A**82**, 033603 (2010)

*Shaking Potentials**

- we consider a harmonically-trapped system in the presence of a time-dependent external potential

$$\hat{H}(t) = \hat{H}_0 + \sum_{i=1}^{N} V_{\rm ext}(\hat{\mathbf{r}}_i - \mathbf{r}_0(t))$$

$$\hat{H}_0 = \sum_{i=1}^{N}\left(\frac{\hat{\mathbf{p}}_i^2}{2m} + V_{\rm trap}(\hat{\mathbf{r}}_i)\right) + \sum_{i<j} v(\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j)$$

- for small displacements $\mathbf{r}_0(t)$,

$$\hat{H}(t) = \hat{H} + \hat{H}'(t)$$

with

$$\hat{H} = \hat{H}_0 + \sum_{i=1}^{N} V_{\rm ext}(\hat{\mathbf{r}}_i)$$

$$\hat{H}'(t) = -\sum_{i=1}^{N} \nabla V_{\rm ext}(\hat{\mathbf{r}}_i)\cdot\mathbf{r}_0(t) = -\int d\mathbf{r}\nabla V_{\rm ext}(\mathbf{r})\cdot\mathbf{r}_0(t)\hat{n}(\mathbf{r}).$$

- the perturbation is seen to couple to the particle density

*Z. Wu and E. Zaremba, Ann. Phys. **342**, 214 (2014)

Perturbative Analysis

- the perturbation can be rewritten as

$$\hat{H}'(t) = \sum_{\mu} r_{0\mu}(t) \frac{m}{i\hbar}[\hat{J}_\mu, \sum_{i=1}^{N} V_{\rm ext}(\hat{\mathbf{r}}_i)] = \sum_{\mu} r_{0\mu}(t) \left[\hat{A}_{1\mu} + \hat{A}_{2\mu}\right]$$

$$\hat{A}_{1\mu} = \frac{m}{i\hbar}[\hat{J}_\mu, \hat{H}], \quad \hat{A}_{2\mu} = Nm\omega_\mu^2 \hat{R}_\mu$$

$$\hat{J}_\mu = \frac{1}{m}\hat{P}_\mu = \frac{1}{m}\sum_{i=1}^{N} \hat{p}_{i\mu}, \quad \hat{R}_\mu = \frac{1}{N}\sum_{i=1}^{N} \hat{r}_{i\mu}$$

- the energy absorption rate is given quite generally by

$$\frac{d\tilde{E}}{dt} = \langle \tilde{\Psi}(t)|\frac{d\hat{H}'(t)}{dt}|\tilde{\Psi}(t)\rangle$$

- in linear response one finds

$$\frac{d\tilde{E}}{dt} = \sum_{i\mu} \dot{r}_{0\mu}(t)\langle \Phi_0|\hat{A}_{i\mu}|\Phi_0\rangle - \sum_{i\mu,j\nu} \dot{r}_{0\mu}(t) \int_{-\infty}^{\infty} dt' \chi_{i\mu.j\nu}(t-t') r_{0\nu}(t')$$

where we have defined the retarded response functions

$$\chi_{i\mu,j\nu}(t-t') \equiv \frac{i}{\hbar}\theta(t-t')\langle \Phi_0|[\hat{A}_{i\mu}(t), \hat{A}_{j\nu}(t')]|\Phi_0\rangle$$

Perturbative Analysis, cont'd

- for a monochromatic displacement of the form

$$r_{0\mu}(t) = \frac{1}{2}(r_{0\mu}e^{-i\omega t} + r_{0\mu}^* e^{i\omega t})$$

the time-averaged energy absorption rate is found to be

$$\overline{\frac{d\tilde{E}}{dt}} = \frac{m^2\omega^3}{2} \sum_{\mu\nu} r_{0\mu}^* r_{0\nu} \left(1 - \frac{\omega_\mu^2}{\omega^2}\right) \left(1 - \frac{\omega_\nu^2}{\omega^2}\right) \mathrm{Im}\Pi_{\mu\nu}(\omega)$$

where $\mathrm{Im}\Pi_{\mu\nu}(\omega)$ is the imaginary part of the Fourier transform of the current-current response function

$$\Pi_{\mu\nu}(t-t') \equiv \frac{i}{\hbar}\theta(t-t')\langle\Phi_0|[\hat{J}_\mu(t), \hat{J}_\nu(t')]|\Phi_0\rangle$$

- the energy absorption rate is directly related to the *optical conductivity* defined as

$$\mathrm{Re}\Sigma_{\mu\nu}(\omega) = \frac{1}{\omega}\mathrm{Im}\Pi_{\mu\nu}(\omega)$$

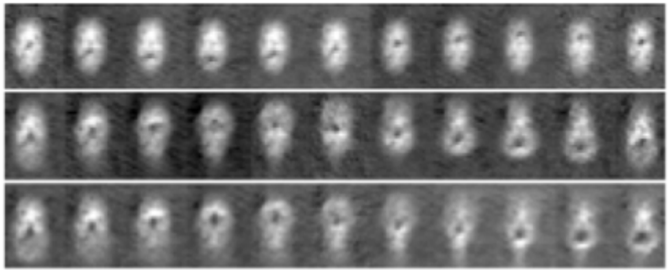
Discussion

- for a displacement of the form $\mathbf{r}_0(t) = z_0 \hat{\mathbf{z}} \sin \omega t$ we have

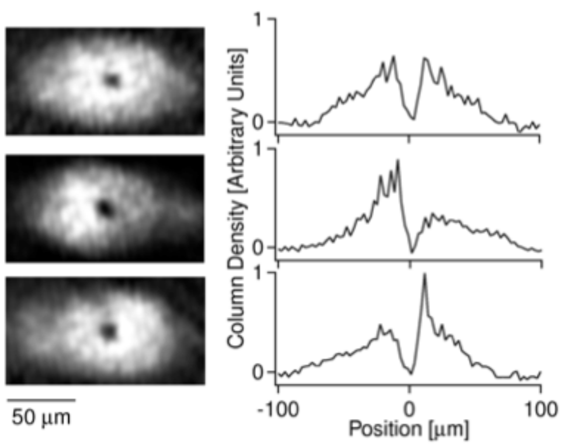
$$\overline{\frac{d\tilde{E}}{dt}} = \frac{1}{2} m^2 \omega^3 z_0^2 \left(1 - \frac{\omega_z^2}{\omega^2}\right)^2 \mathrm{Im}\Pi_{zz}(\omega)$$

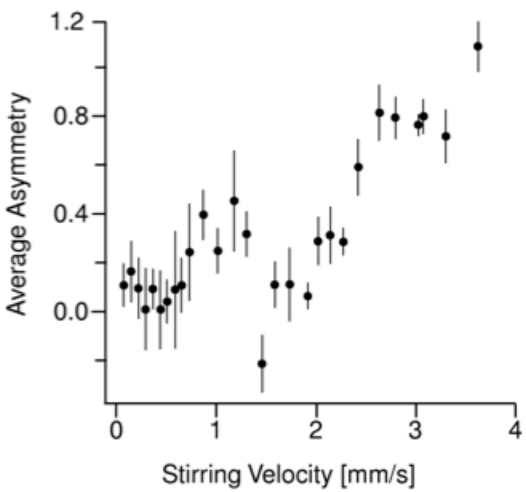
- the current response includes the full effect of the external potential *and* the harmonic trapping potential
- information about the possible excitations in the system is contained in the frequency dependence of the current response function
- in the limit $\omega_z \to 0$ one recovers the result of Tokuno and Giamarchi (PRL 106, 205301 (2011)). It should be noted however that their derivation is not quite right.
- one sees that the energy absorption rate *vanishes* when $\omega = \omega_z$. This is a consequence of the extended harmonic potential theorem (Wu and Zaremba, Ann. Phys. **342**, 214 (2014)) and is not limited to the perturbative analysis.
- the low-frequency spectral density is enhanced in the presence of the trap

MIT Experiment on Superfluidity [*PRL* ***85****, 2228 (2000)*]



- a gaussian potential is rastered periodically through an elongated cylindrical condensate





- the induced density exhibits an asymmetry when dissipation sets in

- the dissipation rate is *small* when the rastering frequency is close to the axial trapping frequency

*Measuring the Optical Conductivity**

- we consider an oscillating trapping potential of the form

$$
\begin{aligned}
V_{\rm trap}(\mathbf{r},t) &= \sum_\mu \frac{1}{2} m\omega_\mu^2 [x_\mu - x_{0\mu}(t)]^2 \\
&= V_{\rm trap}(\mathbf{r}) - \sum_\mu F_\mu(t) x_\mu + \sum_\mu \frac{1}{2} m\omega_\mu^2 x_{0\mu}^2(t)
\end{aligned}
$$

with

$$
F_\mu(t) = m\omega_\mu^2 x_{0\mu}(t)
$$

- the displacement of the trap leads to a time-dependent homogeneous force acting on the system and results in a perturbation that couples to the centre-of-mass coordinate

$$
H'(t) = -\sum_\mu N F_\mu(t) \hat{R}_\mu
$$

- this coupling can be treated perturbatively if $F_\mu(t)$ is small

*Z. Wu, E. Taylor and E. Zaremba, EPL **110**, 26002 (2015)

Perturbation Analysis

- the centre of mass coordinate is the physical quantity of interest
- within linear response theory, one finds

$$\delta\langle \hat{R}_\mu(t)\rangle = \int dt'\, \chi^R_{\mu\nu}(t-t')NF_\nu(t')$$

where the centre of mass response function is

$$\chi^R_{\mu\nu}(t-t') = \frac{i}{\hbar}\theta(t-t')\langle[\hat{R}_\mu(t),\hat{R}_\nu(t')]\rangle$$

- Fourier transforming we have

$$R_\mu(\omega) = \tilde{\chi}^R_{\mu\nu}(\omega)NF_\nu(\omega)$$

- the total current operator is

$$\hat{J}_\mu(t) = N\frac{d\hat{R}_\mu(t)}{dt}$$

and the total induced current is given by

$$J_\mu(\omega) = \Sigma_{\mu\nu}(\omega)F_\nu(\omega)$$

where the global conductivity is

$$\Sigma_{\mu\nu}(\omega) = -i\omega N^2\tilde{\chi}^R_{\mu\nu}(\omega)$$

Observables

- we now choose the force to be in a specific direction $\nu = \beta$; the conductivity tensor is then given by

$$\Sigma_{\alpha\beta}(\omega) = -\frac{i\omega N}{F_\beta(\omega)} R_\alpha(\omega)$$

- furthermore, if the trap displacement is harmonic at the frequency ω_0, we have

$$R_\alpha(t) = A_\alpha(\omega_0)\cos[\omega_0 t - \phi_\alpha(\omega_0)]$$

and

$$\Sigma_{\alpha\beta}(\omega_0) = -\frac{i\omega_0 N}{F_\beta(\omega_0)} A_\alpha(\omega_0) e^{i\phi_\alpha(\omega_0)}$$

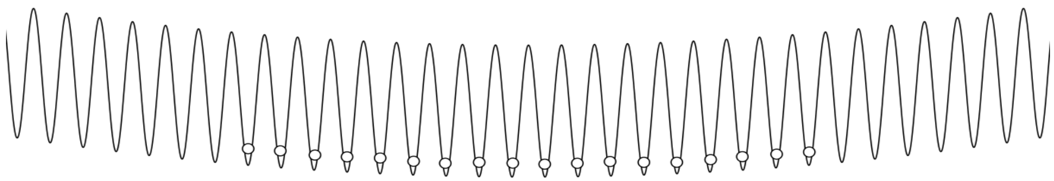
- thus, a measurement of the amplitude and phase of the centre of mass displacement as a function of the oscillation frequency ω_0 determines the optical conductivity
- the optical conductivity is also related to the total current correlations via

$$\Sigma_{\mu\nu}(\omega) = \frac{i}{\omega}\left[\frac{N}{m}\delta_{\mu\nu} - \tilde{\chi}^J_{\mu\nu}(\omega)\right]$$

with

$$\tilde{\chi}^J_{\mu\nu}(\omega) = \mathcal{F}\left[\frac{i}{\hbar}\theta(\tau)\langle[\hat{J}_\mu(\tau), \hat{J}_\nu(0)]\rangle\right] = \mathcal{F}\left[\Pi_{\mu\nu}(\tau)\right]$$

Application: 1D Bose-Hubbard Model



- we consider N atoms in a 1D optical lattice with harmonic confinement; in the tight-binding limit, the Hamiltonian is

$$\hat{H}_{\mathrm{BH}} = \sum_i \epsilon_i \hat{n}_i - t \sum_i \left(\hat{a}_i^\dagger \hat{a}_{i+1} + \hat{a}_{i+1}^\dagger \hat{a}_i \right) + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1)$$

- in this model, the centre-of-mass coordinate is given by

$$\hat{R} = \frac{a}{N} \sum_i i \hat{n}_i$$

and the current operator is

$$\hat{J} \equiv \frac{N}{i\hbar} [\hat{R}, \hat{H}_{\mathrm{BH}}] = -\frac{ta}{i\hbar} \sum_j \left(\hat{a}_j^\dagger \hat{a}_{j-1} - \hat{a}_{j-1}^\dagger \hat{a}_j \right)$$

- the optical conductivity is

$$\Sigma(\omega) = \frac{i}{\omega} \left[-\frac{a^2}{\hbar^2} \langle \hat{T} \rangle - \tilde{\chi}^J(\omega) \right]$$

Calculation of the Current Response

- the imaginary part of the current response function is given by

$$\mathrm{Im}\tilde{\chi}^J(\omega) = \pi \sum_\alpha |\langle \Phi_0 | \hat{J} | \Phi_\alpha \rangle|^2 \left[\delta(\hbar\omega - E_{\alpha 0}) - \delta(\hbar\omega + E_{\alpha 0}) \right]$$

and the real part is obtained from a KK transform

- the eigenstates and eigenenergies are obtained deep in the Mott-insulator regime $t/U \ll 1$; here, the relevant excitations are *single particle-hole* excitations
- we consider a lattice with M sites containing N particles; the ground state is

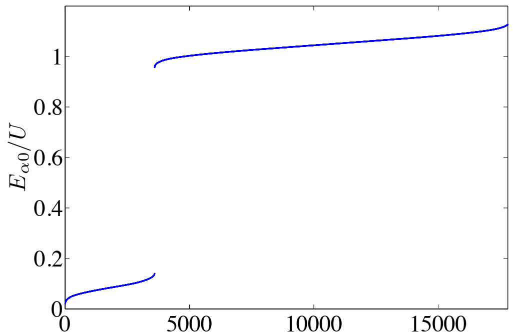
$$|\Phi_0\rangle = \prod_{i \in N} \hat{a}_i^\dagger |0\rangle$$

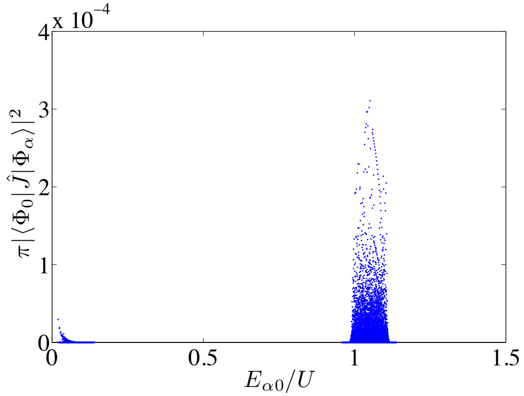
and the set of excited states retained are the particle-hole states

$$|ph\rangle = \hat{a}_p^\dagger \hat{a}_h |\Phi_0\rangle, \quad h \in N, p \in M$$

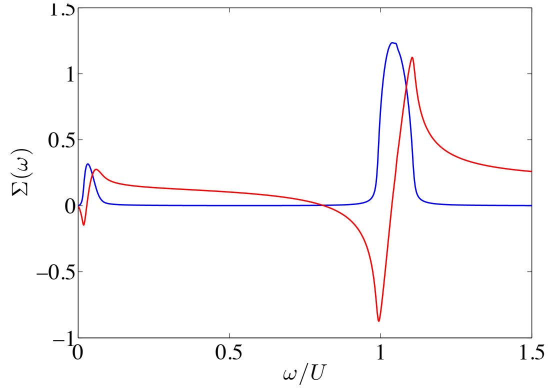
- the B-H Hamiltonian is diagonalized using the above truncated Hilbert space

Numerical Results





- excitation energies *vs.* state index
- current matrix elements *vs.* energy



- the real and imaginary parts of the optical conductivity
- the approximations yield an optical conductivity which accurately satisfies the exact sum rule[1]

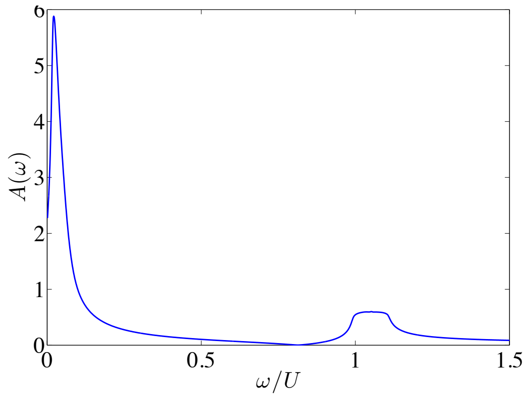
$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \mathrm{Re}\Sigma(\omega) = -\frac{a^2}{\hbar^2}\langle \hat{T} \rangle$$

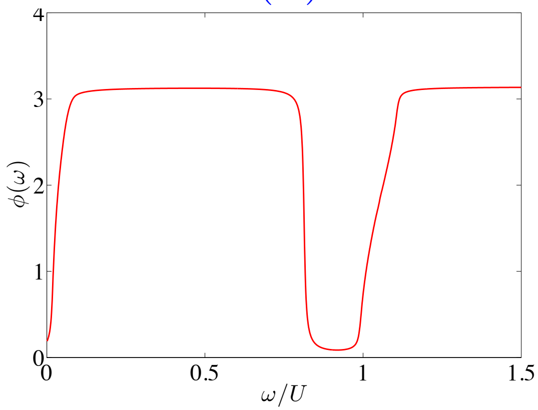
[1]P. F. Maldague, PRB **16**, 2437 (1977)

Behaviour of the Centre of Mass

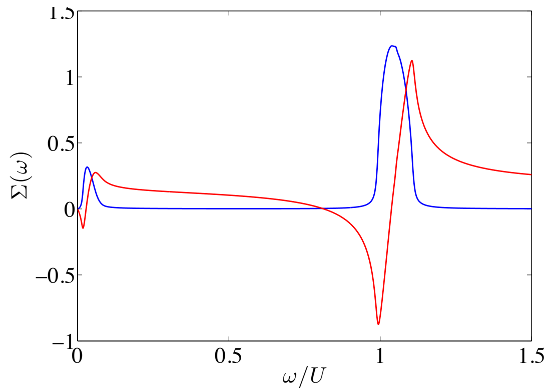
$$R(t) = A(\omega)\cos[\omega t - \phi(\omega)]$$

$$\Sigma(\omega) = -\frac{i\omega N}{F(\omega)} A(\omega) e^{i\phi(\omega)}$$





- amplitude *vs.* frequency
- phase *vs.* frequency



Summary

- we have used the Extended Harmonic Potential Theorem to analyze the disorder-induced damping of the centre-of-mass motion of a Bose condensate

- we have analyzed the energy absorption rate for the case of a shaking external potential in the presence of harmonic confinement; the energy absorption rate is related to the optical conductivity and vanishes when $\omega = \omega_{\text{trap}}$ as a consequence of the extended harmonic potential theorem

- a complementary excitation scheme is to shake the confining harmonic potential keeping other external potentials fixed; this directly yields the optical conductivity

- one advantage of this latter scheme is that the optical conductivity can be determined experimentally by simply measuring the position and phase of the center of mass