Fragmented states for spin-2 Bosons

S.K. Yip

Academia Sinica, Taiwan

H H Jen

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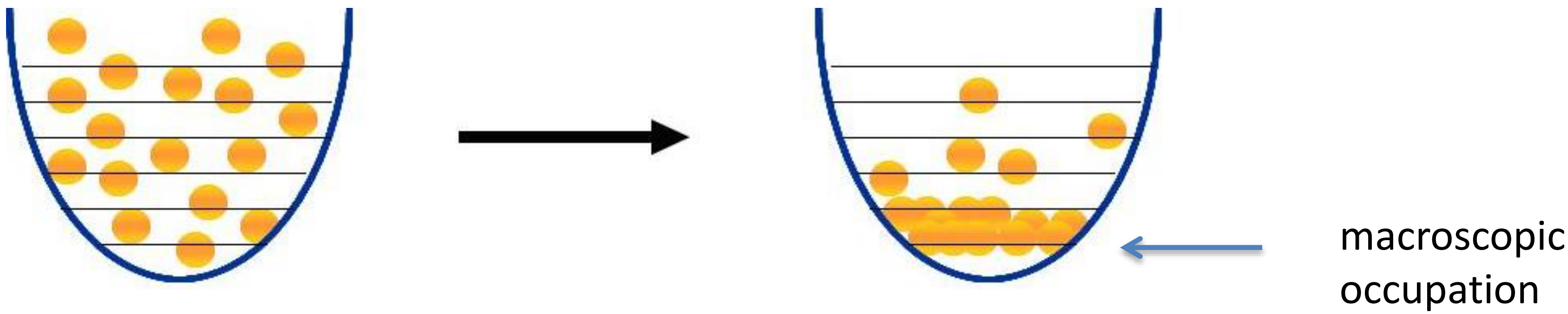
Bose-Einstein Condensation:



Formal definition:

Single particle density matrix $\rho(\vec{r},\vec{r}\,') = \sum_i \lambda_i \psi_i(\vec{r}) \psi_i^*(\vec{r}\,')$

One and only one macroscopically large: $\lambda_1 \sim N \gg \lambda_2 \ldots.$

Expectation values = $\lambda_1 \times$ Value in single particle state $\psi_1(\vec{r})$

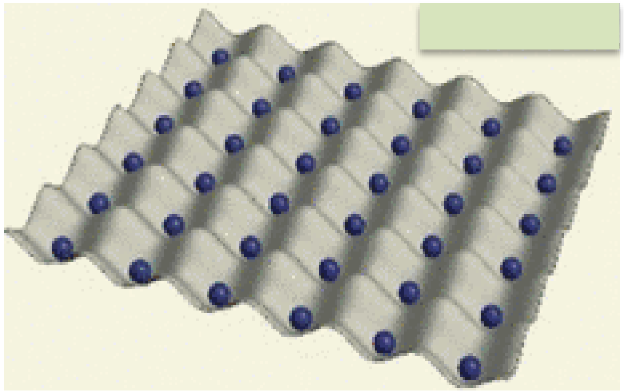
not so obvious!

Nozieres and Saint-James (1982)

Fragmentation when the above does not apply

Examples:

Mott insulating state



$$|\Psi\rangle = a_1^+ a_2^+ \ldots |0\rangle$$

$$\lambda_1 = \lambda_2 = \ldots = 1$$

1D: $n(k) \sim 1/k^{\alpha}$

Many large eigenvalues, none of order N

Two states 1 & 2, equal energy, attractive interaction between 1&2

$$H = \frac{g}{2} a_1^\dagger a_2^\dagger a_2 a_1 = \frac{g}{2} n_1 n_2$$

Ground state: Fock state:

$$|F\rangle = \frac{a_1^{\dagger N/2} a_2^{\dagger N/2}}{(N/2)!} |0\rangle$$

Density matrix:

$$\rho^{(1)} = \langle a_\mu^\dagger a_\nu \rangle = \frac{N}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Coherent state:

$$|\phi_N\rangle = \frac{1}{\sqrt{2^N N!}} (e^{-i\phi/2} a_1^\dagger + e^{i\phi/2} a_2^\dagger)^N |0\rangle$$

$$\rho^{(1)} = \frac{N}{2} \begin{pmatrix} 1 & e^{i\phi} \\ e^{-i\phi} & 1 \end{pmatrix}$$

Eigenvalues: N, 0

$$|F\rangle = \frac{2^{N/2}}{\sqrt{N!}} (N/2)! \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} |\phi_N\rangle$$

Two wells with inter-tunneling t,
repulsive interaction U between particles in same well

Nt >> U, single condensate

Coherent states:
$$|\phi_N\rangle = \frac{1}{\sqrt{2^N N!}}(e^{-i\phi/2}a_1^\dagger + e^{i\phi/2}a_2^\dagger)^N|0\rangle$$

↓ ϕ = phase difference

Density matrix:
$$\rho^{(1)} = \frac{N}{2}\begin{pmatrix} 1 & e^{i\phi} \\ e^{-i\phi} & 1 \end{pmatrix}$$

Eigenvalues: N, 0

U >> Nt: Fock state (fragmented):
$$|F\rangle = \frac{a_1^{\dagger N/2} a_2^{\dagger N/2}}{(N/2)!}|0\rangle \quad \rho^{(1)} = \langle a_\mu^\dagger a_\nu\rangle = \frac{N}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|F\rangle = \frac{2^{N/2}}{\sqrt{N!}}(N/2)!\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}|\phi_N\rangle$$

Fock state =
phase averaged coherent state

Bosons have hyperfine spins:

^{87}Rb lower hyperfine level $f = 1$ c < 0

m_f = -1, 0, 1

upper hyperfine level $f = 2$

^{23}Na lower hyperfine level f = 1 c > 0

upper hyperfine level f = 2

Spin-1:

Interaction = density-density + spin-spin

↓

$$c : \hat{F} \bullet \hat{F} :$$

$$c = (g_2 - g_0)/3$$

ignore spatial parts, mean-field condensate states

c > 0, condensate wavefunction

$$\psi_P = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

or other rotations

$$|\Psi\rangle \propto (a_P^+)^N |0\rangle$$

polar / nematic state

e.g.: $$|\Psi\rangle \propto (a_0^+)^N |0\rangle$$ or $$|\Psi\rangle \propto [(a_1^+ + a_{-1}^+)/\sqrt{2}]^N |0\rangle$$

c < 0

$$\psi_F = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

or rotations

ferro

$$|\Psi\rangle \propto (a_F^+)^N |0\rangle$$

e.g., $$|Y\rangle \mu (a_1^+)^N |0\rangle$$

c > 0

Ground state, N even

$$|\Psi\rangle \propto (\Theta_2^+)^{N/2}\,|0\rangle$$

$$\Theta_2^+ \equiv -2a_1^\dagger a_{-1}^\dagger + a_0^{\dagger 2}$$ create a single pair

Fragmented! $$N_1 = N_0 = N_{-1} = N/3$$

Two particle ground state: - | 1, -1 > - | -1, 1> + | 0 0 >

single condensate restored by spin-non-conserving processes
(Ho and Yip, 2000)

Q: Relationship between condensed and fragmented states ?

(Mueller, Baym, Ho, Ueda 2006; reviews by Ueda 2011)

$$\psi(\Omega) = D(\Omega)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

↓

Rotation by Euler angles $\Omega(\alpha\beta\gamma)$

$$\int_{\Omega} (a^{+}(\Omega))^{N} \, |0\rangle \propto (\Theta_{2}^{+})^{N/2} \, |0\rangle$$

(c.f. previous discussions on two-states)

$\psi(\Omega)$ is a broken symmetry version of the fragmented state

Fragmented state = rotational average of mean-field state

MF

$$\psi_F = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\psi_P = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

c, or a2 – a0

Exact:

$$(\Theta_2^+)^{N/2} \,|\, 0\rangle \propto \int_\Omega (a^+(\Omega))^N \,|\, 0\rangle$$

even

$$|\,\Psi\rangle \propto (a_1^+)^N \,|\, 0\rangle$$

$$a_1^+ (\Theta_2^+)^{N/2} \,|\, 0\rangle$$

odd

•

This is no longer the case for spin-2

$2 \oplus 2 = 0, 1, 2, 3, 4$; Three s-wave scattering lengths $a_0, a_2, a_4 > 0$

Spin dependent part of H:

$$b : \hat{F} \cdot \hat{F} : + g \hat{Q}_2^+ \hat{Q}_2$$

$$\beta = (g_4 - g_2)/7$$

$$\gamma = (g_0 - g_4)/5 - 2(g_2 - g_4)/7$$

$$g_F \equiv 4\pi\hbar^2 a_F / M$$

$$\Theta_2^+ = 2a_2^+ a_{-2}^+ - 2a_1^+ a_{-1}^+ + (a_0^+)^2$$

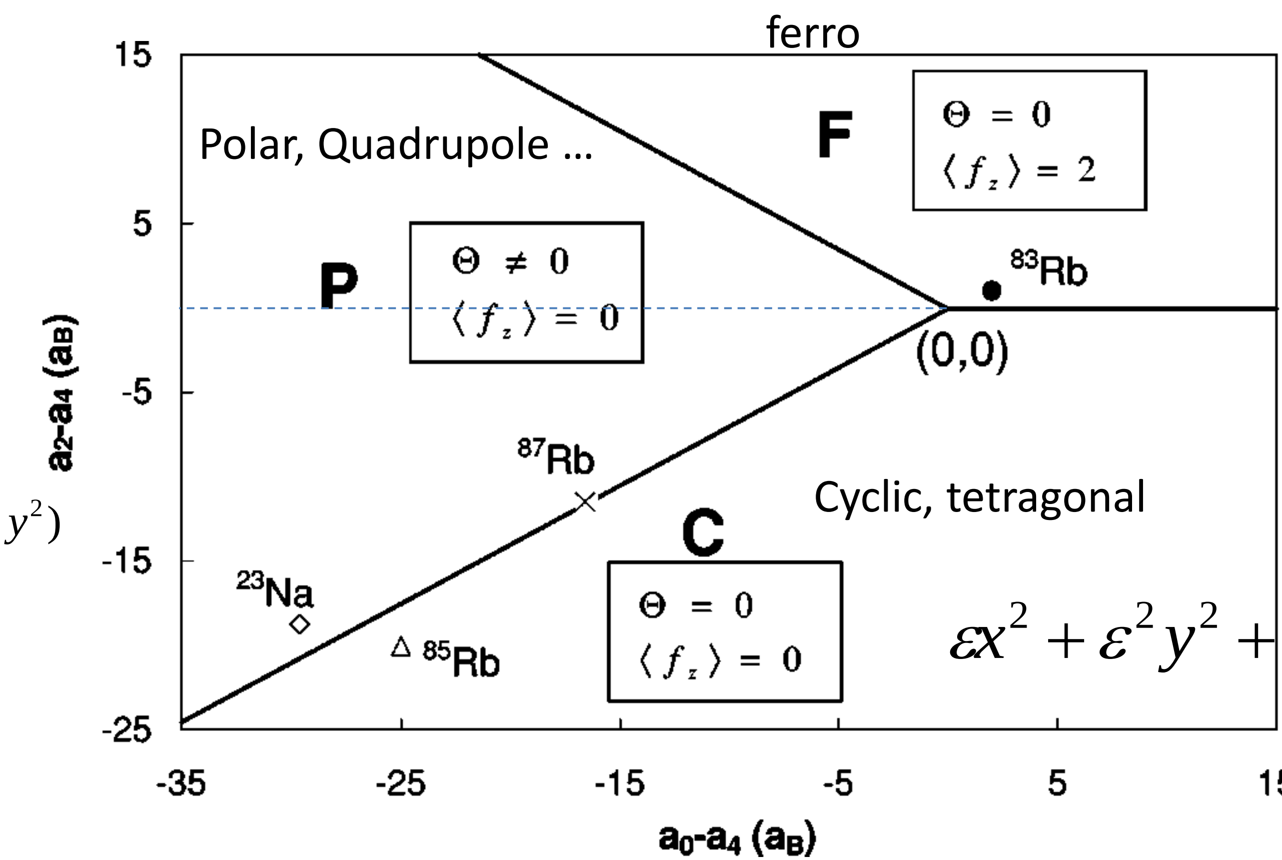
singlet pair

mean-field [Mermin 74; *Ciobanu, Yip, Ho; 2000*; + Ueda/Demler/Zhou …)

Θ, F → c numbers $Y_2^m \leftrightarrow (2z^2 - x^2 - y^2, x^2 - y^2, zx, zy, xy)$

$\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}$ $x^2 - y^2$

$\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$ $2z^2 - (x^2 + y^2)$



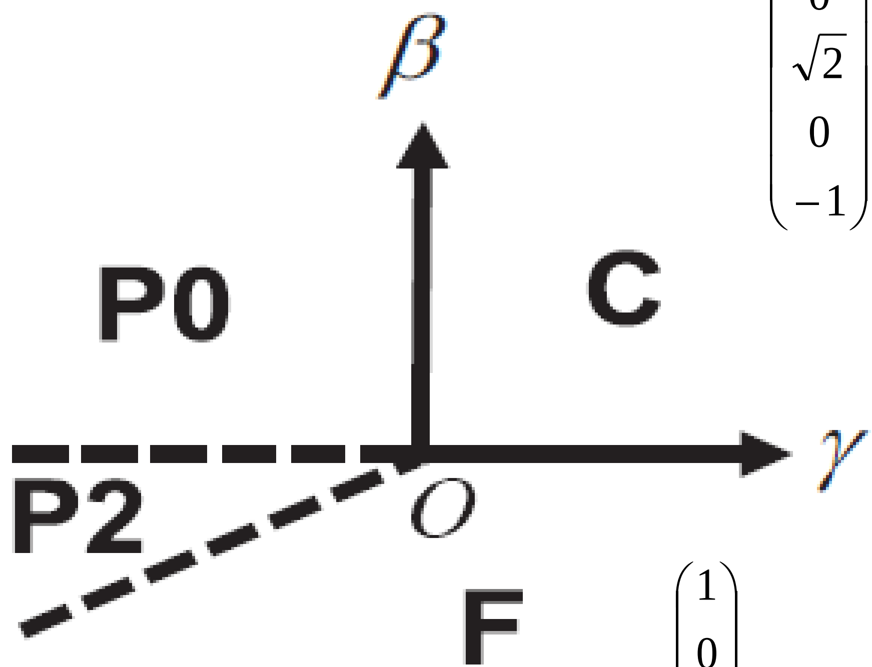
$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$

$\varepsilon x^2 + \varepsilon^2 y^2 + z^2$ $\begin{pmatrix}1\\0\\\sqrt{2}\\0\\-1\end{pmatrix}$

$\varepsilon = e^{i\frac{2\pi}{3}}$

$$2z^2-(x^2+y^2) \qquad \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\\sqrt{2}\\0\\-1\end{pmatrix} \qquad \varepsilon x^2+\varepsilon^2 y^2+z^2$$



$$x^2-y^2 \qquad \begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \qquad x^2-y^2+2ixy$$

Exact many-body states:

(Koashi +Ueda, Ho+ Lan Yin 2000,
Uchino et al 2008)

$$\beta : \hat{F} \bullet \hat{F} + \gamma \hat{\Theta}_2^+ \hat{\Theta}_2$$

F : SO(3)   F(F+1) $C_2(SO(3))$

$$\Theta_2^+ = 2a_2^+ a_{-2}^+ - 2a_1^+ a_{-1}^+ + (a_0^+)^2$$

$$\hat{\Lambda} \equiv \hat{\Theta}_2^\dagger \hat{\Theta}_2$$

$$\Theta_2^+ = \sum_{i=1}^{5} a_i^+ a_i^+$$

$$C_2(SO(5)) = N(N+3) - \hat{\Lambda}$$

$$= \tau(\tau + 3)$$

τ = integer

$$(2z^2 - x^2 - y^2, x^2 - y^2, zx, zy, xy)$$

Energy: $\beta F(F+1) - \gamma \tau(\tau + 3)$

$$\Lambda = N(N+3) - \tau(\tau+3) = (N-\tau)(N+3+\tau)$$

Operators:

$$\Theta_2^+ = 2a_2^+a_{-2}^+ - 2a_1^+a_{-1}^+ + (a_0^+)^2$$ pairs

$$\Theta_3^+ = \frac{1}{\sqrt{6}}(\hat{a}_0^\dagger)^3 - \frac{3}{\sqrt{6}}\hat{a}_1^\dagger\hat{a}_0^\dagger\hat{a}_{-1}^\dagger + \frac{3}{2}(\hat{a}_1^\dagger)^2\hat{a}_{-2}^\dagger + \frac{3}{2}\hat{a}_2^\dagger(\hat{a}_{-1}^\dagger)^2 - \frac{6}{\sqrt{6}}\hat{a}_2^\dagger\hat{a}_0^\dagger\hat{a}_{-2}^\dagger$$ trios

Creation operator for two/three particle singlets

a_2^+ $\quad A_{22}^+ = \sum_m \langle 22 \mid 22, m, 2-m \rangle a_m^+ a_{2-m}^+$ (and rotations) "others"

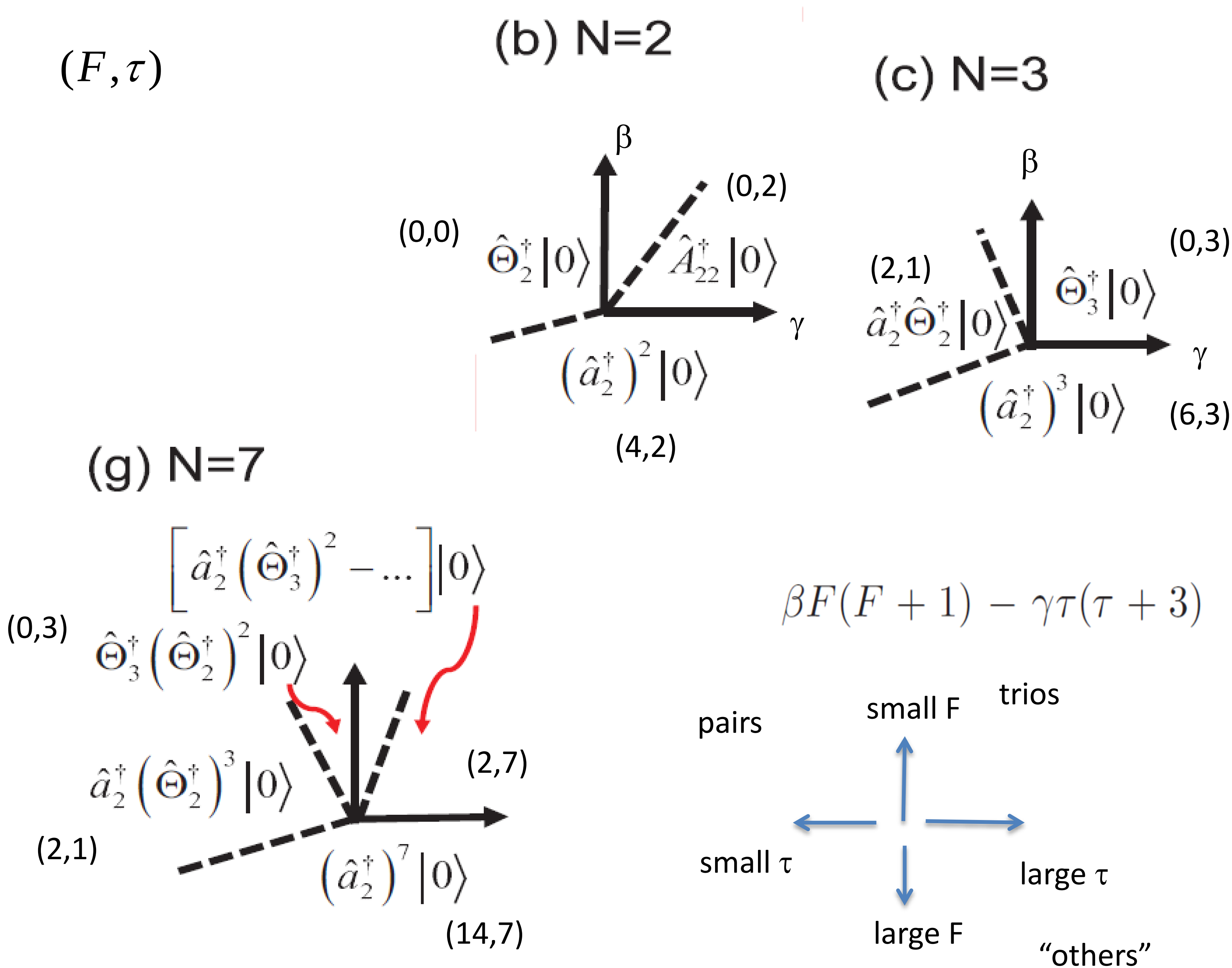
two quantum numbers F and τ

N = 2 x pairs + τ

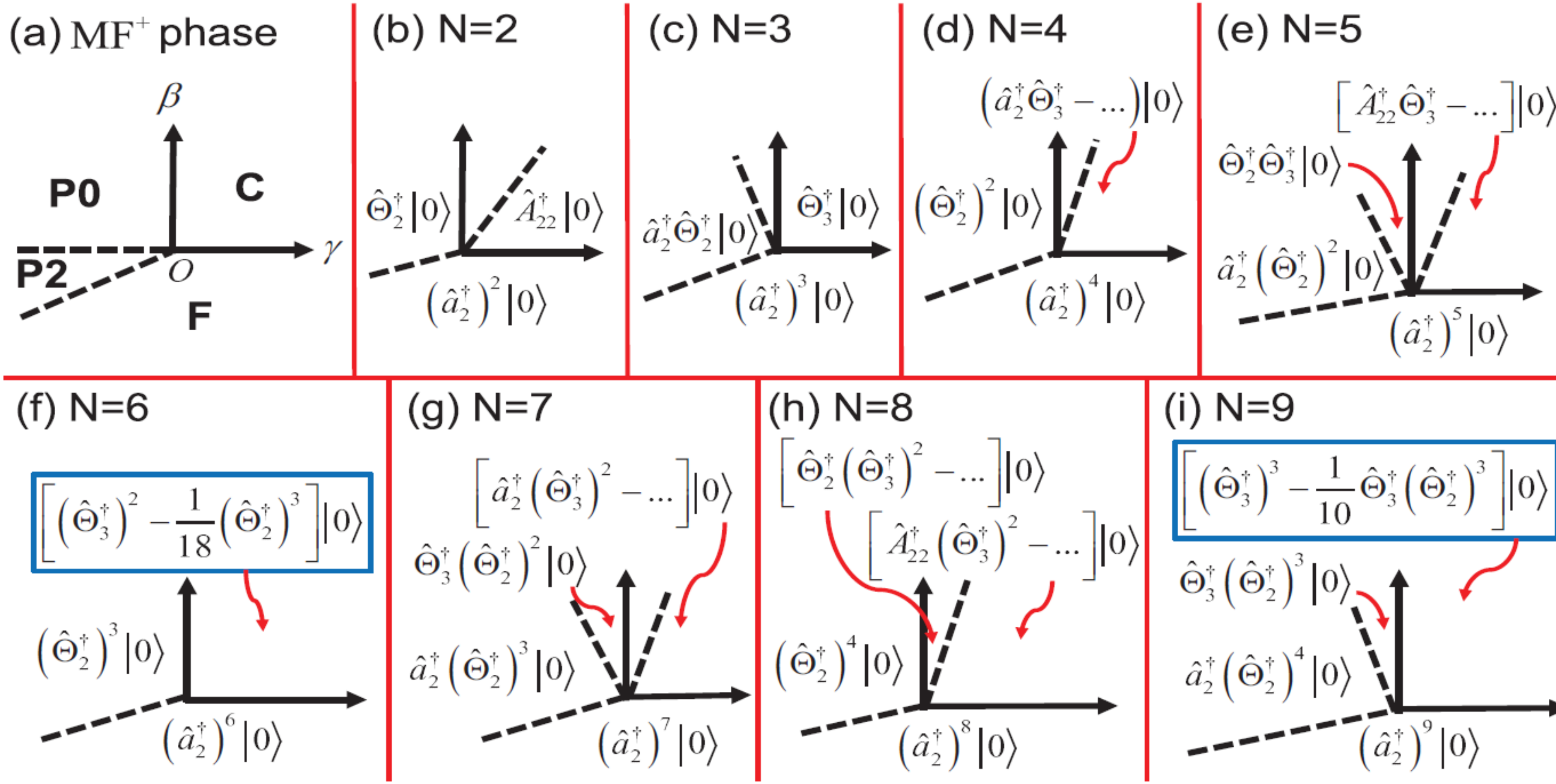
τ = N – 2 x pairs
= 3 x trios + "others"

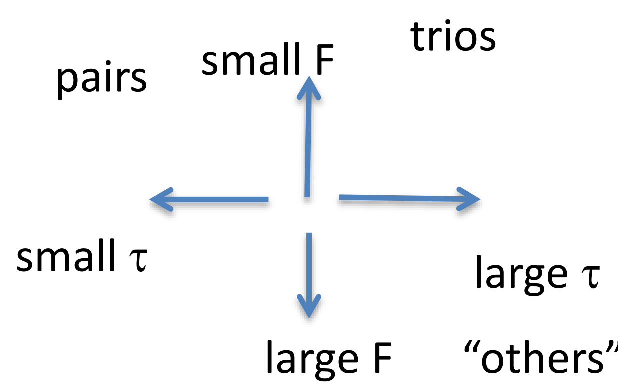
Energy: $\beta F(F+1) - \gamma\tau(\tau+3)$

$$\hat{\Lambda} \equiv \hat{\Theta}_2^\dagger\hat{\Theta}_2 \qquad \Lambda = N(N+3) - \tau(\tau+3) = (N-\tau)(N+3+\tau)$$



(F, τ)
(b) N=2
β
(0,2)
(0,0)
$\hat{\Theta}_2^\dagger |0\rangle$
$\hat{A}_{22}^\dagger |0\rangle$
γ
$(\hat{a}_2^\dagger)^2 |0\rangle$
(4,2)
(c) N=3
β
(0,3)
(2,1)
$\hat{\Theta}_3^\dagger |0\rangle$
$\hat{a}_2^\dagger \hat{\Theta}_2^\dagger |0\rangle$
γ
$(\hat{a}_2^\dagger)^3 |0\rangle$
(6,3)
(g) N=7
$\left[\hat{a}_2^\dagger (\hat{\Theta}_3^\dagger)^2 - \ldots\right] |0\rangle$
(0,3)
$\hat{\Theta}_3^\dagger (\hat{\Theta}_2^\dagger)^2 |0\rangle$
(2,7)
$\hat{a}_2^\dagger (\hat{\Theta}_2^\dagger)^3 |0\rangle$
(2,1)
$(\hat{a}_2^\dagger)^7 |0\rangle$
(14,7)
$\beta F(F+1) - \gamma\tau(\tau+3)$
trios
pairs
small F
small τ
large τ
large F
“others”





$$\beta F(F+1) - \gamma\tau(\tau+3)$$

$$\Theta_2^+\Theta_2[(\Theta_3^+)]|0\rangle = 0 \qquad \tau=3$$

$$\Theta_2^+\Theta_2[(\Theta_3^+)^2 - \frac{1}{18}(\Theta_2^+)^3]|0\rangle = 0 \qquad \tau=6$$

$$\Theta_2^+\Theta_2[(\Theta_3^+)^3 - \frac{1}{10}\Theta_3^+(\Theta_2^+)^3]|0\rangle = 0 \qquad \tau=9$$

mean-field

angular averages:

:

$$\cos\theta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\sin\theta}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\rightarrow a_P^+(\Omega)$$

$$\int_\Omega (a_P^+(\Omega))^N \,|0\rangle$$

$$\begin{pmatrix} 1 \\ 0 \\ \sqrt{2} \\ 0 \\ -1 \end{pmatrix} / 2 \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ \sqrt{2} \\ 0 \end{pmatrix} / \sqrt{3}$$

$$\rightarrow a_C^+(\Omega)$$

$$\int_\Omega (a_C^+(\Omega))^N \,|0\rangle$$

(1,0,0,0,0)

Angular averages:

Ferro always averages to zero

N = 2

$$\int_{\Omega} (a_P^+(\Omega))^2 \,|0\rangle \longrightarrow \Theta_2^+ \,|0\rangle$$

N=3

$$\int_{\Omega} (a_P^+(\Omega))^3 \,|0\rangle$$

$$\int_{\Omega} (a_C^+(\Omega))^3 \,|0\rangle$$

$$\longrightarrow \Theta_3^+ \,|0\rangle$$

N=4

$$\int_{\Omega} (a_P^+(\Omega))^4 \,|0\rangle \longrightarrow (\Theta_2^+)^2 \,|0\rangle$$

$$\int_{\Omega} (a_P^+(\Omega))^6 \,|0\rangle \longrightarrow$$

$$\left[(47 - 2\cos 6\theta)\left(\hat{\Theta}_2^\dagger\right)^3 + (12 + 36\cos 6\theta)\left(\hat{\Theta}_3^\dagger\right)^2\right]|0\rangle$$

in general not even solution of H

$$(\Theta_2^+)^3 \,|0\rangle \qquad [(\Theta_3^+)^2 - \frac{1}{18}(\Theta_2^+)^3]\,|0\rangle$$

$$\int_{\Omega} (a_P^+(\Omega))^8 \,|0\rangle \longrightarrow$$

$$\left[(71 - 8\cos 6\theta)\left(\hat{\Theta}_2^\dagger\right)^4 + 16(3 + 9\cos 6\theta)\hat{\Theta}_2^\dagger\left(\hat{\Theta}_3^\dagger\right)^2\right]|0\rangle$$

$$(\Theta_2^+)^4 \,|0\rangle \qquad \Theta_2^+[(\Theta_3^+)^2 - \frac{1}{18}(\Theta_2^+)^3]\,|0\rangle$$

No correspondence between angular averaged mean-field state and many-body state

Actual picture:

Spin-1:

Singlet state is unique if N even

Yield singlet state if angular average is finite

(ferro averages to zero, so must start with polar)

Spin-2

N = 2	$\int_\Omega (a_P^+(\Omega))^2 \,\vert 0\rangle$	→	$\Theta_2^+ \,\vert 0\rangle$ unique singlet
N=3	$\int_\Omega (a_P^+(\Omega))^3 \,\vert 0\rangle$ $\int_\Omega (a_C^+(\Omega))^3 \,\vert 0\rangle$	→	$\Theta_3^+ \,\vert 0\rangle$ unique singlet
N=4	$\int_\Omega (a_P^+(\Omega))^4 \,\vert 0\rangle$	→	$(\Theta_2^+)^2 \,\vert 0\rangle$ unique singlet

$$\int_{\Omega} (a_P^+(\Omega))^5 \,|\, 0\rangle \longrightarrow \hat{\Theta}_2^\dagger \hat{\Theta}_3^\dagger |0\rangle$$

unique singlet

$$\int_{\Omega} (a_P^+(\Omega))^6 \,|\, 0\rangle \longrightarrow$$

$$\left[(47 - 2\cos 6\theta)\left(\hat{\Theta}_2^\dagger\right)^3 + (12 + 36\cos 6\theta)\left(\hat{\Theta}_3^\dagger\right)^2\right]|0\rangle$$

linear comb. of two singlets

Starts from

$$\cos\theta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{\sin\theta}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Average also over θ, weight factor $\sin(3\theta)$

SO(5) average

Finite only when N is even

$$(\Theta_2^+)^{N/2} \,|\, 0\rangle$$

unique SO(5) invariant

Large N: two particle correlations $\int_{W} (a_P^{+}(W))^{N} \,|\, 0\rangle$

$$\langle a_{m_1}^{\dagger} a_{m_1}^{\dagger} a_{m_3} a_{m_4} \rangle$$

$$= N(N-1)\frac{\int_{\Omega_1,\Omega_2} \psi_{m_1}^{*}(\Omega_1)\psi_{m_2}^{*}(\Omega_1)\psi_{m_3}(\Omega_2)\psi_{m_4}(\Omega_2)\left[\sum_m \psi_m^{*}(\Omega_1)\psi_m(\Omega_2)\right]^{N-2}}{\int_{\Omega_1,\Omega_2}\left[\sum_m \psi_m^{*}(\Omega_1)\psi_m(\Omega_2)\right]^{N}}$$

$$y_m(W) = D(W) \times \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

....

Same as from $\int_{W} (a_P^{+}(W))^{N/2} \,|\, 0\rangle$ up to corrections of 1/N to coef of N(N-1)

Start from Cyclic:

N/3 ≠ integer

$$\int_{\Omega} (a_C^+(\Omega))^N |0\rangle = 0$$

N/3= integer

$$\int_{\Omega} (a_C^+(\Omega))^N |0\rangle = 0$$

$$\hat{\Theta}_2^+ \hat{\Theta}_2 \int_{\Omega} (a_C^+(\Omega))^N |0\rangle = 0$$

τ=0, F=0

$$\sum_{m=-2}^{2} (-1)^m \varphi_m \varphi_{-m} = 0$$

(exact eigenstate in the C region)

Correspondence between angular average mean-field
and exact many-body states are due to mathematical identities
but not physics

Relation between fragmented states and mean-field states
for spin-2 *different* from those in
double wells and spin-1