From Chern-Simons-Maxwell Theory on the Lattice to the Toric Code

Nadiia Vlasii

Institute for Theoretical Physics, Bern University

b UNIVERSITÄT BERN

AEC ALBERT EINSTEIN CENTER FOR FUNDAMENTAL PHYSICS

erc European Research Council

arXiv:1503.07023, T. Z. Olesen, N. D. V., U.-J. Wiese

Abelian Chern-Simons-Maxwell Theory

(2+1)-d Abelian gauge fields

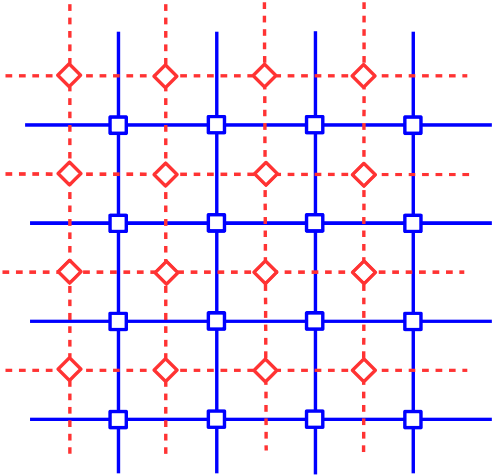
$$A_\mu(x) \in \mathbb{R}$$

$$A'_\mu(x) = A_\mu(x) - \partial_\mu \chi(x),$$

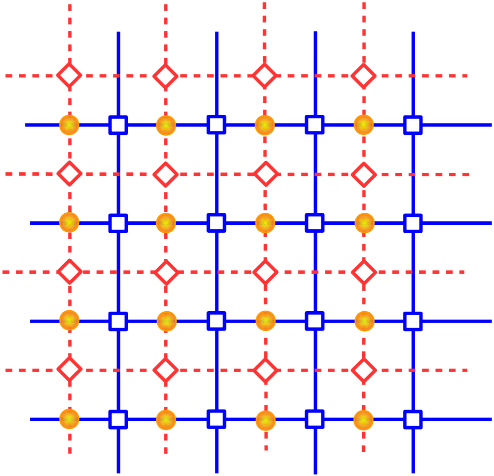
$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

$$S[A] = \int d^3x \left(\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{4\pi} \epsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho \right)$$

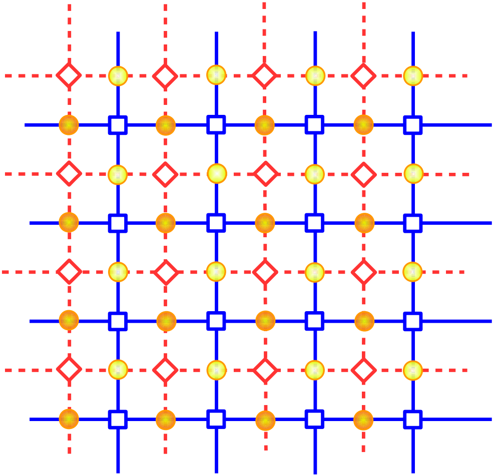
Cross based degrees of freedom



Cross based degrees of freedom

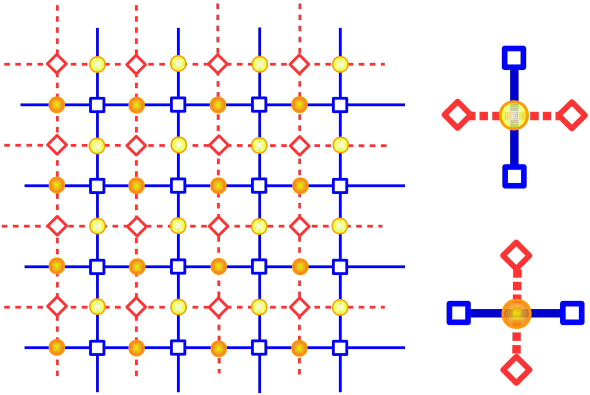


Cross based degrees of freedom



Chern-Simons-Maxwell Theory on the Lattice

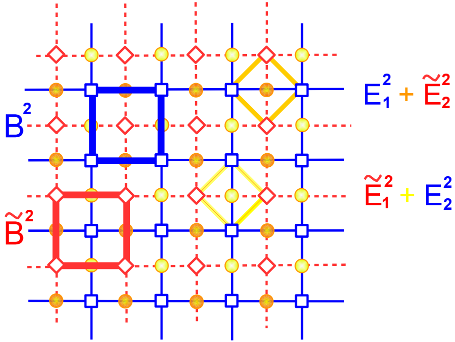
$$S[A, \widetilde{A}] = \int d^3x \left(\frac{1}{4e^2}(F_{\mu\nu}F^{\mu\nu} + \widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu}) + \frac{k}{4\pi}\epsilon_{\mu\nu\rho}(\widetilde{A}^\mu \partial^\nu A^\rho + A^\mu \partial^\nu \widetilde{A}^\rho) \right)$$



The Hamiltonian of doubled Chern-Simons-Maxwell Theory

$$H = \frac{e^2}{2}\sum_{x\in X} a^2(E_{x,1}^2 + \widetilde{E}_{x,2}^2) + \frac{e^2}{2}\sum_{x\in\widetilde{X}} a^2(E_{x,2}^2 + \widetilde{E}_{x,1}^2) +$$

$$+ \frac{1}{2e^2}\sum_{x\in\widetilde{\Lambda}} a^2 B_x^2 + \frac{1}{2e^2}\sum_{x\in\Lambda} a^2 \widetilde{B}_x^2$$



Electric fields of Chern-Simons-Maxwell Theory on the doubled Lattice

$$E_i(x) = -i\partial_{A_i} - \frac{k}{4\pi}\epsilon_{ij}\widetilde{A}_j(x), \quad \widetilde{E}_i(x) = -i\partial_{\widetilde{A}_i} - \frac{k}{4\pi}\epsilon_{ij}A_j(x),$$

$$[E_{x,i}, A_{y,j}] = [\widetilde{E}_{x,i}, \widetilde{A}_{y,j}] = -i\delta_{ij}\frac{1}{a^2}\delta_{xy},$$

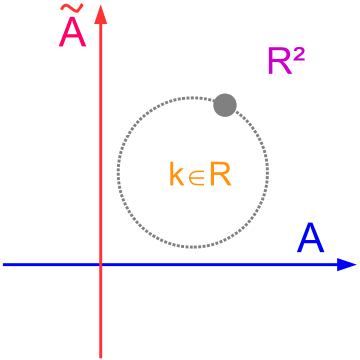
The electric fields of original and dual lattices do not commute

$$[E_i(x), \widetilde{E}_j(y)] = [\widetilde{E}_i(x), E_j(y)] = -i\frac{k}{2\pi}\epsilon_{ij}\delta(x-y)$$

Like for a charged particle in a magnetic field, the commutator of the momenta is non-zero and given by the abstract “magnetic” field $\frac{k}{2\pi}$

"Mechanical" Analog

Noncompact case: 2-d plane $\mathbb{R}^2$



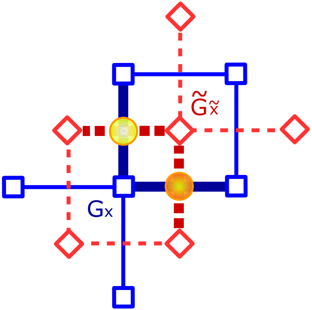
"Particle" moving in the background of an "magnetic" field $\frac{k}{2\pi}$

The Gauss law on the doubled lattice

$$[G_x, \widetilde{G}_{\widetilde{x}}] = 0,$$

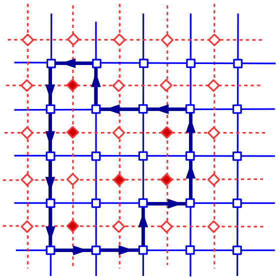
$$[G_x, H] = 0, \qquad [\widetilde{G}_x, H] = 0$$

$$G_x|0\rangle = 0, \qquad \widetilde{G}_x|0\rangle = 0.$$



Mutual Anyonic Statistics

$$G_x|Q,\widetilde{Q}\rangle = Q_x|Q,\widetilde{Q}\rangle,\ x\in\Lambda,\quad \widetilde{G}_x|Q,\widetilde{Q}\rangle = \widetilde{Q}_x|Q,\widetilde{Q}\rangle,\ x\in\widetilde{\Lambda}$$



Wilson loop on the lattice encircles a set of dual charges

$$W_{\mathcal{C}}|Q,\widetilde{Q}\rangle = \prod_{(x,i)\in\mathcal{C}} U_{x,i}|Q,\widetilde{Q}\rangle = \exp(i\Phi_{\mathcal{C}})|Q,\widetilde{Q}\rangle$$

The Compactified Gauge Fields

$A_{x,i}$ and $\widetilde{A}_{x,i}$ turn into phases of the parallel transporters

$$U_{x,i} = \exp(iaA_{x,i}) = \exp(i\varphi_{x,i}) \in U(1)$$
$$\widetilde{U}_{x,i} = \exp(ia\widetilde{A}_{x,i}) = \exp(i\widetilde{\varphi}_{x,i}) \in U(1)$$

The Hamiltonian of a single cross

$$H_{+} = \frac{a^2 e^2}{2}(E^2 + \widetilde{E}^2)$$

$$aE = -i\partial_{\varphi} + a(\varphi, \widetilde{\varphi}) \quad a\widetilde{E} = -i\partial_{\widetilde{\varphi}} + \widetilde{a}(\varphi, \widetilde{\varphi})$$

where $(a, \widetilde{a})$ is an abstract vector potential

$$a(\varphi, \widetilde{\varphi}) = -\frac{k}{4\pi}\widetilde{\varphi}, \quad \widetilde{a}(\varphi, \widetilde{\varphi}) = \frac{k}{4\pi}\varphi,$$

which gives rise to the abstract “magnetic” field on the group space torus

$$b(\varphi, \widetilde{\varphi}) = \partial_{\varphi}\widetilde{a}(\varphi, \widetilde{\varphi}) - \partial_{\widetilde{\varphi}}a(\varphi, \widetilde{\varphi}) = \frac{k}{2\pi}$$

The boudary conditions

Self-adjoint extention of the Hamiltonian

$$H_+ = \frac{a^2 e^2}{2}(E^2 + \widetilde{E}^2)$$

leads to the boundary conditions for the wave function

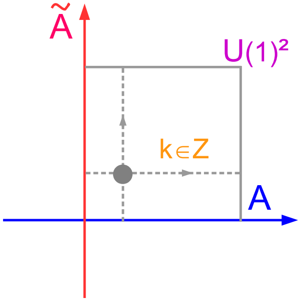
$$\Psi(\varphi + 2\pi, \widetilde{\varphi}) = \exp\left(-i\frac{k}{2}\widetilde{\varphi} + i\widetilde{\theta}\right)\Psi(\varphi, \widetilde{\varphi})$$
$$\Psi(\varphi, \widetilde{\varphi} + 2\pi) = \exp\left(i\frac{k}{2}\varphi + i\theta\right)\Psi(\varphi, \widetilde{\varphi})$$

where θ and $\widetilde{\theta}$ are self-adjoint extension parameters.

Dirac quantization of the level $k \in \mathbb{Z}$

"Mechanical" Analog

Compactified group space: torus $U(1)^2$



"Particle" moving on a torus $U(1)^2$ with "magnetic" flux $2\pi k$, where $k \in \mathbb{Z}$
Polyakov loops break translation invariance $U(1)^2$ to $\mathbb{Z}(k)^2$

Self-adjoint extention background gauge fields

Gauge transformations of the gauge fields on the lattice

$$\exp(i\varphi'_{x,i}) = \exp(i(\varphi_{x,i} + \chi_{x+\frac{a}{2}\hat{i}} - \chi_{x-\frac{a}{2}\hat{i}})),$$
$$\exp(i\widetilde{\varphi}'_{x,i}) = \exp(i(\widetilde{\varphi}_{x,i} + \widetilde{\chi}_{x+\frac{a}{2}\hat{i}} - \widetilde{\chi}_{x-\frac{a}{2}\hat{i}}))$$

Parameters of self-adjoint extention are playing the role of background gauge fields

$$\exp(i\theta'_{x,i}) = \exp(i(\theta_{x,i} + k\chi_{x+\frac{a}{2}\hat{i}} - k\chi_{x-\frac{a}{2}\hat{i}})),$$
$$\exp(i\widetilde{\theta}'_{x,i}) = \exp(i(\widetilde{\theta}_{x,i} + k\widetilde{\chi}_{x+\frac{a}{2}\hat{i}} - k\widetilde{\chi}_{x-\frac{a}{2}\hat{i}}))$$

Gauge transformations which manifest U(1) gauge symmetry reduced to $\mathbb{Z}(k)$

Local Magnetic Translation Group

$\mathbb{Z}(k)$ shift operators T and $\widetilde{T}$ obey local

magnetic translation group

$$\widetilde{T}\,T = \exp\left(\frac{2\pi i}{k}\right) T\,\widetilde{T}$$

$$H_{+}|nl\rangle = M\left(n + \frac{1}{2}\right)|nl\rangle,\ \ T|nl\rangle = \exp\left(\frac{2\pi i l}{k}\right)|nl\rangle.$$

$$\widetilde{T}|nl\rangle = |n(l-1)\rangle$$

where $M = \frac{ke^2}{2\pi}$ is the "photon" mass.

$\mathbb{Z}(k)$ Gauge Symmetry and Gauss Law

The unitary operators of local $\mathbb{Z}(k)$ gauge transformations

$$V_x = \prod_i T_{x+\frac{a}{2}\hat{i},i} T^{\dagger}_{x-\frac{a}{2}\hat{i},i}, \; x \in \Lambda$$

$$\widetilde{V}_x = \prod_i \widetilde{T}_{x+\frac{a}{2}\hat{i},i} \widetilde{T}^{\dagger}_{x-\frac{a}{2}\hat{i},i}, \; x \in \widetilde{\Lambda}$$

must obey $\mathbb{Z}(k)$ Gauss law

$$V_x|\Psi\rangle = |\Psi\rangle, \; x \in \Lambda$$

$$\widetilde{V}_x|\Psi\rangle = |\Psi\rangle, \; x \in \widetilde{\Lambda}$$

The effective low-energy physics

In the pure Chern-Simons limit ($e^2 \to \infty$) the spectrum is restricted to the lowest Landau level, which is k times degenerate.

The effective low-energy Hamiltonian is given by

$$H_{\mathrm{eff}} = -\frac{C^4}{2e^2a^2} \sum_{x \in \widetilde{\Lambda}} \left[\exp\left(-\frac{i\eta_x}{k}\right) \widetilde{V}_x + \exp\left(\frac{i\eta_x}{k}\right) \widetilde{V}_x^\dagger \right]$$

$$-\frac{C^4}{2e^2a^2} \sum_{x \in \Lambda} \left[\exp\left(\frac{i\widetilde{\eta}_x}{k}\right) V_x + \exp\left(-\frac{i\widetilde{\eta}_x}{k}\right) V_x^\dagger \right]$$

where operators of $\mathbb{Z}(k)$ gauge transformations are

$$V_x = \prod_i T_{x+\frac{a}{2}\hat{i},i} T^\dagger_{x-\frac{a}{2}\hat{i},i}, \; x \in \Lambda,$$

$$\widetilde{V}_x = \prod_i \widetilde{T}_{x+\frac{a}{2}\hat{i},i} \widetilde{T}^\dagger_{x-\frac{a}{2}\hat{i},i}, \; x \in \widetilde{\Lambda}$$

Relation to the Toric Code

The $\mathbb{Z}(2)$ toric code is characterized by the Hamiltonian

$$H = -J \sum_{x \in \widetilde{\Lambda}} U_x - G \sum_{x \in \Lambda} V_x,$$

where

$$V_x = \prod_i \exp\left(i\pi S^3_{x+\frac{a}{2}\hat{i}} \right) \exp\left(-i\pi S^3_{x-\frac{a}{2}\hat{i}} \right) \quad x \in \Lambda$$

$$U_x = S^1_{x-\frac{a}{2}\hat{2},1} S^1_{x+\frac{a}{2}\hat{1},2} S^1_{x+\frac{a}{2}\hat{2},1} S^1_{x-\frac{a}{2}\hat{1},2}, \quad x \in \widetilde{\Lambda}$$

Conclusions

The $\mathbb{Z}(k)$ toric code emerges from the doubled Chern-Simons-Maxwell theory in the limit of infinite "photon" mass, $ke^2/2\pi \to \infty$.

The toric code has a large variety of hidden self-adjoint extension parameters θ and $\widetilde{\theta}$, which transform as lattice gauge fields.