Simulating Quantum Fluids

Thomas Schaefer, North Carolina State University

Why simulate (nearly perfect) quantum fluids?

Hard computational problem: Have to determine real time correlation functions. Achieve quantum supremacy?

Fluid dynamics is the universal effective description of non-equilibrium many body systems. Description is "most effective" in nearly perfect fluids.

Fluid-gravity correspondence: Can (strongly coupled) fluids teach us something about quantum gravity?

Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2$$

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \qquad \omega < T$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad \omega < T \frac{s}{\eta}$$

Effective theories (Strong coupling)

$$\mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \ldots \Leftrightarrow S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\mathcal{R} + \ldots$$

$$SO(d+2,2) \to Schr^2_d \qquad AdS_{d+3} \to Schr^2_d$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = 0 \quad (\omega < T)$$

Outline

I. EFT: Gradient expansion

II. EFT: Fluctuations

III. Models of fluids: Kinetic theory & QFT

IV. Models of fluids: Holography

V. Analyzing fluids: How to measure η/s

I. Gradient expansion (simple non-relativistic fluid)

Simple fluid: Conservation laws for mass, energy, momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\vec{\jmath}^{\,\rho} = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla}\vec{\jmath}^{\,\epsilon} = 0$$

$$\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial x_j}\Pi_{ij} = 0$$

Ward identity: mass current = momentum density

$$\vec{\jmath}^{\,\rho} \equiv \rho\vec{v} = \vec{\pi}$$

Constitutive relations: Gradient expansion for currents

Energy momentum tensor

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j + \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v_k\right) + O(\partial^2)$$

Conformal fluid dynamics: Symmetries

Symmetries of a conformal non-relativistic fluid

Galilean boost	$\vec{x}' = \vec{x} + \vec{v}t$	$t' = t$
Scale trafo	$\vec{x}' = e^s \vec{x}$	$t' = e^{2s} t$
Conformal trafo	$\vec{x}' = \vec{x}/(1+ct)$	$1/t' = 1/t + c$

This is known as the Schrödinger algebra (= the symmetries of the free Schrödinger equation)

Generators: Mass, momentum, angular momentum

$$M = \int dx\, \rho \quad P_i = \int dx\, \jmath_i \quad J_{ij} = \int dx\, \epsilon_{ijk} x_j \jmath_k$$

Boost, dilations, special conformal

$$K_i = \int dx\, x_i \rho \quad D = \int dx\, x \cdot \jmath \quad C = \int dx\, x^2 \rho/2$$

Spurion method: Local symmetries

Diffeomorphism invariance $\delta x_i = \xi_i(x,t)$

$$\delta g_{ij} = -\mathcal{L}_\xi g_{ij} = -\xi^k \partial_k g_{ij} + \ldots$$

Gauge invariance $\delta\psi = i\alpha(x,t)\psi$

$$\begin{aligned} \delta A_0 &= -\dot{\alpha} - \xi^k \partial_k A_0 - A_k \dot{\xi}^k \\ \delta A_i &= -\partial_i \alpha - \xi^k \partial_k A_i - A_k \partial_i \xi^k + m g_{ik} \dot{\xi}^k \end{aligned}$$

Conformal transformations $\delta t = \beta(t)$

$$\delta O = -\beta \dot{O} - \frac{1}{2}\Delta_O \dot{\beta} O$$

More recent work: Newton-Cartan geometry

Example: Stress tensor

Determine transformation properties of fluid dynamic variables

$$\delta\rho = -\mathcal{L}_\xi \rho \qquad \delta s = -\mathcal{L}_\xi s \qquad \delta v = -\mathcal{L}_\xi v + \dot{\xi}$$

Stress tensor: Ideal fluid dynamics

$$\Pi^0_{ij} = P g_{ij} + \rho v_i v_j , \qquad\qquad P = \frac{2}{3}\mathcal{E}$$

First order viscous hydrodynamics

$$\delta^{(1)}\Pi_{ij} = -\eta\sigma_{ij} - \zeta g_{ij}\langle\sigma\rangle \qquad \zeta = 0$$

$$\sigma_{ij} = \left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3} g_{ij}\langle\sigma\rangle\right) \qquad\qquad \langle\sigma\rangle = \nabla\cdot v + \frac{\dot{g}}{2g}$$

Son (2007)

Simple application: Kubo formula

Consider background metric $g_{ij}(t,x) = \delta_{ij} + h_{ij}(t,x)$. Linear response

$$\delta\Pi^{xy} = -\frac{1}{2} G_R^{xyxy} h_{xy}$$

Harmonic perturbation $h_{xy} = h_0 e^{-i\omega t}$

$$G_R^{xyxy} = P - i\eta\omega + \ldots$$

Kubo relation: $$\eta = -\lim_{\omega \to 0} \left[\frac{1}{\omega} \mathrm{Im} G_R^{xyxy}(\omega, 0) \right]$$

Gradient expansion: $\omega \leq \frac{P}{\eta} \simeq \frac{s}{\eta} T$.

Second order conformal hydrodynamics

Second order gradient corrections to stress tensor

$$\delta^{(2)}\Pi^{ij} \;=\; \eta\tau_\pi \left[{}^{\langle}D\sigma^{ij\rangle} + \frac{2}{3}\,\sigma^{ij}(\nabla\cdot v)\right] + \lambda_1 \sigma^{\langle i}{}_k \sigma^{j\rangle k} + \lambda_2 \sigma^{\langle i}{}_k \Omega^{j\rangle k} + \lambda_3 \Omega^{\langle i}{}_k \Omega^{j\rangle k} + O(\nabla^2 T)$$

$$D = \partial_0 + v\cdot\nabla \quad A^{\langle ij\rangle} = \frac{1}{2}\left(A_{ij} + A_{ji} - \frac{2}{3} g_{ij} A^k{}_k\right) \quad \Omega^{ij} = \left(\nabla_i v_j - \nabla_j v_i\right)$$

New transport coefficients $\tau_\pi, \lambda_i, \gamma_i$

Can be written as a relaxation equation for $\pi^{ij} \equiv \delta\Pi^{ij}$

$$\pi^{ij} = -\eta\sigma^{ij} - \tau_\pi \left[{}^{\langle}D\pi^{ij\rangle} + \frac{5}{3}(\nabla\cdot v)\pi^{ij}\right] + \ldots$$

Chao, Schaefer (2011)

Second order fluid dynamics: Causality

"Speed" of diffusive wave in Navier-Stokes theory

$$v_D = \frac{\partial|\omega|}{\partial k} = \frac{2\eta}{\rho}\,k$$

May encounter $v_D \gg c_s$

Not a fundamental problem (should impose $k < \Lambda$), but a nuisance in simulations.

Second order fluid dynamics, relaxation type

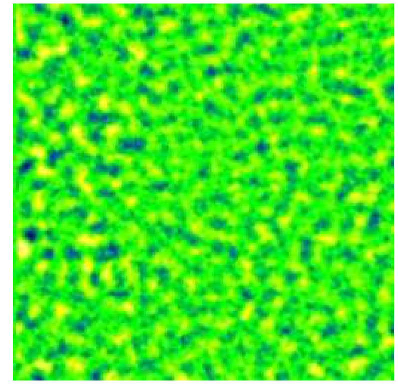
$$i\omega = \frac{\nu k^2}{1 - i\omega\tau_\pi} \qquad \text{("resummed hydro")}$$

Limiting speed $v_D^\infty \sim \sqrt{\eta/(\rho\tau_\pi)}$

Find $v_D^\infty \sim c_s$ for $\tau_\pi = \eta/P$.

II. Beyond gradients: Hydrodynamic fluctuations

Hydrodynamic variables fluctuate



$$\langle \delta v_i(x,t)\delta v_j(x',t)\rangle = \frac{T}{\rho}\delta_{ij}\delta(x-x')$$

Linearized hydrodynamics propagates fluctuations as shear or sound

$$\langle \delta v_i^T \delta v_j^T\rangle_{\omega,k} = \frac{2T}{\rho}(\delta_{ij} - \hat{k}_i\hat{k}_j)\frac{\nu k^2}{\omega^2 + (\nu k^2)^2} \qquad \textit{shear}$$

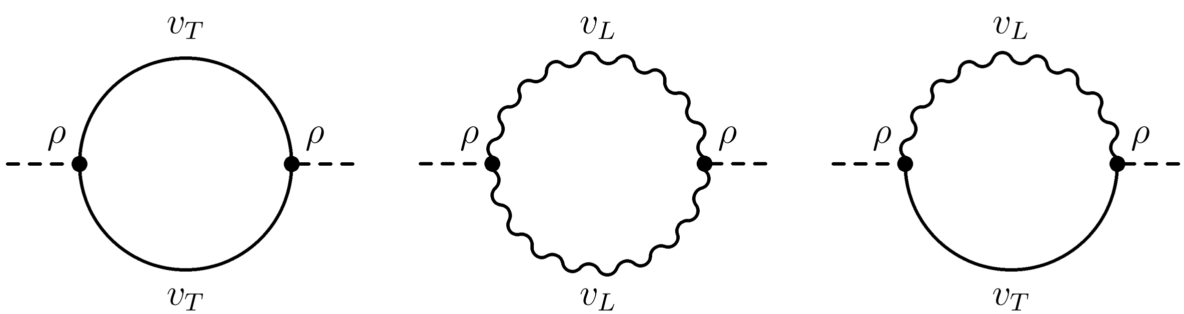
$$\langle \delta v_i^L \delta v_j^L\rangle_{\omega,k} = \frac{2T}{\rho}\hat{k}_i\hat{k}_j\frac{\omega k^2\Gamma}{(\omega^2 - c_s^2k^2)^2 + (\omega k^2\Gamma)^2} \qquad \textit{sound}$$

$v = v_T + v_L$: $\nabla \cdot v_T = 0$, $\nabla \times v_L = 0$ $\qquad$ $\nu = \eta/\rho$, $\Gamma = \frac{4}{3}\nu + \ldots$

Hydro Loops: "Breakdown" of second order hydro

Correlation function in hydrodynamics

$$G_S^{xyxy} = \langle\{\Pi^{xy}, \Pi^{xy}\}\rangle_{\omega,k} \simeq \rho_0^2 \langle\{v_x v_y, v_x v_y\}\rangle_{\omega,k}$$



Match to response function in $\omega \to 0$ (Kubo) limit

$$G_R^{xyxy} = P + \delta P - i\omega[\eta + \delta\eta] + \omega^2 \left[\eta\tau_\pi + \delta(\eta\tau_\pi)\right]$$

with

$$\delta P \sim T\Lambda^3 \qquad \delta\eta \sim \frac{T\rho\Lambda}{\eta} \qquad \delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}} \frac{T\rho^{3/2}}{\eta^{3/2}}$$

Hydro Loops: RG and "breakdown" of 2nd order hydro

Cutoff dependence can be absorbed into bare parameters. Non-analytic terms are cutoff independent.

Fluid dynamics is a "renormalizable" effective theory.

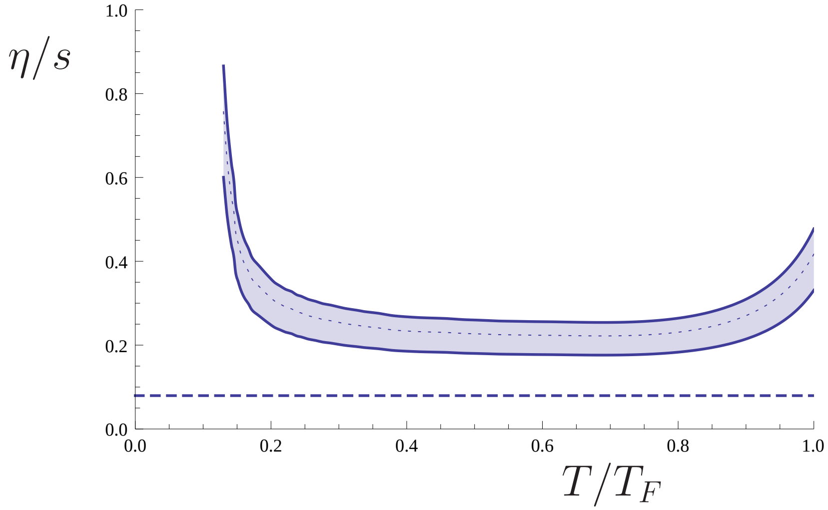
Small η enhances fluctuation corrections: $\delta\eta \sim T \left(\frac{\rho}{\eta}\right)^2 \left(\frac{P}{\rho}\right)^{1/2}$

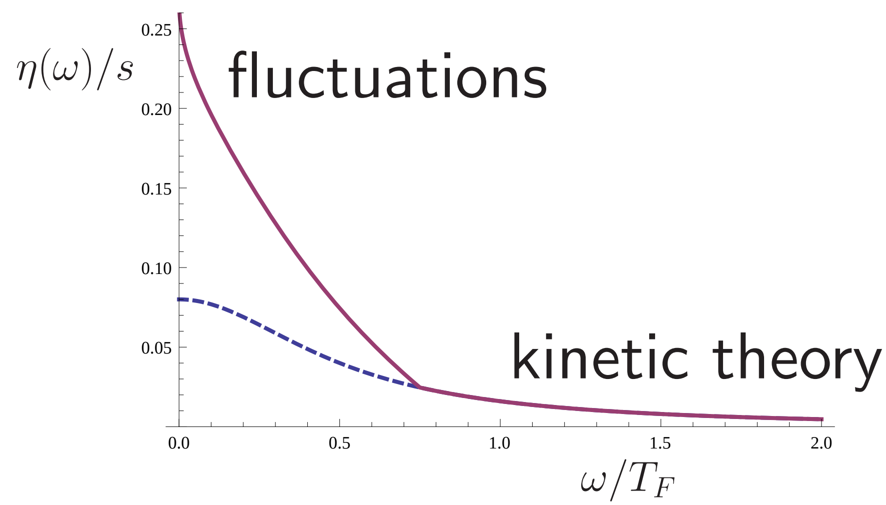
Small η leads to large $\delta\eta$: There must be a bound on η/n.

Relaxation time diverges: $\delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}} \left(\frac{\rho}{\eta}\right)^{3/2}$

2nd order hydro without fluctuations inconsistent.

Fluctuation induced bound on η/s





$$(\eta/s)_{min} \simeq 0.2$$

spectral function

non-analytic $\sqrt{\omega}$ term

Schaefer, Chafin (2012), see also Kovtun, Moore, Romatschke (2011)

III. Kinetic theory

Microscopic picture: Quasi-particle distribution function $f_p(x,t)$

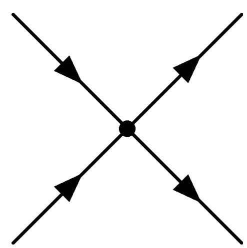
$$\rho(x,t) = \int d\Gamma_p \sqrt{g} m f_p(x,t) \qquad \pi_i(x,t) = \int d\Gamma_p \sqrt{g} p_i f_p(x,t)$$

$$\Pi_{ij}(x,t) = \int d\Gamma_p \sqrt{g} p_i v_j f_p(x,t)$$

Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{m} \frac{\partial}{\partial x^i} - \left(g^{il} \dot{g}_{lj} p^j + \Gamma^i_{jk} \frac{p^j p^k}{m} \right) \frac{\partial}{\partial p^i} \right) f_p(t,x,) = C[f]$$

$$C[f] =$$



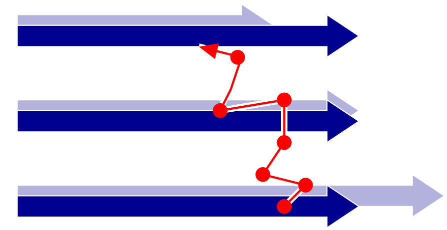
Solve order-by-order in Knudsen number $Kn = l_{mfp}/L$

Kinetic theory: Knudsen expansion

Chapman-Enskog expansion $f = f_0 + \delta f_1 + \delta f_2 + \ldots$

Gradient exp. $\delta f_n = O(\nabla^n)$

$\equiv$ Knudsen exp. $\delta f_n = O(Kn^n)$



First order result

Bruun, Smith (2005)

$$\delta^{(1)}\Pi^{ij} = -\eta\sigma^{ij} \qquad\qquad \eta = \frac{15}{32\sqrt{\pi}}(mT)^{3/2}$$

Second order result

Chao, Schaefer (2012), Schaefer (2014)

$$\delta^{(2)}\Pi^{ij} = \frac{\eta^2}{P}\left[\langle D\sigma^{ij}\rangle + \frac{2}{3}\sigma^{ij}(\nabla\cdot v)\right] + \frac{\eta^2}{P}\left[\frac{15}{14}\sigma^{\langle i}{}_k\sigma^{j\rangle k} - \sigma^{\langle i}{}_k\Omega^{j\rangle k}\right] + O(\kappa\eta\nabla^i\nabla^j T)$$

relaxation time $\quad \tau_\pi = \eta/P$

Frequency dependence, breakdown of kinetic theory

Consider harmonic perturbation $h_{xy}e^{-i\omega t+ikx}$. Use schematic collision term $C[f_p^0+\delta f_p]=-\delta f_p/\tau$.

$$\delta f_p(\omega,k)=\frac{1}{2T}\frac{-i\omega p_x v_y}{-i\omega+i\vec{v}\cdot\vec{k}+\tau_0^{-1}}\,f_p^0\,h_{xy}\,.$$

Leads to Lorentzian line shape of transport peak

$$\eta(\omega)=\frac{\eta(0)}{1+\omega^2\tau_0^2}$$

Pole at $\omega=i\tau_0^{-1}$ $(\tau_0=\eta/(sT))$ controls range of convergence of gradient expansion.

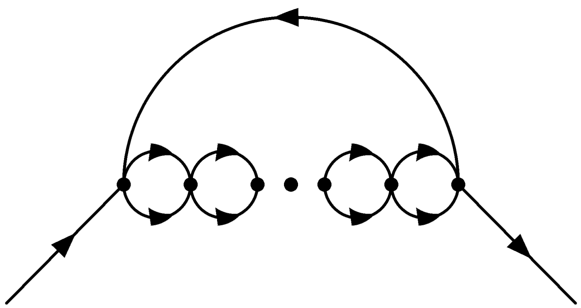
High frequency behavior misses short range correlations for $\omega>T$.

Bulk viscosity and conformal symmetry breaking

Conformal symmetry breaking (thermodynamics)

$$1 - \frac{2\mathcal{E}}{3P} = \frac{\langle \mathcal{O}_C \rangle}{12\pi m a P} \sim \frac{1}{6\pi} n\lambda^3 \frac{\lambda}{a}$$

How does this translate into $\zeta \neq 0$? Momentum dependent $m^*(p)$.



$$Im\,\Sigma(k) \sim zT\sqrt{\frac{T}{\epsilon_k}} Erf\left(\sqrt{\frac{\epsilon_k}{T}}\right) \ll T$$

$$Re\,\Sigma(k) \sim zT\frac{\lambda}{a}\sqrt{\frac{T}{\epsilon_k}} F_D\left(\sqrt{\frac{\epsilon_k}{T}}\right)$$

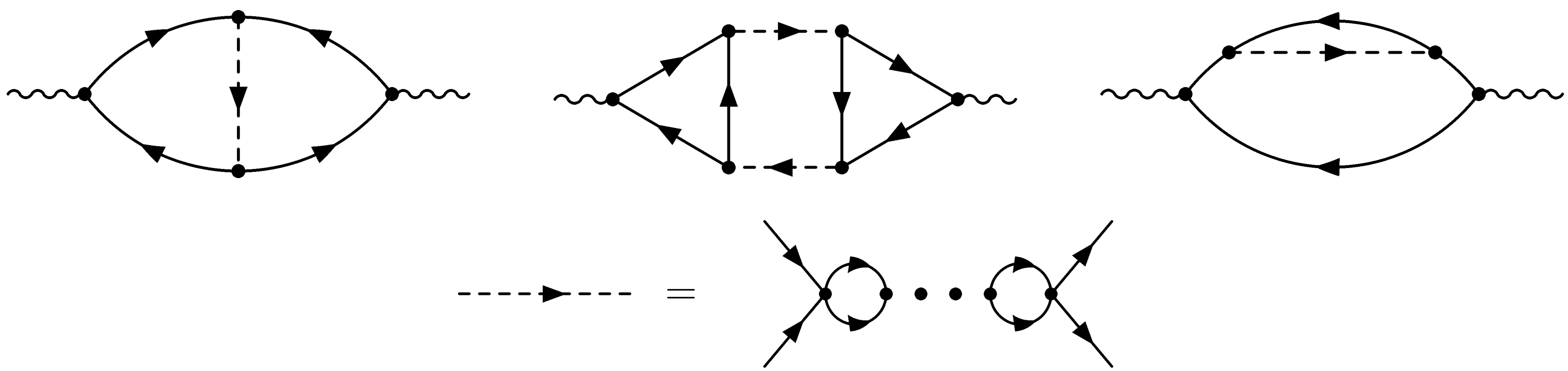
Bulk viscosity

$$\zeta = \frac{1}{24\sqrt{2}\pi}\lambda^{-3}\left(\frac{z\lambda}{a}\right)^2$$

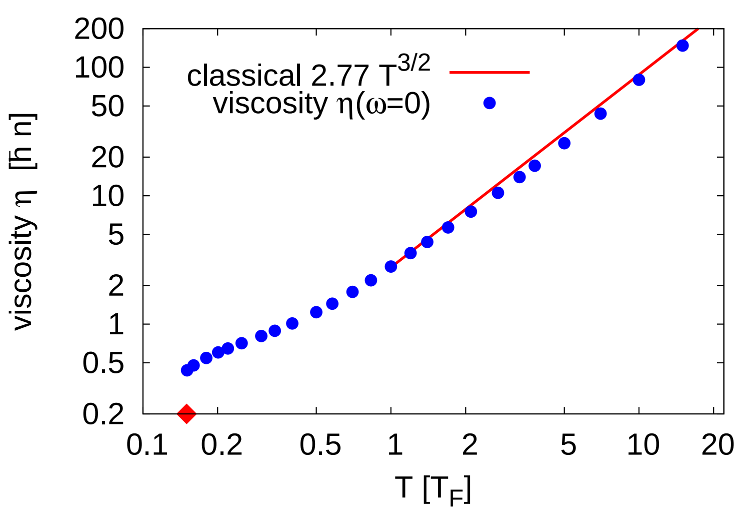
$$\zeta \sim \left(1 - \frac{2\mathcal{E}}{3P}\right)^2 \eta$$

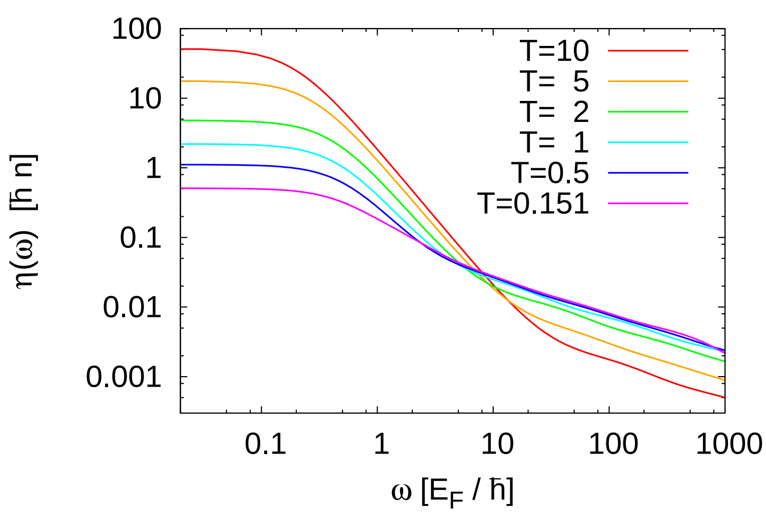
IIIb. Quantum Field Theory

The diagrammatic content of the Boltzmann equation is known: Kubo formula with "Maki-Thompson" + "Azlamov-Larkin" + "Self-energy"



Can be used to extrapolate Boltzmann result to $T \sim T_F$





Enss, Zwerger (2011), see also Levin (2014)

Short time behavior: OPE

Operator product expansion (OPE)

$$\eta(\omega) = \sum_n \frac{\langle \mathcal{O}_n \rangle}{\omega^{(\Delta_n - d)/2}} \qquad \mathcal{O}_n(\lambda^2 t, \lambda x) = \lambda^{-\Delta_n} \mathcal{O}_n(t, x)$$

Leading operator: Contact density (Tan)

$$\mathcal{O}_{\mathcal{C}} = C_0^2 \psi\psi\psi^\dagger\psi^\dagger = \Phi\Phi^\dagger \qquad \Delta_{\mathcal{C}} = 4$$

$\eta(\omega) \sim \langle \mathcal{O}_{\mathcal{C}} \rangle / \sqrt{\omega}$. Asymptotic behavior + analyticity gives sum rule

$$\frac{1}{\pi} \int dw \left[\eta(\omega) - \frac{\langle \mathcal{O}_{\mathcal{C}} \rangle}{15\pi\sqrt{m\omega}} \right] = \frac{\mathcal{E}}{3}$$

Randeria, Taylor (2010), Enss, Zwerger (2011), Hoffman (2013)

IV. Holography

DLCQ idea: Light cone compactification of relativistic theory in d+2

$$p_\mu p^\mu = 2p_+p_- - p_\perp^2 = 0 \qquad p_- = \frac{p_\perp^2}{2p_+} \qquad p_+ = \frac{2n+1}{L}$$

Galilean invariant theory in d+1 dimensions.

String theory embedding: Null Melvin Twist

$$AdS_{d+3} \xrightarrow{\text{NMT}} Schr_d^2$$

$$Iso(AdS_{d+3}) = SO(d+2,2) \supset Schr(d)$$

Son (2008), Balasubramanian et al. (2008)

Other ideas: Horava-Lifshitz (Karch, 2013)

Schrödinger Metric

Coordinates $(u, v, \vec{x}, r)$, periodic in v, $\vec{x} = (x, y)$

$$ds^2 = \frac{r^2}{k(r)^{2/3}} \left\{ \left[\frac{1 - f(r)}{4\beta^2} - r^2 f(r) \right] du^2 + \frac{\beta^2 r_+^4}{r^4}\, dv^2 - [1 + f(r)]\, du\, dv \right\}$$

$$+ k(r)^{1/3} \left\{ r^2\, d\vec{x}^2 + \frac{dr^2}{r^2 f(r)} \right\}$$

Fluctuations $\delta g_x^y = e^{-i\omega u} \chi(\omega, r)$ satisfy $(u = (r_+/r)^2)$

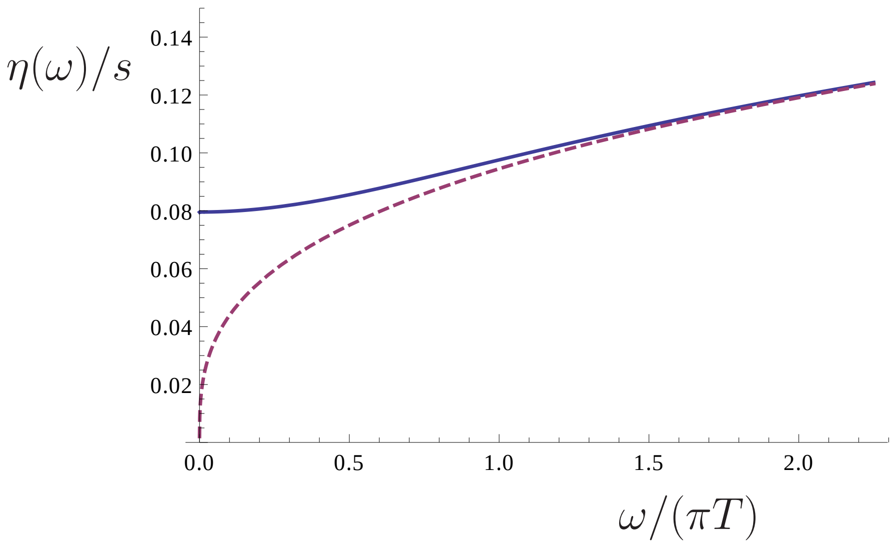
$$\chi''(\omega, u) - \frac{1 + u^2}{f(u) u} \chi'(\omega, u) + \frac{u}{f(u)^2} \mathfrak{w}^2 \chi(\omega, u) = 0$$

Retarded correlation function

$$G_R(\omega) = \frac{\beta r_+^3 \Delta v}{4\pi G_5} \left. \frac{f(u)\chi'(\omega, u)}{u\chi(\omega, u)} \right|_{u \to 0} .$$

Adams et al. (2008), Herzog et al. (2008)

Spectral function



$$\eta(0)/s = 1/(4\pi)$$

$$\eta(\omega \to \infty) \sim \omega^{1/3}$$

Kubo relation (incl. τ_π): $G_R(\omega) = P - i\eta\omega + \tau_\pi \eta \omega^2 + \kappa_R k^2$

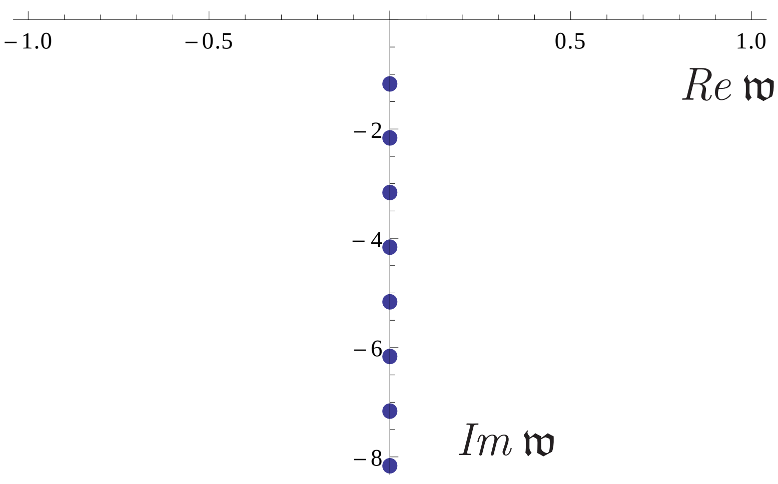
$$\tau_\pi T = -\frac{\log(2)}{2\pi} \qquad AdS_5: \ \tau_\pi T = \frac{2-\log(2)}{2\pi}$$

Range of validity of fluid dynamics: $\omega < T$

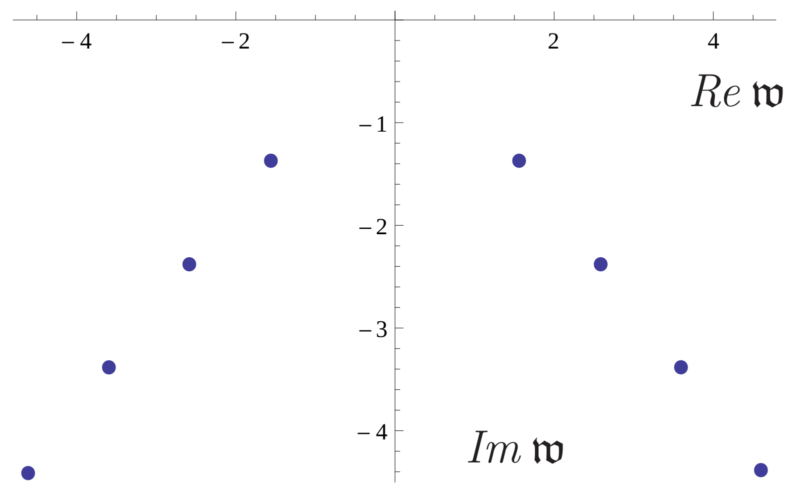
Sch_2: Cannot be matched to relaxation type hydro?

Schaefer (2014), BRSSS (2008)

Quasi-normal modes



Sch_2^2



AdS_5

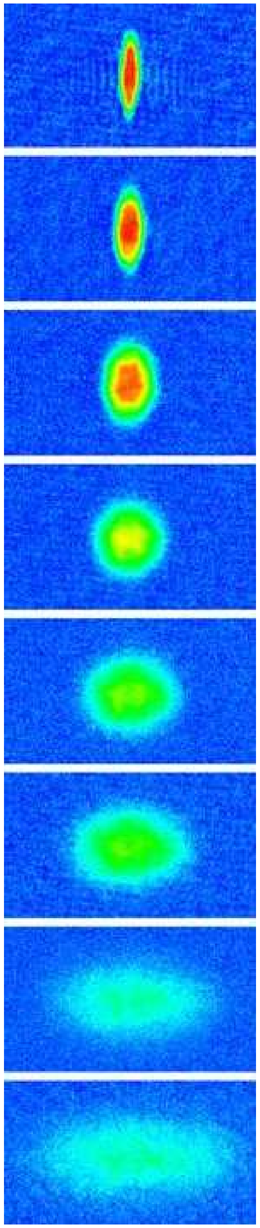
QNM's are stable, $\mathrm{Im}\,\lambda < 0$.

Pole at $\omega \sim iT$ limits convergence of fluid dynamics.

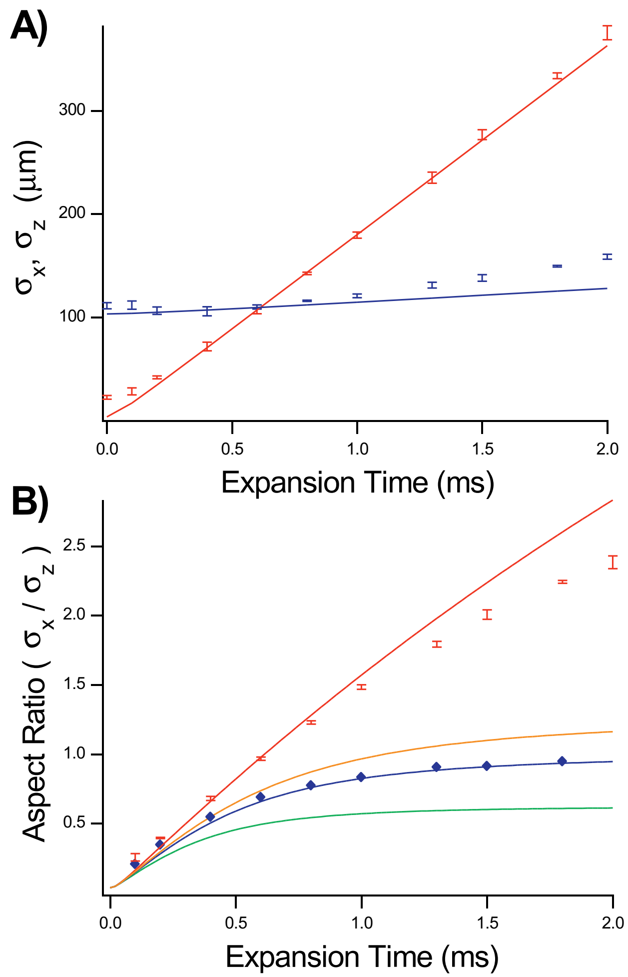
Modes overdamped in Sch_2^2.

Schaefer (2014), Starinets (2002), Heller (2012)

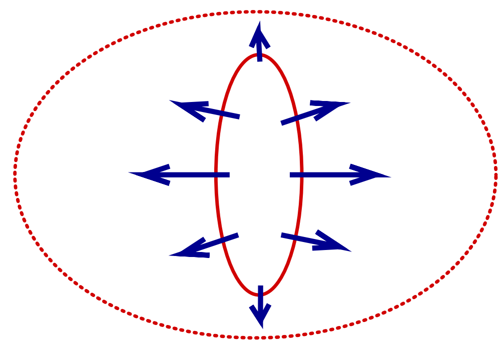
V. Experiments: Elliptic flow



O'Hara et al. (2002)

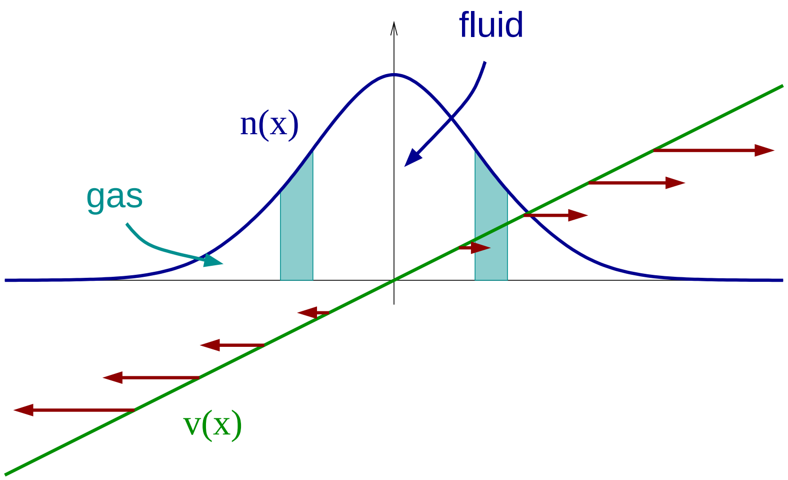


Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy



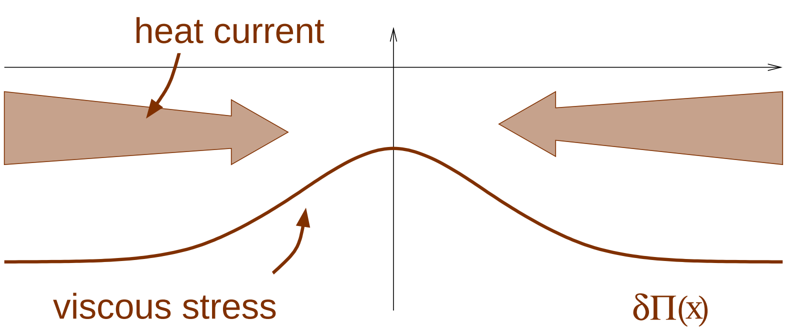
Determination of $\eta(n,T)$

Measurement of $A_R(t, E_0)$ determines $\eta(n,T)$. But:



The whole cloud is not a fluid.

Can we ignore this issue?



No. Hubble flow & low density viscosity $\eta \sim T^{3/2}$ lead to paradoxical fluid dynamics.

Possible Solutions

Combine hydrodynamics & Boltzmann equation. Not straightforward.

Hydrodynamics + non-hydro degrees of freedom ($\mathcal{E}_a$; $a = x, y, z$)

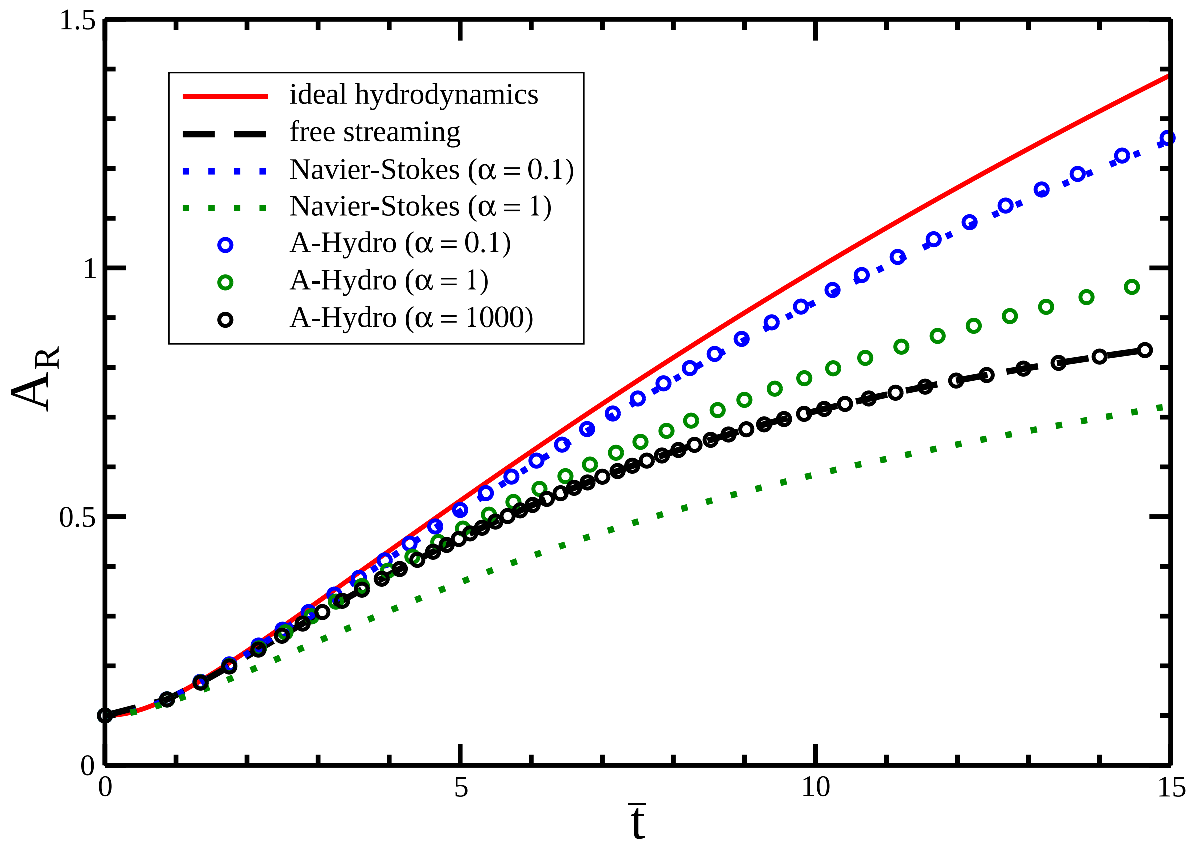
$$\frac{\partial \mathcal{E}_a}{\partial t} + \vec{\nabla} \cdot \vec{\jmath}^{\,\epsilon}_a = -\frac{\Delta P_a}{2\tau} \qquad \Delta P_a = P_a - P$$

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{\nabla} \cdot \vec{\jmath}^{\,\epsilon} = 0 \qquad \mathcal{E} = \sum_a \mathcal{E}_a$$

τ small: Fast relaxation to Navier-Stokes with $\tau = \eta/P$

τ large: Additional conservation laws. Ballistic expansion.

Anisotropic Hydrodynamics: Aspect ratio



Consider $\eta = \alpha n$ and $\alpha \in [0, \infty)$

Navier-Stokes: Ideal hydro $\rightarrow$ very viscous hydro.

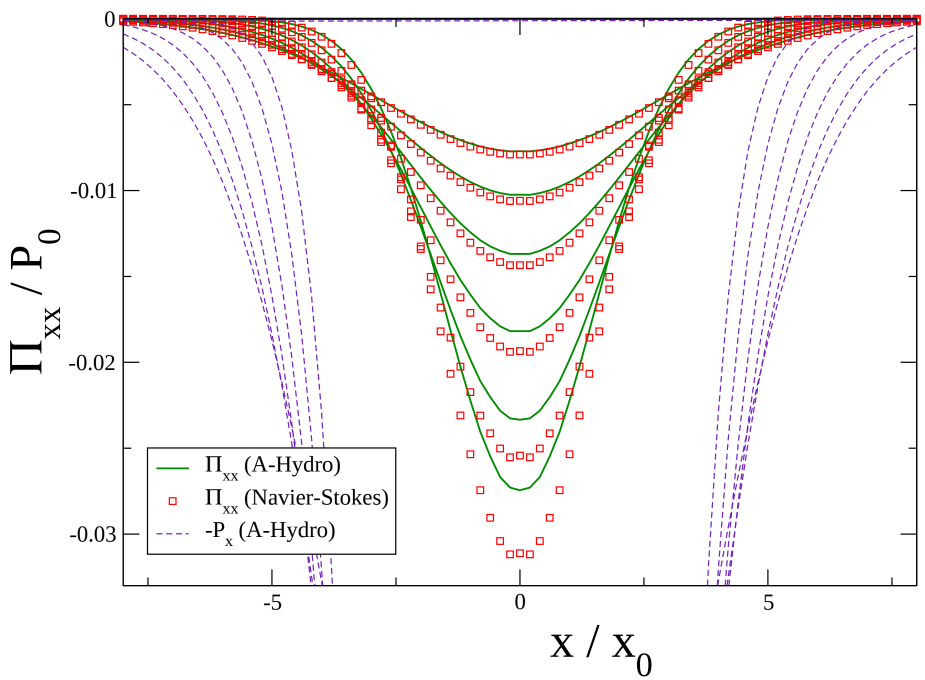
A-hydro: Ideal hydro $\rightarrow$ ballistic expansion.

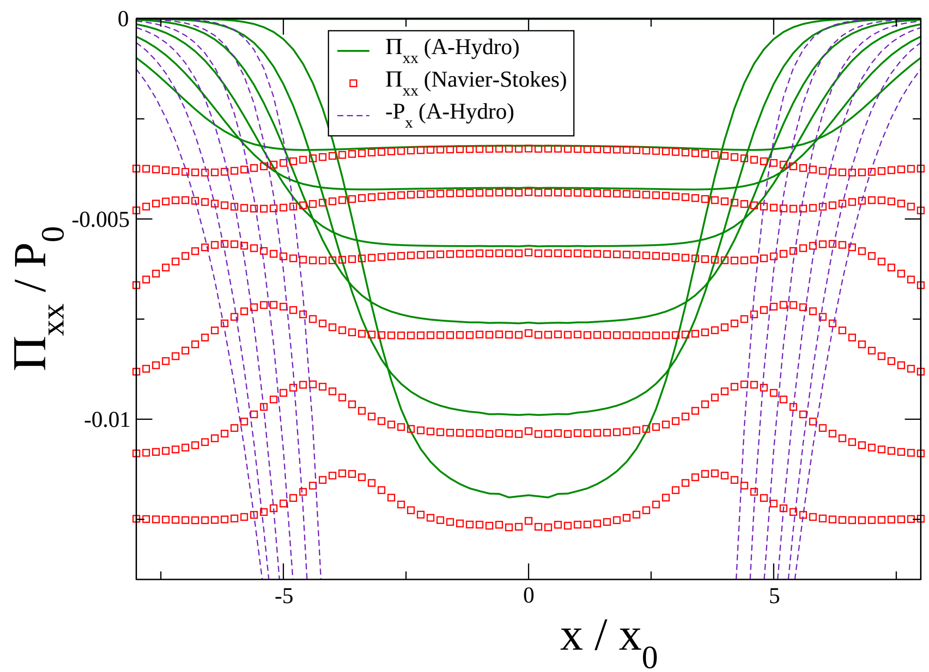
AVH1 hydro code, M. Bluhm & T.S. (2015)

Anisotropic Hydrodynamics

$$\eta = \alpha_n n$$

$$\eta = \alpha_T (mT)^{3/2}$$





Π_{xx}(Navier-Stokes) Π_{xx}(A-Hydro)

AVH1 hydro code, M. Bluhm & T.S. (2015)

Outlook

Fluid dynamics as an E(F)T: Many interesting questions remain.

Experiment: Main issue is temperature, density dependence of η/s. How to unfold?

Need hydro codes that exit "gracefully" (anisotropic hydro, hydro+cascade, or LBE)

Quasi-particles vs quasi-normal modes (kinetics vs holography) unresolved. Need better holographic models, improved lattice calculations.