Wilson ratio: universal nature of quantum fluids

Xi-Wen Guan

中国科学院武汉物理与数学研究所
wuhan Institute of Physics and Mathematics ,Chinese Academy of Sciences

key collaborators: Yi-Cong Yu, Yang-Yang Chen, Hai-Qing Lin

Washington University, April 2015

Fermi liquids vs Non-Fermi liquids

60 years of Fermi liquid theory — quasiparticles

Electronic metal, Kondo impurities, Helium-3, heavy fermions, quantum gases etc.

35 years of Luttinger liquids — collective motion of bosons

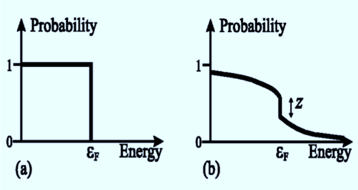
1D correlated electronic systems, spin liquids, quantum gases, quantum wires, etc.

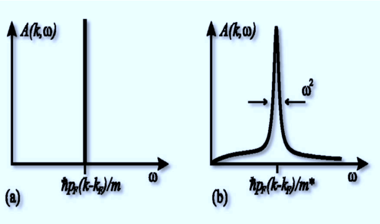
Non-Fermi liquids

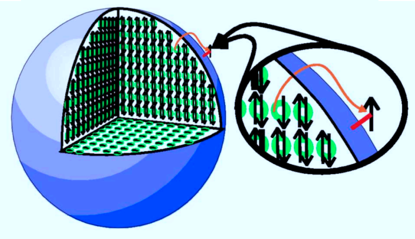
Metal at criticality, 1D quantum liquids, multi-channel Kondo models, quantum gases close to a critical temperature, compounds with spin-orbit coupling, etc.

Fermi Liquids

Quasiparticles







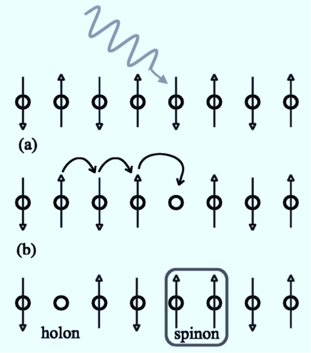
Spectral function $A(k, \omega)$: The probability that an electron with momentum k may be found with a given energy.

Particle excitations near the Fermi surface adiabatically evolve into long- lived quasiparticles, with the same charge, spin and momentum.

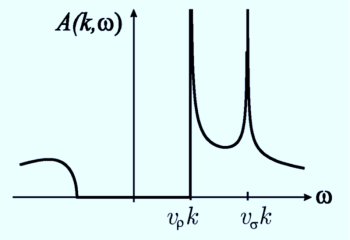
$$\varepsilon_P - \varepsilon_F = (K_F/m^*)(P - K_F)$$

Luttinger liquids

Collective motion of bosons – Bose quasiparticles

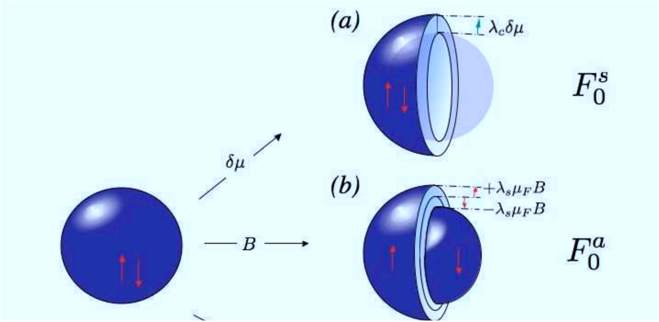


Spin-charge separation



The two singular features corresponding to the spinon and holon with different velocities.

Giamarchi, "Quantum Physics in one dimension" , 2004
A. J. Schofield, Contemporary Physics, 2010



polarization of Fermi surface

3D $c_v = \frac{1}{3}\frac{m^* k_F k_B^2 T}{\hbar^3}, \quad \chi = \frac{m^* k_F}{\pi^2\hbar}\frac{\mu_F^2 g^2}{1+F_0^a}, \quad R_W = \frac{1}{1+F_0^a}$

1D $c_{\rm v} = \frac{\pi k_B^2 T}{3\hbar}\left(\frac{1}{v_{\rm s}} + \frac{1}{v_{\rm c}}\right), \quad \chi = \frac{1}{\hbar\pi v_{\rm s}}, \quad R_W = \frac{2v_{\rm c}}{v_{\rm c}+v_{\rm s}}$

$$R_W = \frac{4}{3}\left(\frac{\pi k_B}{\mu_B g}\right)^2 \frac{\chi}{c_v/T}$$

Wilson ratios

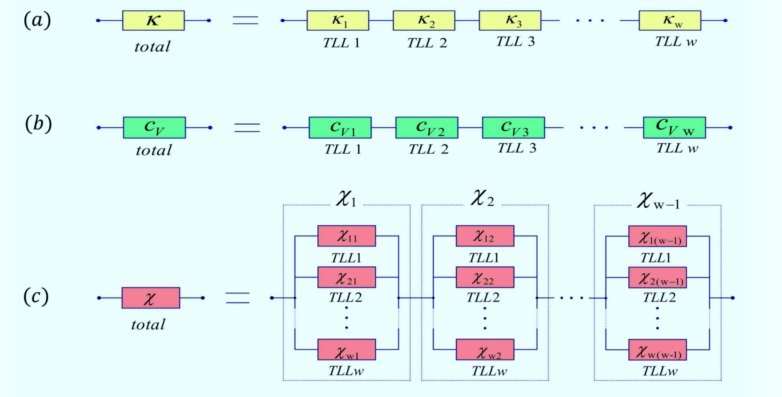
The Wilson ratios, defined as the ratios of the magnetic susceptibility/compressibility to specific heat divided by temperature are dimensionless constants at the renormalization fixed point of these systems. The values of the ratio indicate interaction effects and quantifies spin/particle number fluctuations.

$$G = E - N\mu - MH - TS$$

$$\langle \delta M^2 \rangle = \Delta^D k_B T \chi, \quad \langle \delta N^2 \rangle = \Delta^D k_B T \kappa$$

$$R_W^s = \frac{4}{3}\left(\frac{\pi k_B}{\mu_B g}\right)^2 \frac{\chi}{c_v/T}, \quad R_W^c = \frac{\pi^2 k_B^2}{3} \frac{\kappa}{c_v/T}$$

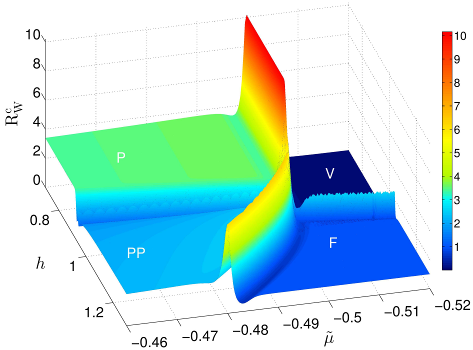
Wilson ratio: essence of Fermi and Luttinger liquids



$$\kappa = \kappa_1 + \kappa_2 + \cdots + \kappa_w, \qquad 1/\kappa_r = \hbar\pi v^r_{\rm Nc}/r^2$$

$$\frac{1}{\chi_\ell} = \frac{1}{\chi_{1\ell}} + \frac{1}{\chi_{1\ell}} + \cdots + \frac{1}{\chi_{w\ell}}, \qquad 1/\chi_{r\ell} = \hbar\pi v^{r\ell}_{\rm Ns}/r^2$$

$$c_{\rm v} = c_{\rm v}^1 + c_{\rm v}^2 + \cdots c_{\rm v}^{\rm w}, \qquad c_{\rm v}^r = \pi k_B^2 T \,/\, \left(3\hbar v_s^r\right)$$



$$R_{\mathrm{W}}^{\mathrm{s,c}} = \frac{\text{quantum fluctuation}}{\text{thermal fluctuation}} = \begin{cases} \text{const.} & \text{for FL, SL, TLL,} \\ \text{QC} & \text{for, Non FL.} \end{cases}$$

$$R_{\mathrm{W}}^{s\ell} = \frac{1}{\left(\frac{v_{\mathrm{Ns}}^{w\ell}}{w^2} + \dots \frac{v_{\mathrm{Ns}}^{2\ell}}{4} + v_{\mathrm{Ns}}^{1\ell}\right)\left(\frac{1}{v_{\mathrm{s}}^{w}} + \dots \frac{1}{v_{\mathrm{s}}^{2}} + \frac{1}{v_{\mathrm{s}}^{1}}\right)}$$

$$R_{\mathrm{W}}^{\mathrm{c}} = \frac{\left(\frac{w^2}{v_{\mathrm{Nc}}^{w}} + \dots \frac{4}{v_{\mathrm{Nc}}^{2}} + \frac{1}{v_{\mathrm{Nc}}^{1}}\right)}{\left(\frac{1}{v_{\mathrm{s}}^{w}} + \dots \frac{1}{v_{\mathrm{s}}^{2}} + \frac{1}{v_{\mathrm{s}}^{1}}\right)}, \quad R_{\mathrm{W}}^{\mathrm{c,s}} = \mathcal{F}_{\mathrm{c,s}}\left(\frac{(\tilde{\mu} - \tilde{\mu}_c)}{t^{\frac{1}{\nu z}}}\right) + \lambda_0 t^{-\beta} \mathcal{G}_{\mathrm{c,s}}\left(\frac{(\tilde{\mu} - \tilde{\mu}_c)}{t^{\frac{1}{\nu z}}}\right)$$

I. Two-component Fermi gas

II. Wilson ratios of SU(N) Fermi gases

III. Conclusion

Group of Quantum Integralable System
量子可积系统研究组

Exactly solved models finding their way into the lab

- 2004-2014, Lieb-Liniger Bose gas
- 2008, Yang-Yang thermodynamics of Bose gas
- 2009, super Tonks-Girardeau gas
- 2010, Yang-Gaudin spin-1/2 Fermi gas
- 2010, quantum criticality of the 1D quantum Ising model
- 2012, quantum phonon fluctuations in the 1D Bose gas
- 2012, Wilson ratio in a spin-1/2 Heisenberg ladder
- 2012-2013, testing 1D interacting fermions: from few to many
- 2013, quantum impurity and magnons in 1D Hubbard model of cold atoms
- 2014, 1D liquid of multi-component fermions with tunable spins
- 2014, measuring excitations energies in 1D Bose gas
- 2014, Experimental Observation of a Generalized Gibbs Ensemble

See a review: Cazalilla, et al. , Review of Modern Physics, 83, 1405 (2011)
Guan, Batchelor, Lee, Review of Modern Physics, 85, 1633 (2013)

I. Two-component Fermi gas

Yang-Gaudin model

$$\mathcal{H} = \sum_{j=\downarrow,\uparrow} \int_0^L \phi_j^\dagger(x) \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right) \phi_j(x) dx + g_{1D} \int_0^L \phi_\downarrow^\dagger(x) \phi_\uparrow^\dagger(x) \phi_\uparrow(x) \phi_\downarrow(x) dx - \frac{H}{2} \int_0^L \left(\phi_\uparrow^\dagger(x) \phi_\uparrow(x) - \phi_\downarrow^\dagger(x) \phi_\downarrow(x) \right) dx$$

many-body problem

$$\mathcal{H} = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g_{1\mathrm{D}} \sum_{1 \le i < j \le N} \delta(x_i - x_j) - H m^z$$

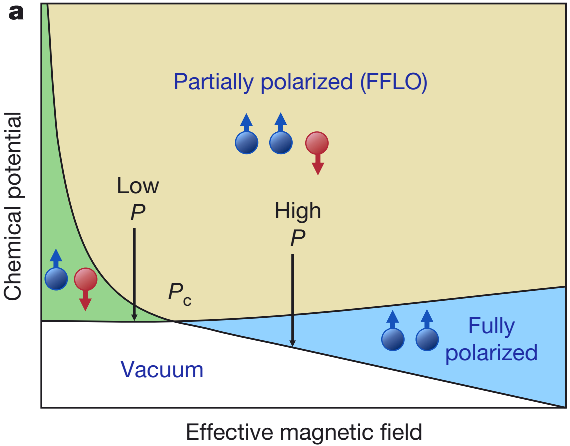
- H: effective magnetic field
- $g_{1\mathrm{D}} = -\frac{\hbar^2 c}{m}$, $c = -2/a_{1\mathrm{D}}$, $a_{1\mathrm{D}} = -\frac{a_\perp^2}{a_{3\mathrm{D}}} + A a_\perp$

C. N. Yang,1967; M. Gaudin, 1967

Olshanii M., Phys. Rev. Lett., 81, 938 (1998)

Guan, Batchelor, Lee, Rev. Mod. Phys., Rev. Mod. Phys. 85, 1633 (2013)

many others ...



Experimental measurement of phase diagram of two-component ultracold ^{6}Li atoms trapped in an array of 1D tubes through the finite temperature density profiles.
Liao *et al*, Nature **467**, 567 (2010)

Orso, PRL 07; Hu, Liu & Drummond, PRL 07
Guan, Batchelor, Lee & Bortz, PRB 07
Zhao, Guan, Liu, Batchelor & Oshikawa, PRL 09

$$\frac{1}{\chi} = \frac{1}{\chi_{\mathrm{u}}} + \frac{1}{\chi_{\mathrm{b}}}, \quad c_V = \frac{\pi^2 k_{\mathrm{B}}^2 T}{3}\left(\frac{1}{\hbar\pi v_{\mathrm{s}}^{\mathrm{u}}} + \frac{1}{\hbar\pi v_{\mathrm{s}}^{\mathrm{b}}}\right)$$

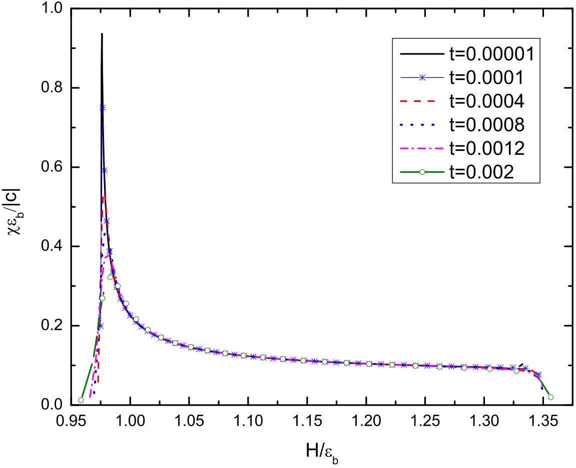
$$R_W^s = \frac{4}{3}\left(\frac{\pi k_B}{\mu_B g}\right)^2 \frac{\chi}{c_V/T} = \frac{4}{\left(v_N^{\mathrm{b}} + 4v_N^{\mathrm{u}}\right)\left(\frac{1}{v_S^{\mathrm{b}}} + \frac{1}{v_S^{\mathrm{u}}}\right)}$$

$$H = 2(\mu_1 - \mu_2) + \frac{1}{2}c^2, \quad n = 2n_2 + n_1$$

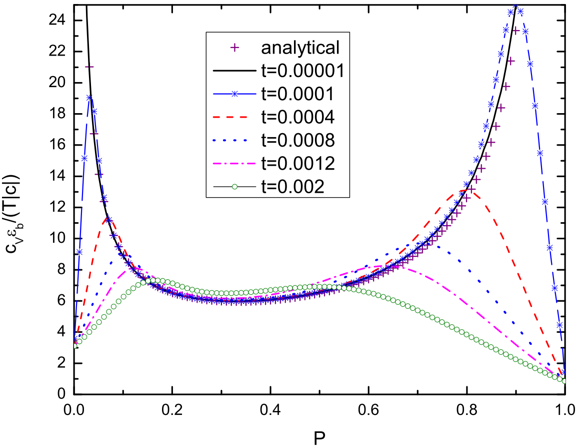
$$\chi^{-1} = \frac{1}{2}\left(\frac{\partial H}{\partial m}\right)_n = \frac{\Delta\mu_1 - \Delta\mu_2}{\Delta n_1} = \frac{\Delta\mu_1}{\Delta n_1} + \frac{1}{2}\frac{\Delta\mu_2}{\Delta n_2}$$

$$\chi_u = \left(\frac{\partial n_1}{\partial \mu_1}\right)_n, \quad \chi_b = 2\cdot\left(\frac{\partial n_2}{\partial \mu_2}\right)_n$$

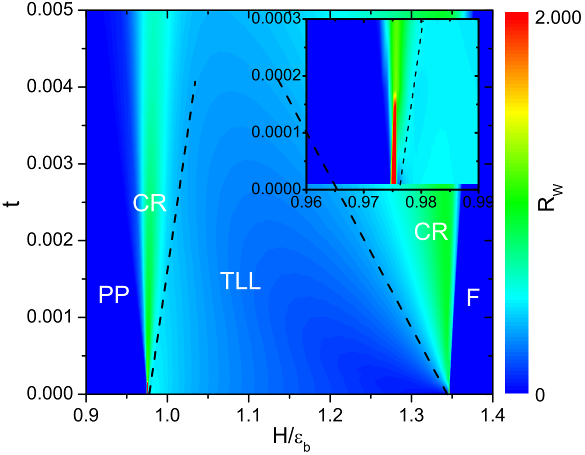
Guan, Yin, Foerster, Batchelor, Lee, Lin, PRL 111, 130401(2013)



$$F(T) = E_0 - \frac{\pi C T^2}{6\hbar}\left(\frac{1}{v_{\mathrm{b}}} + \frac{1}{v_{\mathrm{u}}}\right)$$



$$c_V = \frac{\pi K_B^2 T}{3\hbar}\left(\frac{1}{v_{\mathrm{b}}} + \frac{1}{v_{\mathrm{u}}}\right),$$



$$R_W^s = \frac{4}{\left(v_N^{\mathrm{b}} + 4v_N^{\mathrm{u}}\right)\left(\frac{1}{v_S^{\mathrm{b}}} + \frac{1}{v_S^{\mathrm{u}}}\right)}$$

$$\kappa = \kappa_b + \kappa_u, \qquad \langle \delta N^2 \rangle = \Delta^D k_B T \kappa$$

$$\kappa_{\rm b}^{-1} = \frac{\hbar\pi v_{Nc}^{\rm b}}{2}, \qquad , \quad \kappa_{\rm u}^{-1} = \hbar\pi v_{Nc}^{\rm u}$$

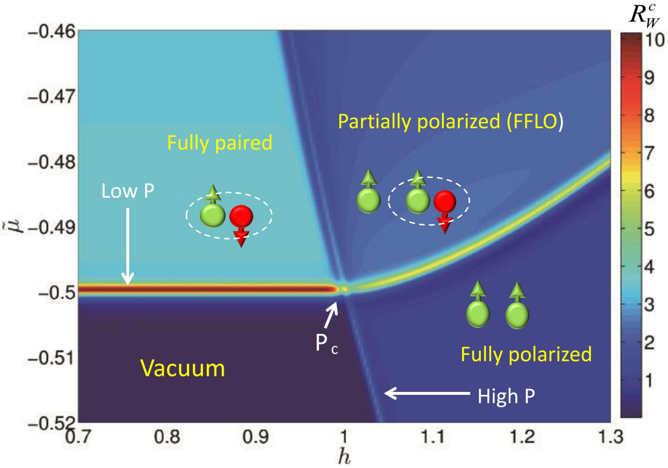
$$v_{Nc}^{\rm b} = \frac{1}{\hbar\pi}\left(\frac{\partial\mu_b}{\partial n_b}\right)_{H,c}, \quad v_{Nc}^{\rm u} = \frac{1}{\hbar\pi}\left(\frac{\partial\mu_u}{\partial n_u}\right)_{H,c}$$

$$\kappa_{\rm b}^{-1} \approx \frac{\pi^2 n_2}{4}\left\{1 + \frac{6n_1}{c} + \frac{4n_2}{c} + \frac{n_2^2}{2cn_1} + \frac{24n_1^2}{c^2} + \frac{24n_1n_2}{c^2} + \frac{17n_2^2}{c^2} - \frac{2n_2^3}{c^2n_1} + \frac{n_2^4}{4c^2n_1^2}\right\}$$

$$\kappa_{\rm u}^{-1} \approx 2\pi^2 n_1\left\{1 + \frac{12n_2}{c} + \frac{16n_1^2}{cn_2} + \frac{96n_2^2}{c^2} + \frac{384n_1^2}{c^2} - \frac{8n_1n_2}{c^2} - \frac{96n_1^3}{c^2n_2} + \frac{256n_1^4}{c^2n_2}\right\}$$

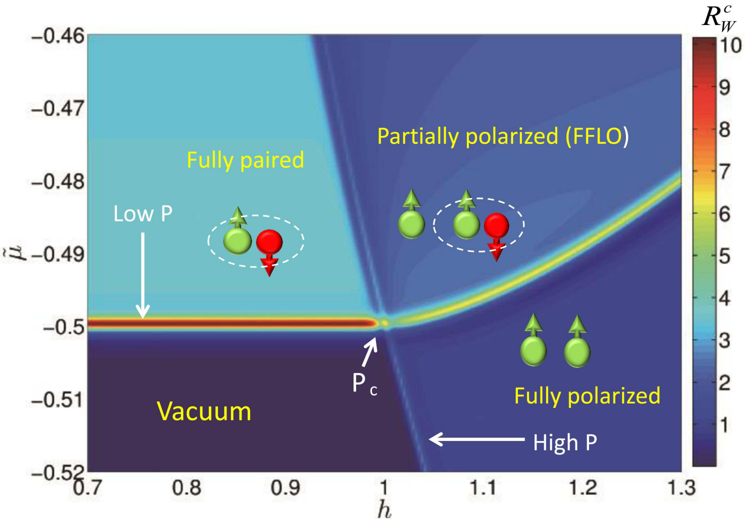
$$v_{\rm u} \approx \frac{\hbar}{2m} 2\pi n_1\left(1 + 2A_1/|c| + 3A_1^2/c^2\right)$$

$$v_s^{\rm b} \approx \frac{\hbar}{2m}\pi n_2\left(1 + 2A_2/|c| + 3A_2^2/c^2\right)$$



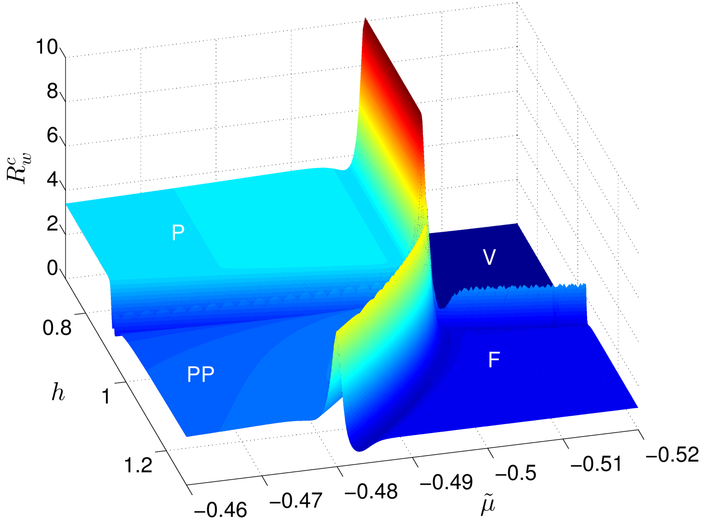
$$R_W^c = \frac{\pi^2 k_B^2}{3} \frac{\kappa}{c_v/T}$$

$$R_W^c = \left(\frac{2}{v_{Nc}^{\mathrm{b}}} + \frac{1}{v_{Nc}^{\mathrm{u}}}\right) \Big/ \left(\frac{1}{v_s^{\mathrm{b}}} + \frac{1}{v_s^{\mathrm{u}}}\right)$$



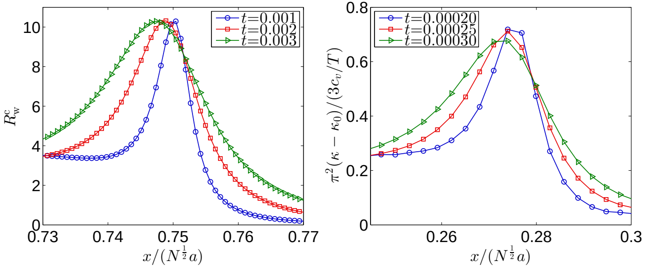
The second Wilson ratio at $t \sim 0.001\epsilon_b$

$$R_W^c = \frac{\pi^2 k_B^2}{3} \frac{\kappa}{c_v/T}, \quad R_W^c = K, \quad TTL_F : R_W^c = 1$$



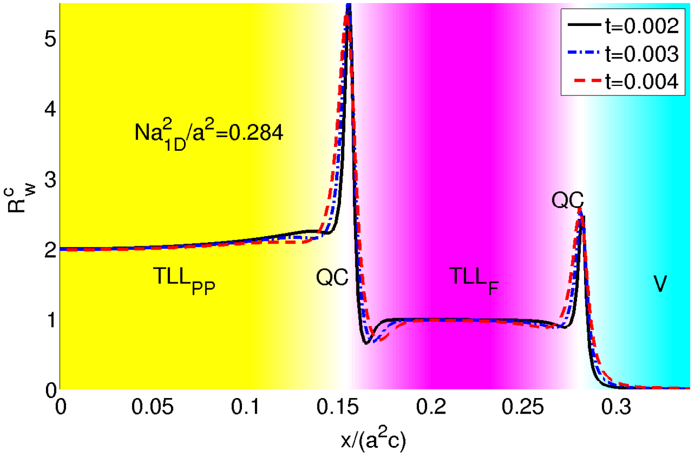
The second Wilson ratio at $t \sim 0.001\epsilon_b$

$$R_{\rm W}^{\rm c,s} \quad = \quad \mathcal{F}_{\rm c,s}\left(\frac{(\tilde{\mu}-\tilde{\mu}_c)}{t^{\frac{1}{\nu z}}}\right)+\lambda_0 t^{-\beta}\mathcal{G}_{\rm c,s}\left(\frac{(\tilde{\mu}-\tilde{\mu}_c)}{t^{\frac{1}{\nu z}}}\right)$$



$$R_{\mathrm{W}}^{\mathrm{c,s}} = \mathcal{F}_{\mathrm{c,s}}\left(\frac{(\tilde{\mu}-\tilde{\mu}_c)}{t^{\frac{1}{\nu z}}}\right) + \lambda_0 t^{-\beta}\mathcal{G}_{\mathrm{c,s}}\left(\frac{(\tilde{\mu}-\tilde{\mu}_c)}{t^{\frac{1}{\nu z}}}\right)$$

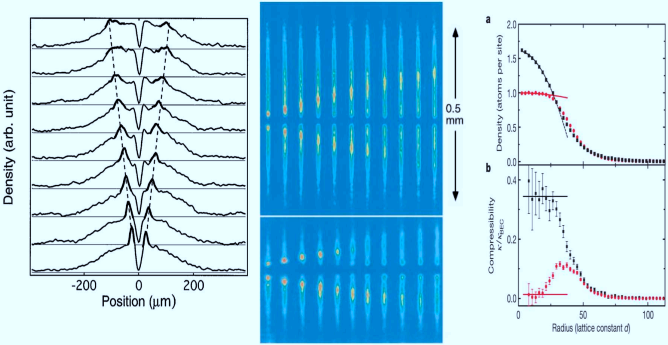
$$\frac{Na_{1\mathrm{d}}^2}{a^2} = 4\int_{-\infty}^{\infty} n(y)/c\,dy, \quad Ma_{1\mathrm{D}}^2/a^2 = 4\int_{-\infty}^{\infty} m_z(y)/c\,dy$$



The second Wilson ratio in an harmonic trap with $P = 0.47$

$$R_W^c = \left(\frac{2}{v_{Nc}^{\mathrm{b}}} + \frac{1}{v_{Nc}^{\mathrm{u}}}\right) / \left(\frac{1}{v_s^{\mathrm{b}}} + \frac{1}{v_s^{\mathrm{u}}}\right)$$

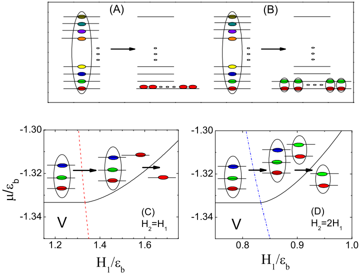
Experimental tests



C. Chin's group, Nature 460,995 (2009)
Ketterle's group, PRL 79, 553 (1997)

$$R_W^c = \frac{\pi^2 k_B^2}{3} \frac{\kappa}{c_v/T}, \quad \kappa = \partial n/\partial\mu = -n'(x)/(xm\omega_x^2)$$

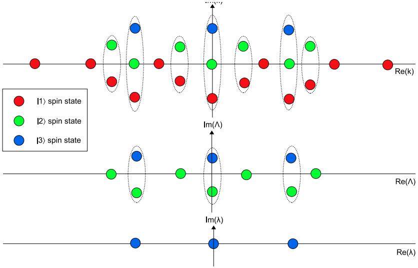
II. Wilson ratios of SU(N) Fermi gases

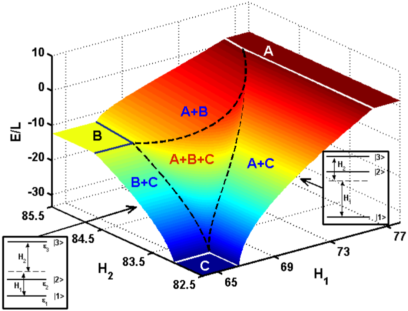


Sutherland, PRL 20, 98 (1968)
Ho, Yi, PRL, 82, 247 (1999)
Wu, Hu, Zhang, PRL, 91 186402 (2003)
Controzzi, Tsvelik, PRL 96, 097205 (2006)
Guan, Batchelor, et al PRL 100, 200401 (2008)
Guan, Lee, et al PRA 82, 021606(R) (2010)
Guan, Ma, Wilson, PRA 85, 033633 (2012)
Cazalilla, Rey, Rep. Prog. Phys. 77, 124401 (2014)
....

Atom Species	Mass (u)	Nuclear Spin (I)	Symmetry Group	Scattering Length (nm)
^{171}Yb	170.93	$\frac{1}{2}$	SU(2)	$-0.15(19)$ [^{171}Yb], $-30.6(3.2)$ [^{173}Yb] [57]
^{173}Yb	172.94	$\frac{5}{2}$	SU(6)	$10.55(11)$ [^{173}Yb], $-30.6(3.2)$ [^{171}Yb] [57]
^{87}Sr	86.91	$\frac{9}{2}$	SU(10)	$5.09(10)$ [^{87}Sr] [58]

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N_f}\frac{\partial^2}{\partial x_i^2} + g_{1\mathrm{D}}\sum_{1\le i<j\le N_f}\delta(x_i - x_j)$$

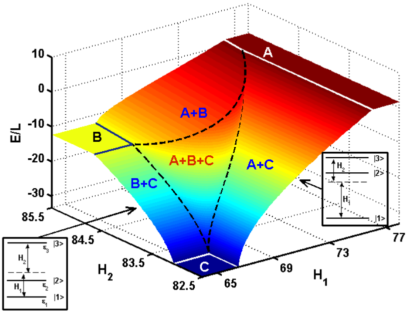


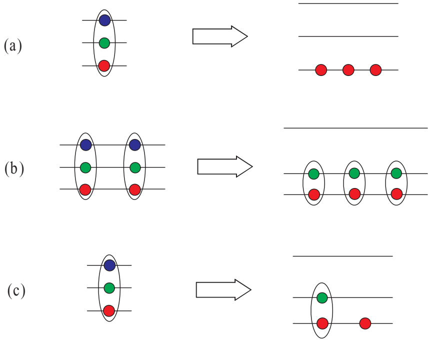


$$\text{pairs}: \; k_i = \Lambda_i \pm \mathrm{i}c/2; \quad \text{trions}: \; k_i = \lambda_i \pm \mathrm{i}c, \; \lambda_i$$

$$\mu = -(\epsilon_Z^1 + \epsilon_Z^2 + \epsilon_Z^3)/3, \quad H_1 = -\mu - \epsilon_Z^1, \quad H_2 = \epsilon_Z^3 + \mu$$

Guan, Batchelor, et al PRL 100, 200401 (2008)
Guan, Ma, Wilson, PRA 85, 033633 (2012)
He, Lee, et al JPA 44, 405005 (2011)





$$n = -\frac{\partial G}{\partial \mu}, \quad n_1 = -\frac{\partial G}{\partial H_1}, \quad n_2 = -\frac{\partial G}{\partial H_2}$$

$$n = n_1 + 2n_2 + 3n_3, \quad \chi_1 = \frac{\partial M}{\partial H_1}, \quad \chi_2 = \frac{\partial M}{\partial H_2}$$

$$H_r/r = (\mu_r - \mu_w) + \Delta_{wr}$$

Experimental realization of three-component ^{6}Li Fermi gases:
Ottenstein et al, PRL (2008); Huckans et al, PRL (2009)

$$
\begin{aligned}
\varepsilon_1(k) &= k^2 - \mu - H_1 + Ta_1 * \ln(1 + e^{-\varepsilon_2/T})(k) + Ta_2 * \ln(1 + e^{-\varepsilon_3/T})(k) \\
&\quad -T\sum_{n=1}^{\infty} a_n * \ln(1 + \xi_n^{-1}) \\
\varepsilon_2(k) &= 2k^2 - 2\mu - H_2 - \frac{1}{2}c^2 + Ta_1 * \ln(1 + e^{-\varepsilon_1/T})(k) + Ta_2 * \ln(1 + e^{-\varepsilon_2/T})(k) \\
&\quad -T(a_1 + a_3) * \ln(1 + e^{-\varepsilon_3/T})(k) - T\sum_{n=1}^{\infty} a_n * \ln(1 + \varsigma_n^{-1})(k) \\
\varepsilon_3(k) &= 3k^2 - 3\mu - 2c^2 + Ta_2 * \ln(1 + e^{-\varepsilon_1/T})(k) + T(a_1 + a_3) * \ln(1 + e^{-\varepsilon_2/T})(k) \\
&\quad +T(a_2 + a_4) * \ln(1 + e^{-\varepsilon_3/T})(k) \\
\ln \xi_n(k) &= \frac{n(2H_1 - H_2)}{T} + a_n * \ln(1 + e^{-\varepsilon_1/T})(k) + \sum_{m=1}^{\infty} T_{mn} * \ln(1 + \xi_m^{-1})(k) \\
&\quad -\sum_{m=1}^{\infty} S_{mn} * \ln(1 + \varsigma_m^{-1})(k) \\
\ln \varsigma_n(k) &= \frac{n(2H_2 - H_1)}{T} + a_n * \ln(1 + e^{-\varepsilon_2/T})(k) + \sum_{m=1}^{\infty} T_{mn} * \ln(1 + \varsigma_m^{-1})(k) \\
&\quad -\sum_{m=1}^{\infty} S_{mn} * \ln(1 + \xi_m^{-1})(k)
\end{aligned}
$$

$$\varepsilon^{(r)} = V_r - \sum_{s=1}^{w} A_{\mathrm{rs}} * \varepsilon_{-}^{(s)}, \quad (r = 1, 2, \cdots, w).$$

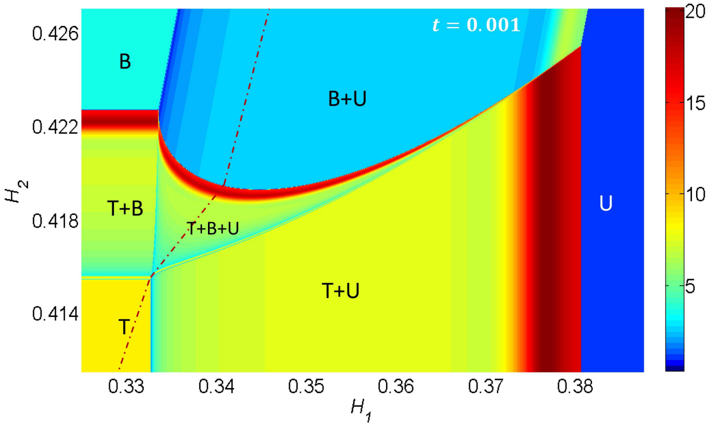
$$\tilde{\eta}^{(r)} = \tilde{V}^{(r)} - \sum_{s=1}^{w-1} \hat{B}_{\mathrm{rs}} * \tilde{\eta}_{-}^{(s)}, \quad (r = 1, 2, \cdots, w-1)$$

Mapping out quantum liquids of the SU(3) Fermi gas from the W_R^c

$$
\begin{aligned}
v_{\mathrm{Nc}}^1 &= \frac{L}{\hbar\pi}\left(\frac{\partial\mu_1}{\partial n_1}\right)_{H_1,H_2}, \; v_{\mathrm{Nc}}^2 = \frac{2L}{\hbar\pi}\left(\frac{\partial\mu_2}{\partial n_2}\right)_{H_1,H_2} \\
v_{\mathrm{Nc}}^3 &= \frac{3L}{\hbar\pi}\left(\frac{\partial\mu_3}{\partial n_3}\right)_{H_1,H_2}
\end{aligned}
$$

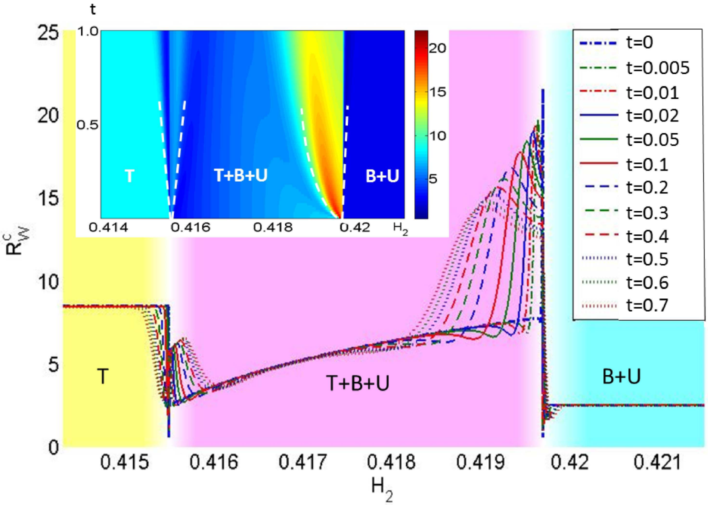
$$
\begin{aligned}
\frac{d\mu_1}{dn_1} &= \frac{\partial\mu_1}{\partial n_1} + \frac{\partial\mu_1}{\partial n_2}\frac{\frac{\partial(H_1H_2)}{\partial(n_3n_1)}}{\frac{\partial(H_1H_2)}{\partial(n_2n_3)}} + \frac{\partial\mu_1}{\partial n_3}\frac{\frac{\partial(H_1H_2)}{\partial(n_1n_2)}}{\frac{\partial(H_1H_2)}{\partial(n_2n_3)}} \\
\frac{d\mu_2}{dn_2} &= \frac{\partial\mu_2}{\partial n_1}\frac{\frac{\partial(H_1H_2)}{\partial(n_2n_3)}}{\frac{\partial(H_1H_2)}{\partial(n_3n_1)}} + \frac{\partial\mu_2}{\partial n_2} + \frac{\partial\mu_2}{\partial n_3}\frac{\frac{\partial(H_1H_2)}{\partial(n_1n_2)}}{\frac{\partial(H_1H_2)}{\partial(n_3n_1)}} \\
\frac{d\mu_3}{dn_3} &= \frac{\partial\mu_3}{\partial n_1}\frac{\frac{\partial(H_1H_2)}{\partial(n_2n_3)}}{\frac{\partial(H_1H_2)}{\partial(n_1n_2)}} + \frac{\partial\mu_3}{\partial n_2}\frac{\frac{\partial(H_1H_2)}{\partial(n_3n_1)}}{\frac{\partial(H_1H_2)}{\partial(n_1n_2)}} + \frac{\partial\mu_3}{\partial n_3}
\end{aligned}
$$

Mapping out quantum liquids of the SU(3) Fermi gas from the W_R^c

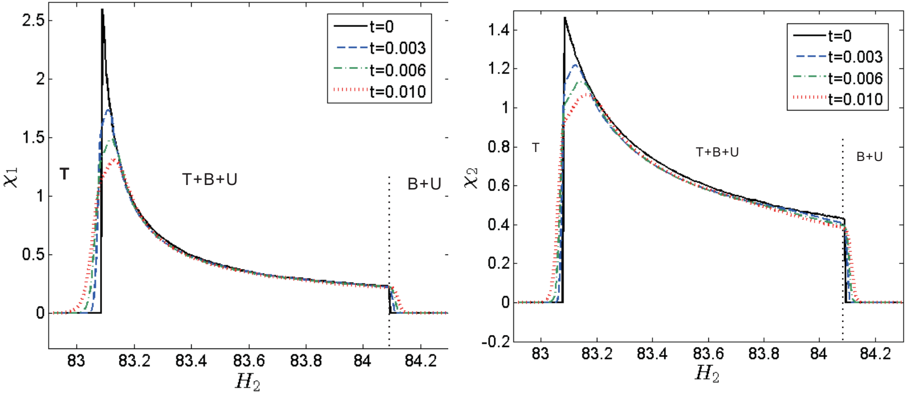


$$R_W^c = \frac{\pi^2 k_B^2}{3}\frac{\kappa}{c_v/T}, \quad \text{unpaired}: R_w^c = K^u, \ \text{pair}: R_w^c = K^b, \ \text{trion}: R_w^c = K^t$$

Yu, Chen, Zhou, Lin, Guan, in preparation

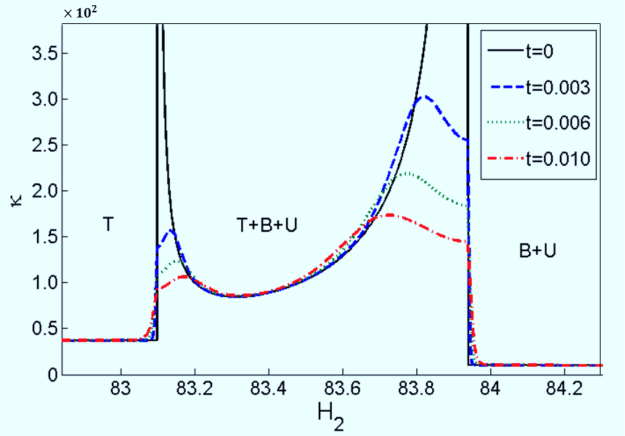


$$W_{\mathrm{R}}^{\mathrm{c}} = \left(\frac{9}{v_{\mathrm{Nc}}^{1}} + \frac{4}{v_{\mathrm{Nc}}^{2}} + \frac{1}{v_{\mathrm{Nc}}^{3}}\right) / \left(\frac{1}{v_{\mathrm{s}}^{1}} + \frac{1}{v_{\mathrm{s}}^{2}} + \frac{1}{v_{\mathrm{s}}^{3}}\right)$$



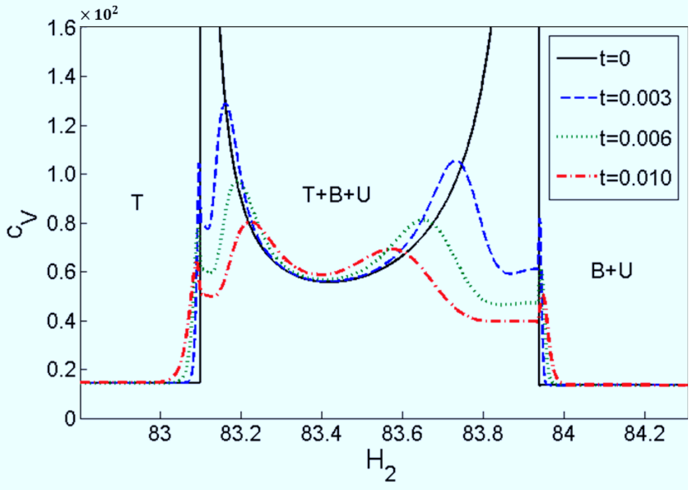
$$\frac{1}{\chi_{r\ell}} = \frac{\hbar\pi}{r^2} v_{\mathrm{Ns}}^{r\ell} = \left(\frac{\lambda}{\lambda+1}\widehat{O}_1 + \frac{1}{\lambda+1}\widehat{O}_2 \right) \widehat{O}_\ell E_r, \quad \widehat{O}_r = \frac{\partial}{\partial n_r} - \frac{r}{3}\frac{\partial}{\partial n_3}$$

$$\frac{1}{\chi_1} = \frac{1}{\chi_{11}} + \frac{1}{\chi_{21}} + \frac{1}{\chi_{31}}, \quad \frac{1}{\chi_2} = \frac{1}{\chi_{12}} + \frac{1}{\chi_{22}} + \frac{1}{\chi_{32}}$$



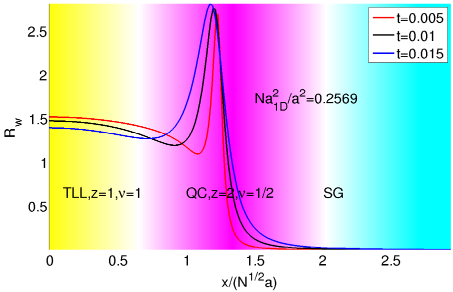
$$\kappa = \kappa_{\rm t} + \kappa_{\rm b} + \kappa_{\rm u}$$

$$\kappa_t^{-1} = \frac{1}{3}\left(\frac{\partial \mu_3}{\partial n_3}\right)_{H_1,H_2,c}, \quad \kappa_b^{-1} = \frac{1}{2}\left(\frac{\partial \mu_2}{\partial n_2}\right)_{H_1,H_2,c}, \quad \kappa_u^{-1} = \left(\frac{\partial \mu_1}{\partial n_1}\right)_{H_1,H_2,c}$$



$$c_{\rm v} = c_{\rm v}^1 + c_{\rm v}^2 + \cdots c_{\rm v}^{\rm w}$$

$$c_{\rm v}^r = \frac{\pi k_B^2 T}{3\hbar}\frac{1}{v_{\rm s}^r}, \quad r = 1, 2, 3$$

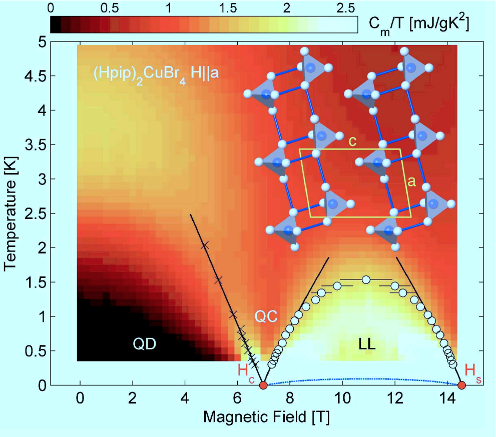


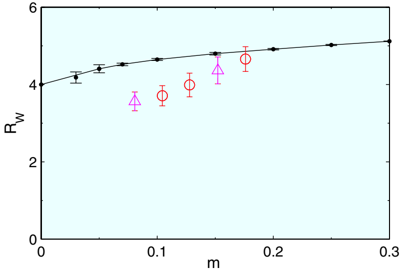
- The Lieb-Linger Bose gas

$$H = \int dx \left(\frac{\pi v_s K}{2} \Pi^2 + \frac{v_s}{2\pi K} \left(\partial_x \phi\right)^2 \right)$$

- Luttinger liquid vs Fermi liquid

$$\kappa = \frac{1}{\hbar \pi v_N}, \qquad c_v = \frac{\pi k_B^2 T}{3} \frac{1}{\hbar v_s} \qquad \mathbf{R_w^c = K}$$





Thielemann et al, PRL, 2008; Ruegg et al. PRL, 2008
Thielemann Ninios et al, PRL, 2012

IV. Conclusion

1. **Wilson ratio presents the essence of the quantum liquids at the renormalization fixed point.**

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