

# Towards Constraining the Nucleon-Nucleon Interaction from Neutron Star Observations

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July 31, 2014

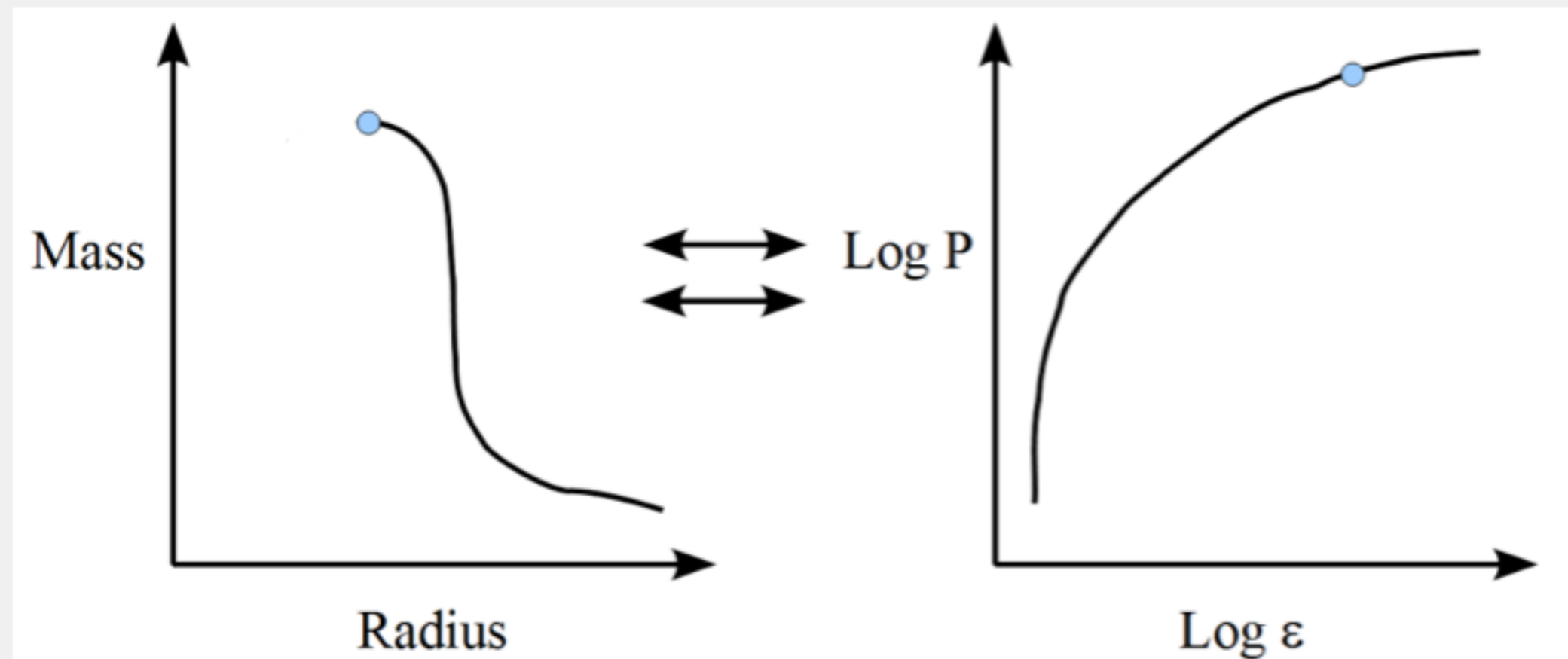
With: Edward F. Brown (MSU), Farrukh J. Fattoyev (TAMU-Commerce), Tobias Fischer (Wroclaw), Stefano Gandolfi (Los Alamos), Matthias Hempel (Basel), James M. Lattimer (Stony Brook), William G. Newton (TAMU-Commerce) and Madappa Prakash (Ohio)

# Outline

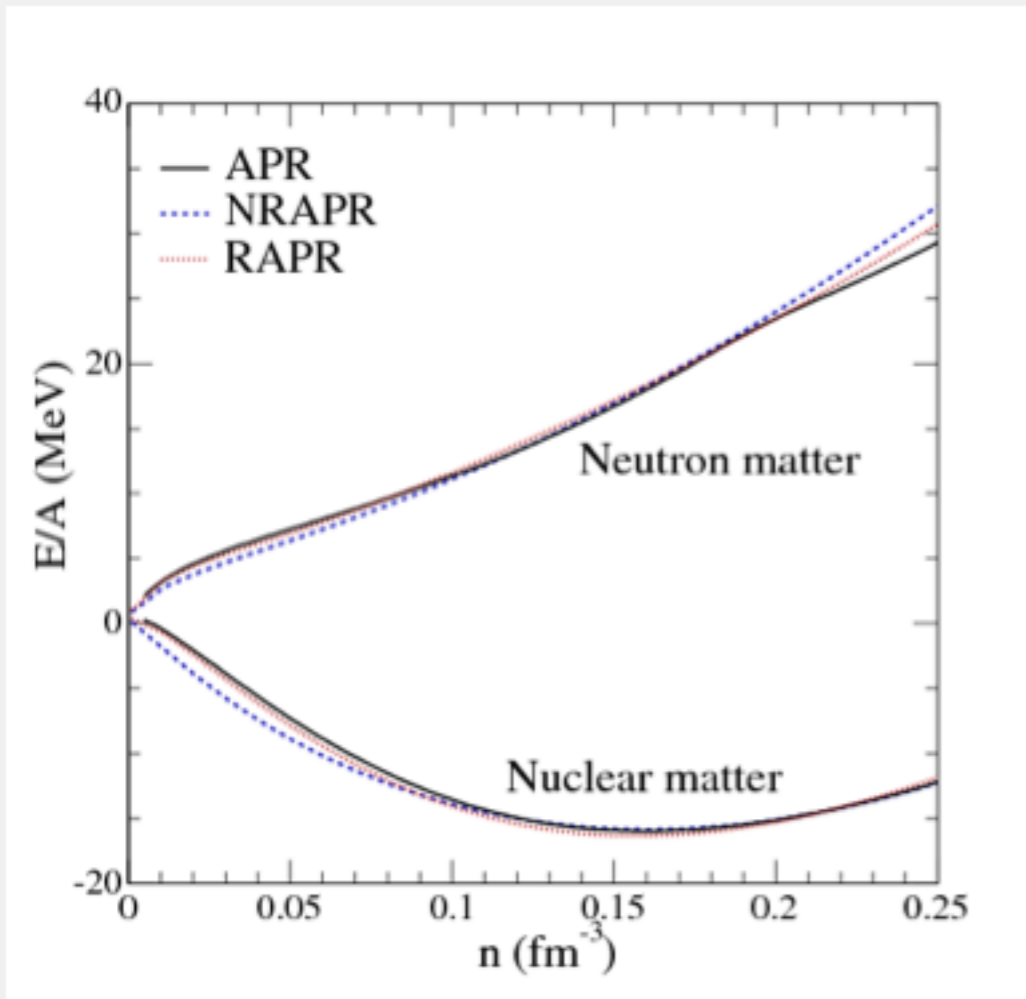
- Neutron star masses and radii and the EOS
- Nucleon-nucleon interaction and the symmetry energy
- Systematic uncertainties and self-consistency
- Finite-temperature EOS tables
- Tidal deformabilities

# Neutron Star Masses and Radii and the EOS

- Neutron stars (to better than 10%) all lie on one universal mass-radius curve  
(Largest correction is rotation - work in progress)
- Recent measurement of two  $2 M_{\odot}$  neutron stars  
Demorest et al. (2010), Antoniadis et al. (2013)
- As of 2007 neutron star radii constrained to 8-15 km, now 10-13 km  
Lattimer and Prakash (2007); Steiner, Lattimer and Brown (2013)



- Einstein's field equations provide a 1-1 correspondence
- Formally an underconstrained problem, but effectively over constrained if you have enough precise data (we don't yet)

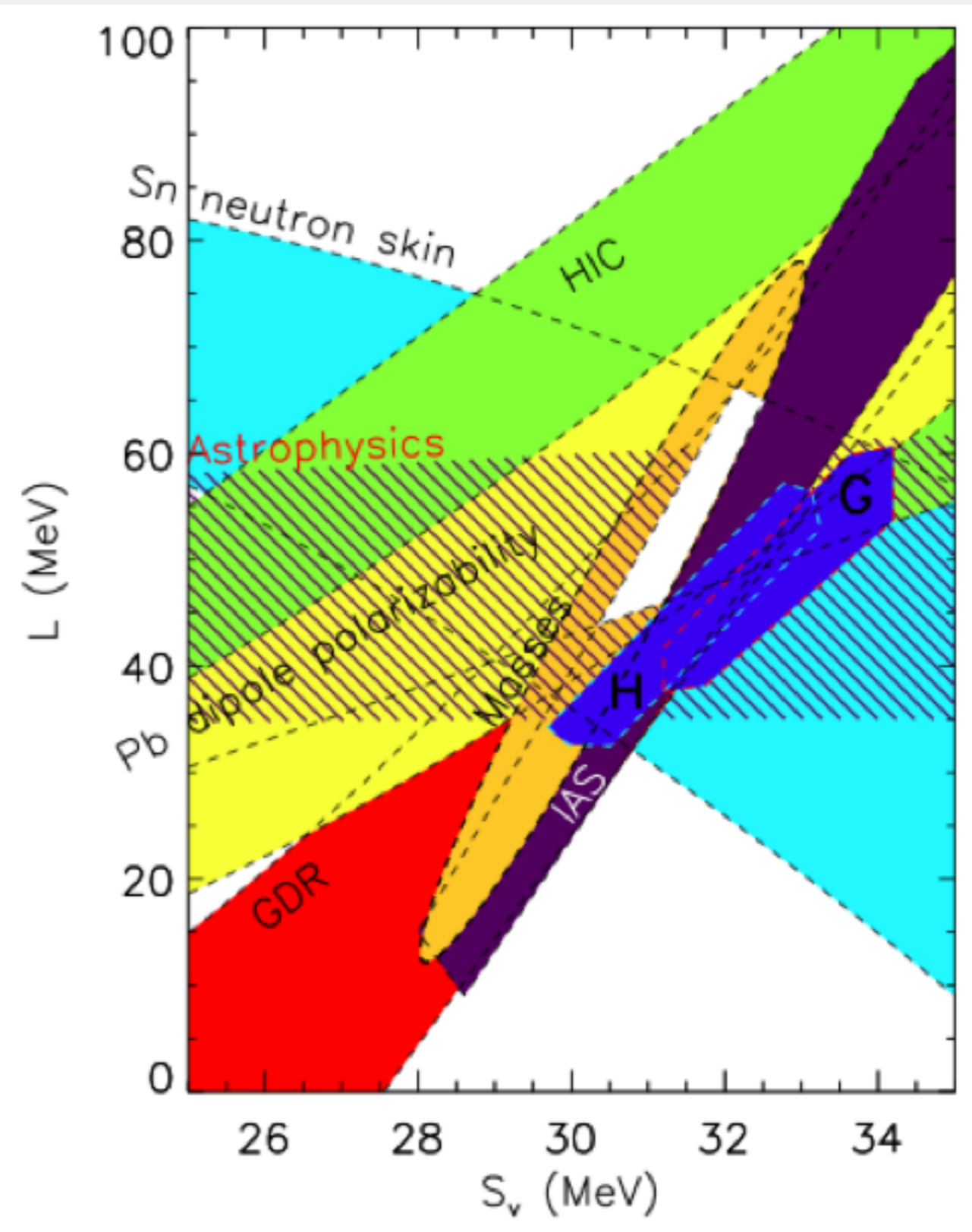


Steiner, Prakash, Lattimer, and Ellis (2005)

Nucleonic matter:

$$\varepsilon = B + \frac{K}{18n_0^2} (n - n_0)^2 + (1 - 2x)^2 S(n)$$

- $S(n_B) \equiv E_{\text{neut}}(n_B) - E_{\text{nuc}}(n_B)$
- $S = S(n_0); L = 3n_0 S'(n_0)$
- Not just EOS: finite-temperature, transport properties, etc.
- Not necessarily applicable at high densities



Lattimer and Steiner (2014)

# Bridging Nuclear and Astro-physics

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## Isospin Dependence of Strong Interactions

Heavy Ion Collisions

Multi-Fragmentation  
Flow  
Isospin Fractionation  
Isoscaling  
Isospin Diffusion

Nuclear Masses

Neutron Skin Thickness

Isovector Giant Dipole Resonances

Fission

Nuclei Far from Stability

Rare Isotope Beams

## Many-Body Theory

Symmetry Energy  
(Magnitude and Density Dependence)

Supernovae

Weak Interactions  
Early Rise of  $L_{\nu e}$   
Bounce Dynamics  
Binding Energy

Proto-Neutron Stars

 $\nu$  Opacities  
 $\nu$  Emissivities  
SN r-Process  
Metastability

Neutron Stars

Observational  
Properties

Binary Mergers

Decompression/Ejection  
of Neutron-Star Matter  
r-Process

QPO's

Mass  
Radius

NS Cooling

Temperature  
 $R_{\infty, z}$   
Direct Urca  
Superfluid Gaps

X-ray Bursters

 $R_{\infty, z}$ 

Gravity Waves

Mass/Radius  
 $dR/dM$ 

Pulsars

Masses  
Spin Rates  
Moments of Inertia  
Magnetic Fields  
Glitches - Crust

Maximum Mass, Radius

Composition:

Hyperons, Deconfined Quarks  
Kaon/Pion Condensates

# Radius Measurements in qLMXBs

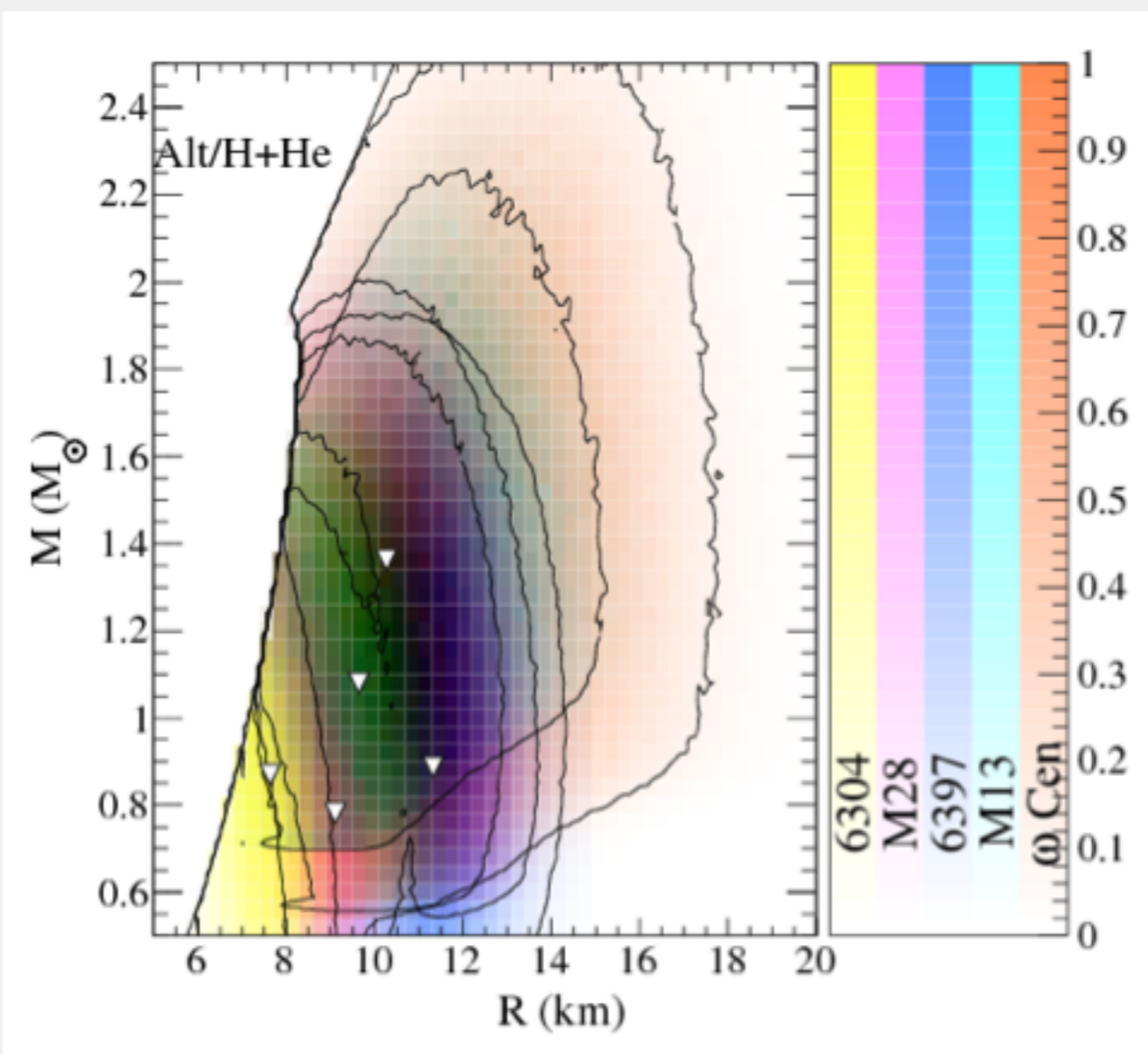
## Quiescent LMXBs

- Measure flux of photons and their energy distribution
- Know distance if in a globular cluster
- Implies radius measurement

$$F \propto T_{\text{eff}}^4 \left( \frac{R_{\infty}}{D} \right)^2$$

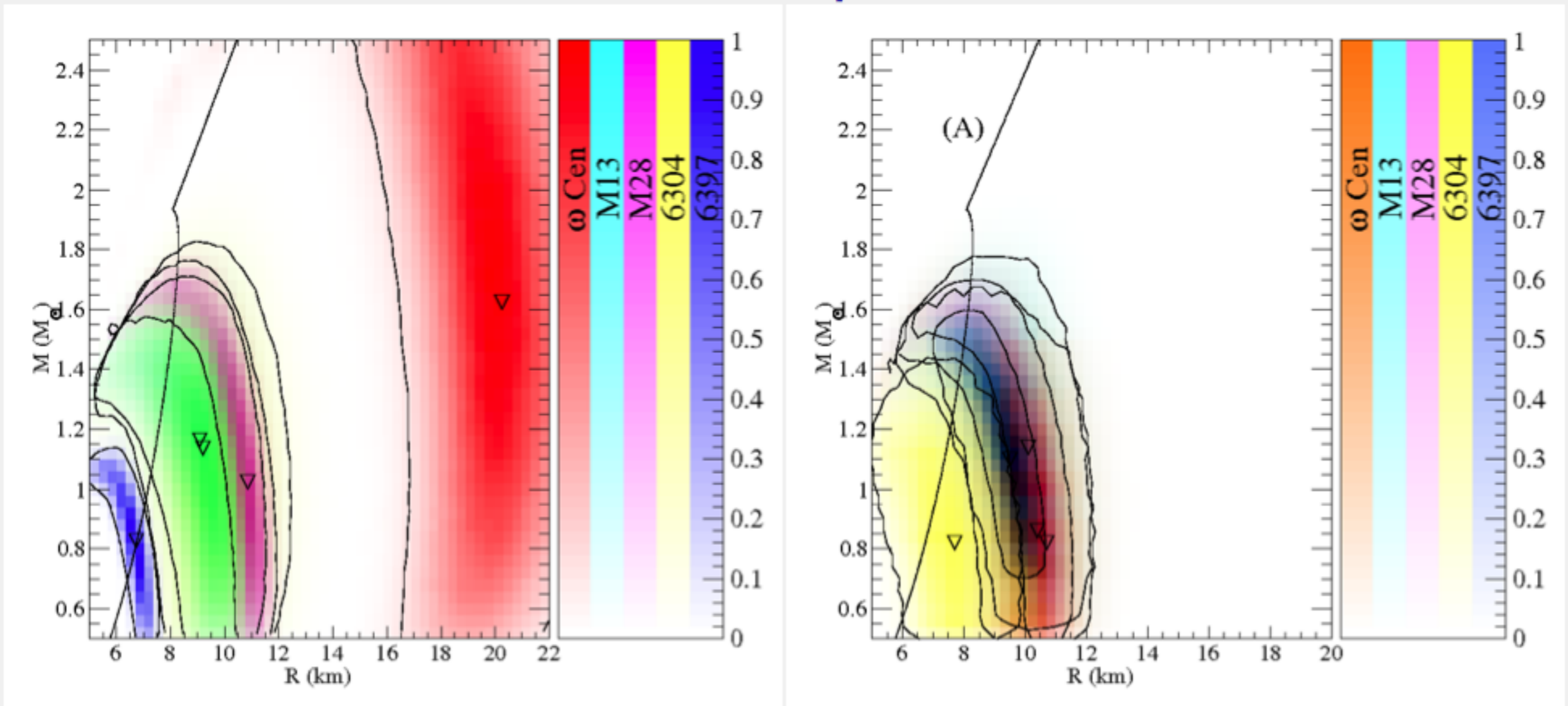
i.e. Rutledge, Bildsten, and Brown (1999)

- Many important unresolved systematics
- Need to understand X-ray absorption between source and observer
- Need information about the atmosphere, including composition
- Also need X-ray absorption and absolute flux calibration
- Inevitably give small radii for some low-mass stars



Lattimer and Steiner (2014) - Probability distributions for five neutron stars, colors added together

# As of last year...



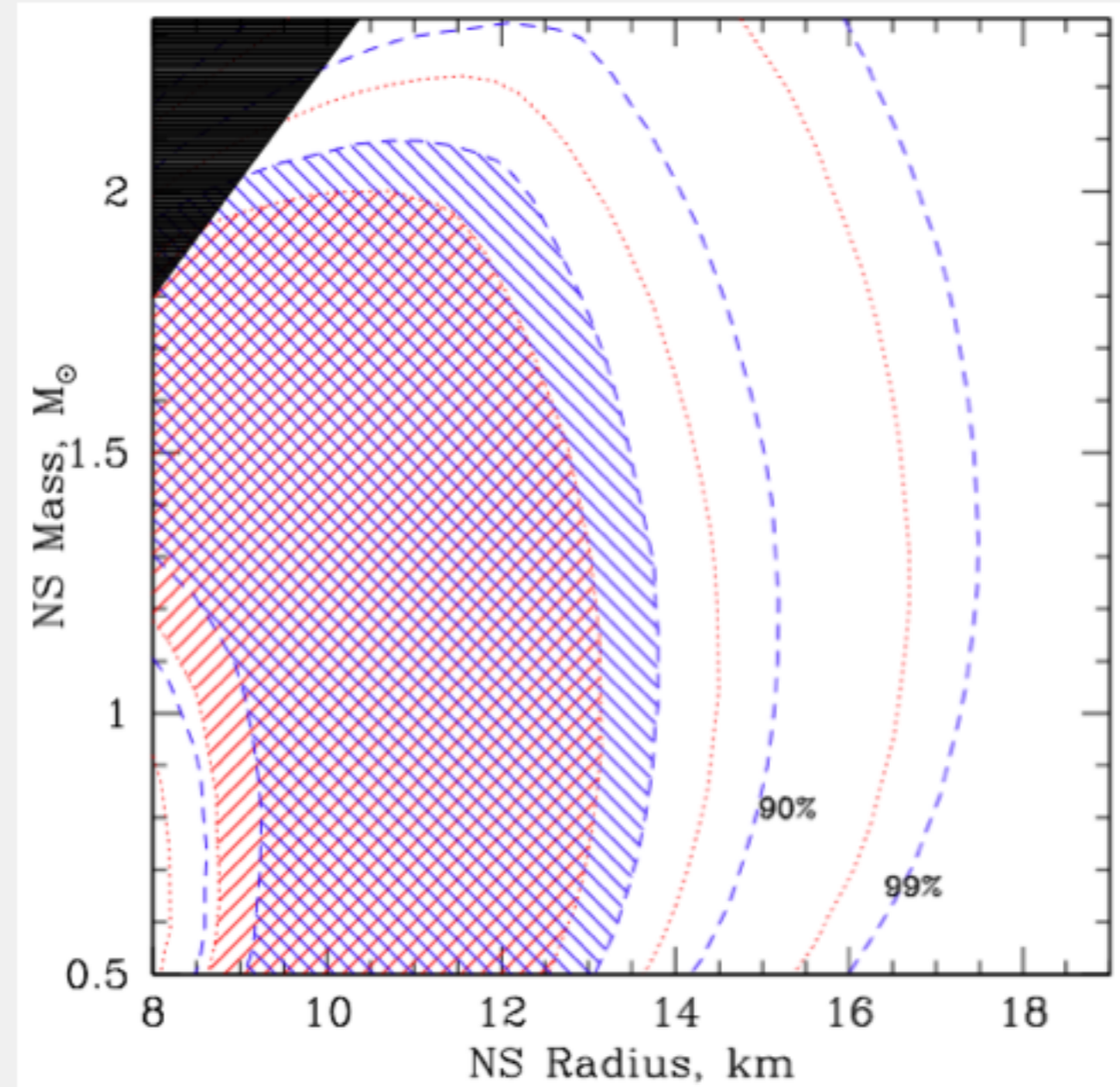
Results from Guillot, Servillat, Webb, and Rutledge (2013) slightly adapted for Lattimer and Steiner (2014) before any assumption about the  $M$ - $R$  curve

Lattimer and Steiner (2014), relative Bayes factor of 1200

- $R_{\text{NS}}$  in  $\omega$  Cen : 11 km or 20 km!
- $R_{\text{NS}}$  in NGC 6397  $\sim$  7 km?
- We tried different  $N_H$  values, different distances, and Helium atmospheres
- 36 separate models, millions of parameterizations

# Recent Updates

- Smaller  $N_H$  values for  $\omega$  Cen!
- "This is a  $\sim 25\%$  change in the best-fit NS radius (or a 20% change in the radius lower limit) due to a change in the ISM model, reinforcing the critical importance of accurate modeling of the ISM"  
Heinke et al. (2014)
- Changed ISM abundances from Anders & Ebihara (1982) to Wilms, Allen & McCray (2000) (XSPEC options)
- Confirmation of expectations from nuclear physics

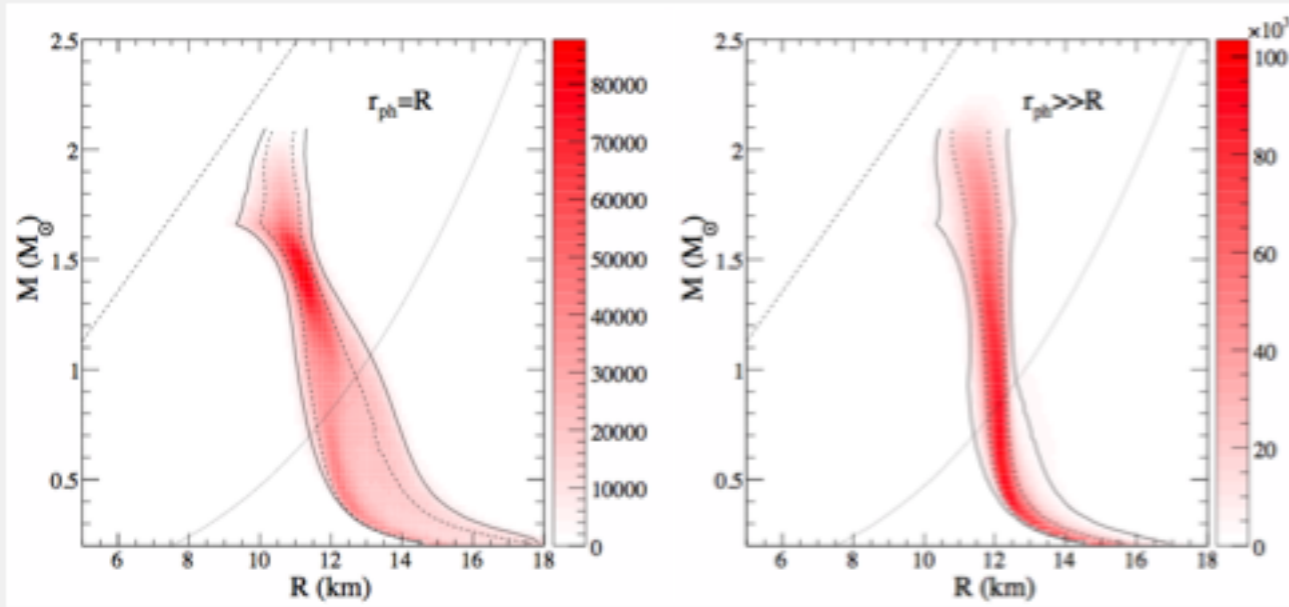


Heinke et al. (2014)

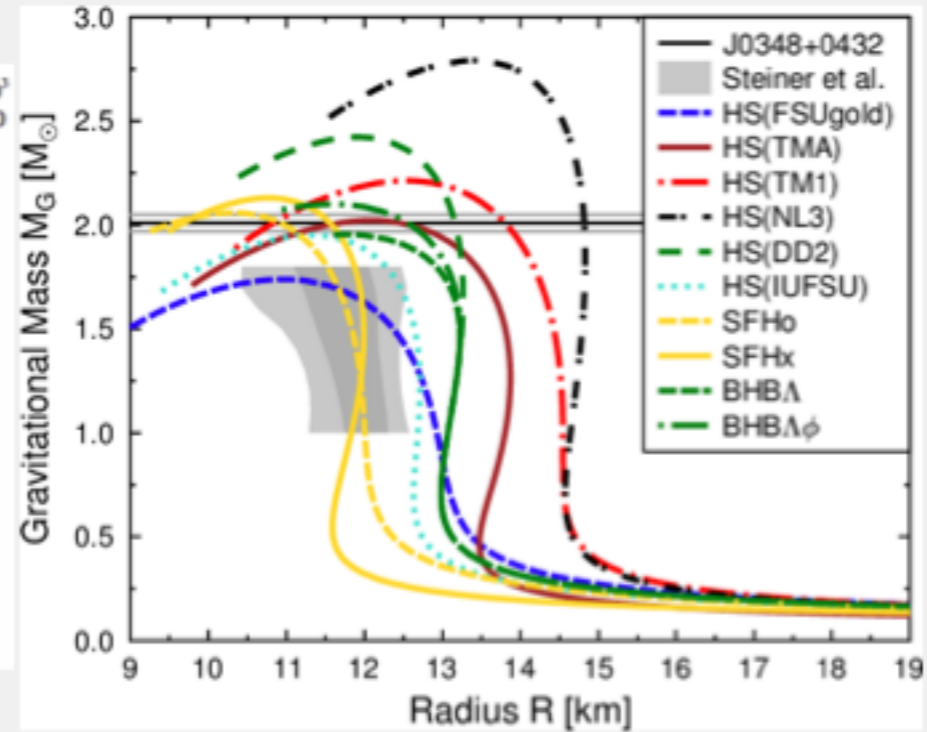
- Radius ranges won't change that much from Steiner, Lattimer, and Brown (2013)



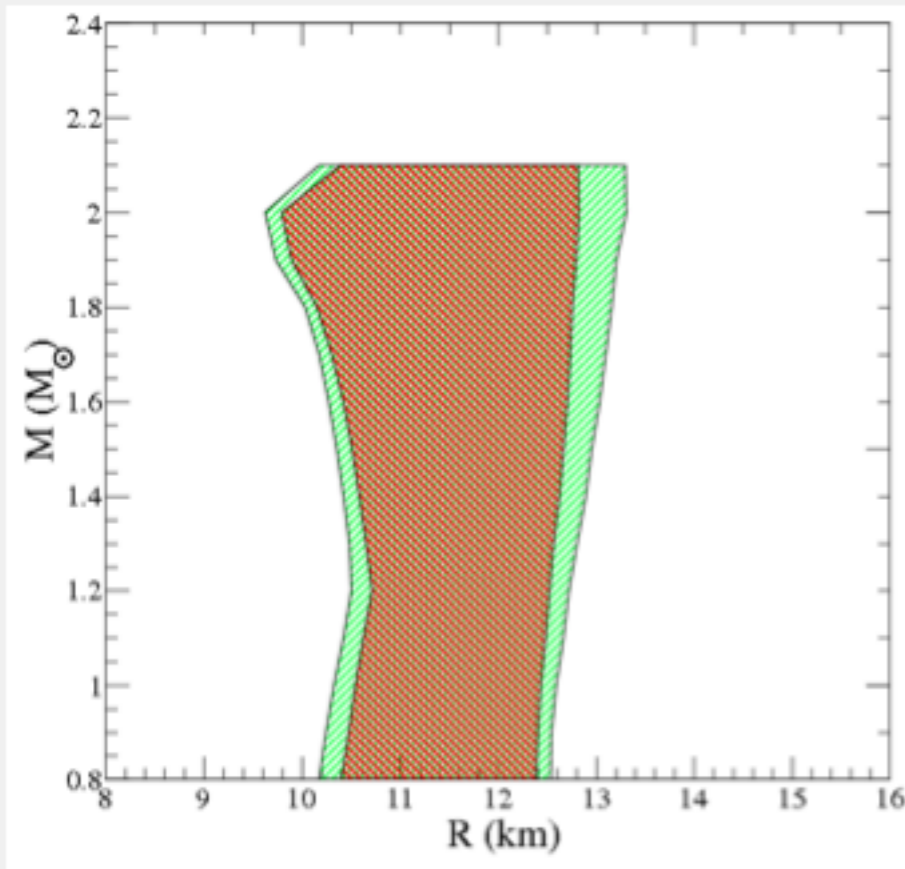
# Evolution of Radius Constraints



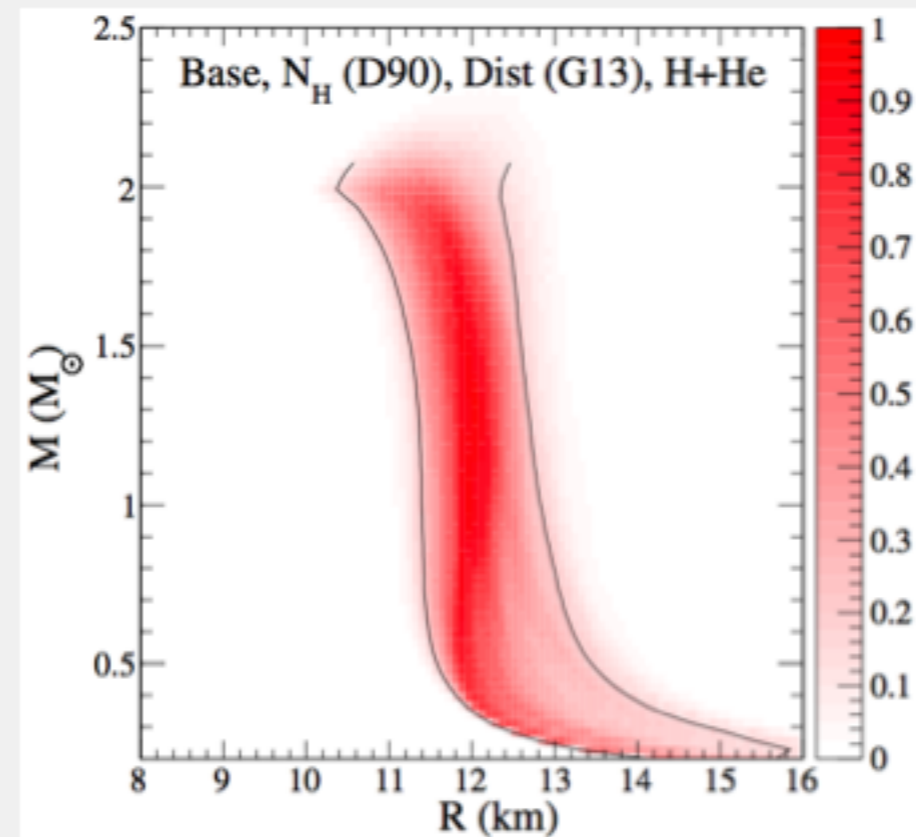
Steiner, Lattimer, and Brown (2010)



Steiner, Hempel and Fischer (2013)



Steiner, Lattimer, and Brown (2013)



Lattimer and Steiner (2014)

# Varying Models and Data Assumptions

EOS Model	Data modifications	$R_{95\%>}$	$R_{68\%>}$	$R_{68\%<}$	$R_{95\%<}$
(km)					
Variations in the EOS model					
A	-	11.18	11.49	12.07	12.33
B	-	11.23	11.53	12.17	12.45
C	-	10.63	10.88	11.45	11.83
D	-	11.44	11.69	12.27	12.54
Variations in the data interpretation					
A	I	11.82	12.07	12.62	12.89
A	II	10.42	10.58	11.09	11.61
A	III	10.74	10.93	11.46	11.72
A	IV	10.87	11.19	11.81	12.13
A	V	10.94	11.25	11.88	12.22
A	VI	11.23	11.56	12.23	12.49
Global limits		10.42	10.58	12.62	12.89

Steiner, Lattimer, and Brown (2013)

- Each model represents  $10^5 - 10^6$  individual EOSs
- Critical component: trying different EOS parameterizations and different interpretations of the data
- Model C allows for strong phase transitions
- Try several different models to assess systematics

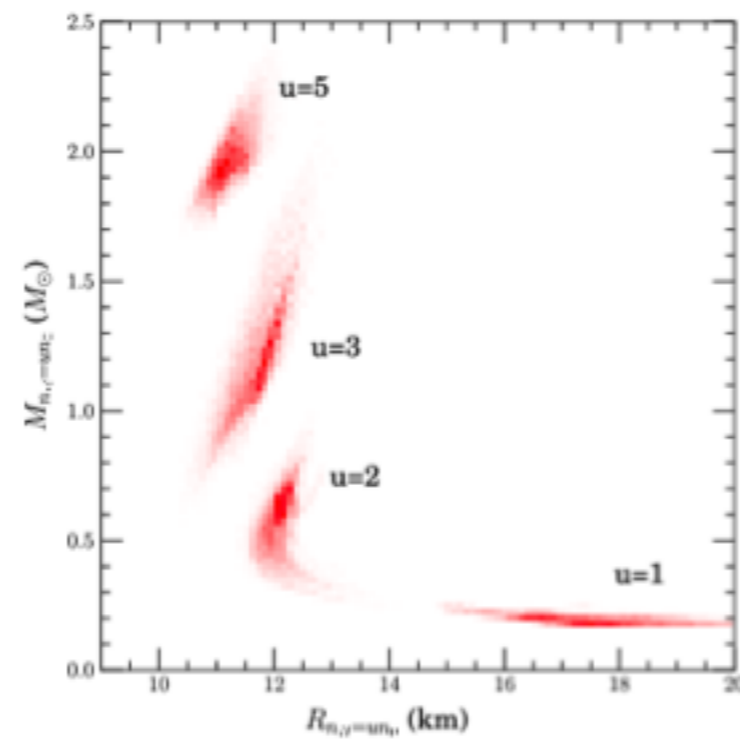
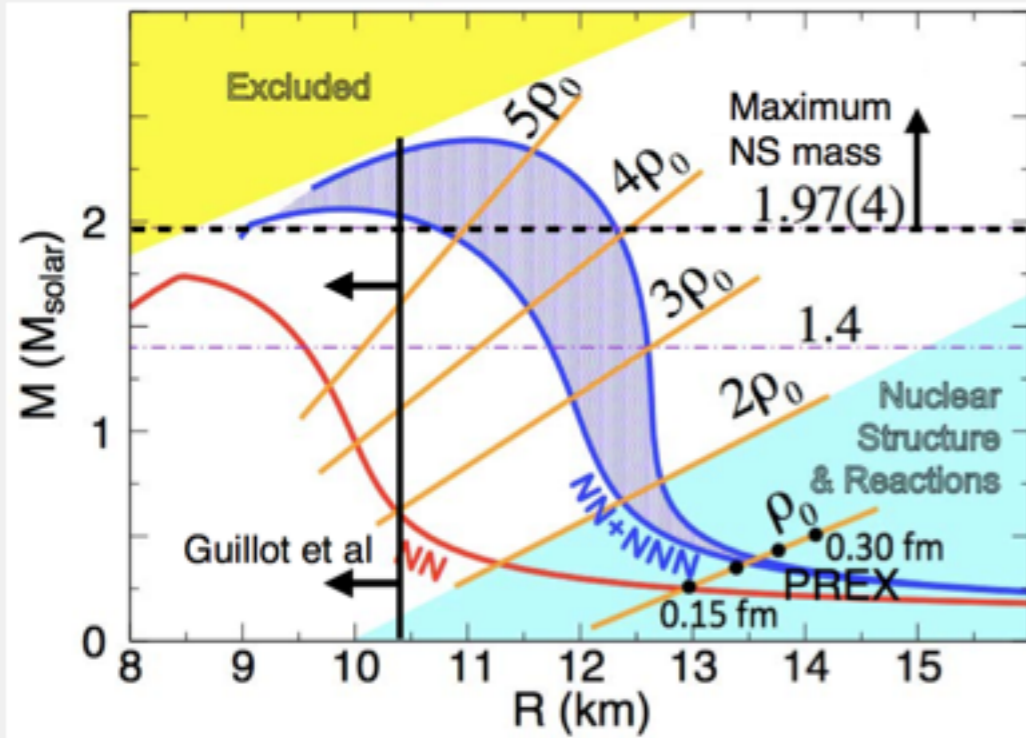
**Table 5**  
Posterior Confidence Ranges and Evidence Integrals

Model	$N_H$	Dist.	Comp.	$R_{1.4}$ (km)	$I$
Base	G13	G13	H	11.11–11.88	$(1.77 \pm 0.09) \times 10^{-8}$
Base	G13	G13	H+He	11.36–12.84	$(4.50 \pm 0.21) \times 10^{-3}$
Base	G13	Alt	H	10.73–11.65	$(1.86 \pm 0.18) \times 10^{-6}$
Base	G13	Alt	H+He	11.45–13.32	$(3.71 \pm 0.21) \times 10^{-1}$
Base	G13	H10	H	10.77–11.71	$(1.23 \pm 0.09) \times 10^{-7}$
Base	G13	H10	H+He	11.36–13.44	$(4.28 \pm 0.35) \times 10^{-3}$
Base	D90	G13	H	10.67–11.51	$(4.65 \pm 0.48) \times 10^{-3}$
Base	D90	G13	H+He	11.31–12.64	$(2.14 \pm 0.19) \times 10^{+2}$
Base	D90	Alt	H	10.85–11.79	$(9.40 \pm 1.22) \times 10^{-3}$
Base	D90	Alt	H+He	11.37–12.61	$(4.06 \pm 0.36) \times 10^{+2}$
Base	D90	H10	H	10.78–11.70	$(4.78 \pm 0.73) \times 10^{-3}$
Base	D90	H10	H+He	11.23–12.62	$(1.57 \pm 0.07) \times 10^{+2}$
Base	H10	G13	H	10.87–11.82	$(1.04 \pm 0.08) \times 10^{+0}$
Base	H10	G13	H+He	11.15–12.38	$(1.84 \pm 0.12) \times 10^{+2}$
Base	H10	Alt	H	11.03–12.07	$(1.39 \pm 0.20) \times 10^{+2}$
Base	H10	Alt	H+He	11.04–12.31	$(1.44 \pm 0.10) \times 10^{+2}$
Base	H10	H10	H	10.78–11.95	$(7.52 \pm 0.65) \times 10^{+1}$
Base	H10	H10	H+He	11.31–12.66	$(5.30 \pm 0.22) \times 10^{+2}$
Exo	G13	G13	H	9.15–10.81	$(7.32 \pm 0.63) \times 10^{-6}$
Exo	G13	G13	H+He	10.52–11.77	$(4.46 \pm 0.38) \times 10^{-2}$
Exo	G13	Alt	H	10.42–11.39	$(1.21 \pm 0.19) \times 10^{-3}$
Exo	G13	Alt	H+He	10.88–12.59	$(7.33 \pm 0.78) \times 10^{-1}$
Exo	G13	H10	H	10.61–11.41	$(2.23 \pm 0.48) \times 10^{-5}$
Exo	G13	H10	H+He	10.76–12.38	$(1.67 \pm 0.16) \times 10^{-2}$
Exo	D90	G13	H	9.39–10.97	$(5.46 \pm 1.74) \times 10^{-1}$
Exo	D90	G13	H+He	10.53–12.45	$(2.29 \pm 0.13) \times 10^{+1}$
Exo	D90	Alt	H	9.86–11.44	$(3.04 \pm 0.42) \times 10^{-1}$
Exo	D90	Alt	H+He	10.90–12.31	$(4.46 \pm 0.22) \times 10^{+1}$
Exo	D90	H10	H	9.60–11.38	$(2.27 \pm 0.50) \times 10^{-1}$
Exo	D90	H10	H+He	10.61–12.28	$(2.59 \pm 0.15) \times 10^{+1}$
Exo	H10	G13	H	9.87–11.49	$(5.15 \pm 0.51) \times 10^{+0}$
Exo	H10	G13	H+He	10.60–11.99	$(4.67 \pm 0.46) \times 10^{+1}$
Exo	H10	Alt	H	10.45–11.74	$(5.17 \pm 0.64) \times 10^{+1}$
Exo	H10	Alt	H+He	10.53–11.81	$(7.49 \pm 0.75) \times 10^{+1}$
Exo	H10	H10	H	10.42–11.72	$(2.83 \pm 0.21) \times 10^{+1}$
Exo	H10	H10	H+He	10.74–12.39	$(8.93 \pm 0.47) \times 10^{+1}$

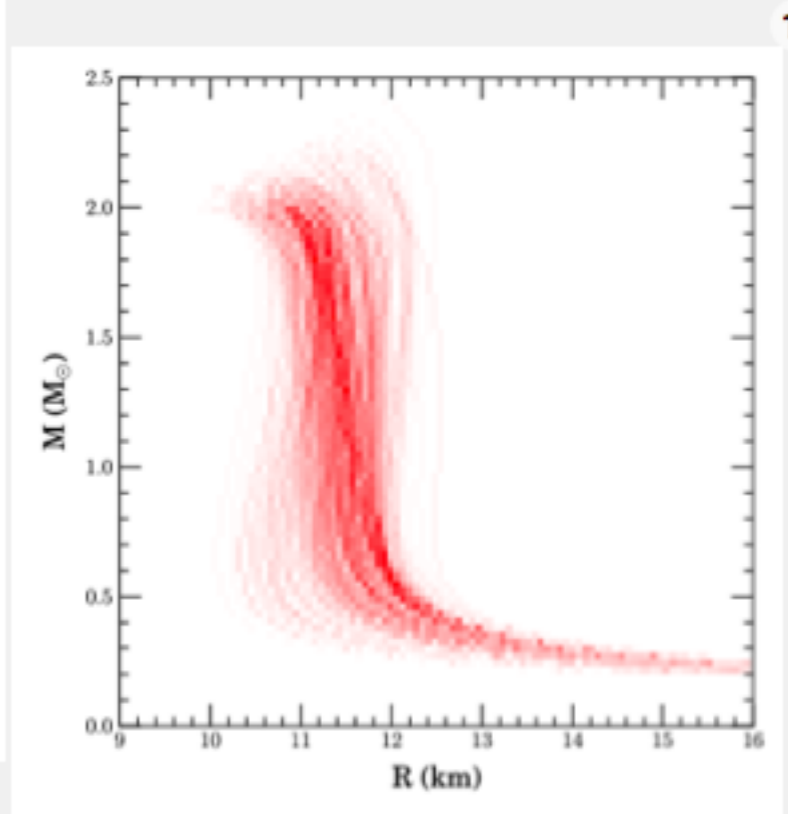
## Varying Models and Data Assumptions

- In this case, 36 separate combinations, each with  $10^5 - 10^6$  individual EOSs
- Not all systematic uncertainties are included

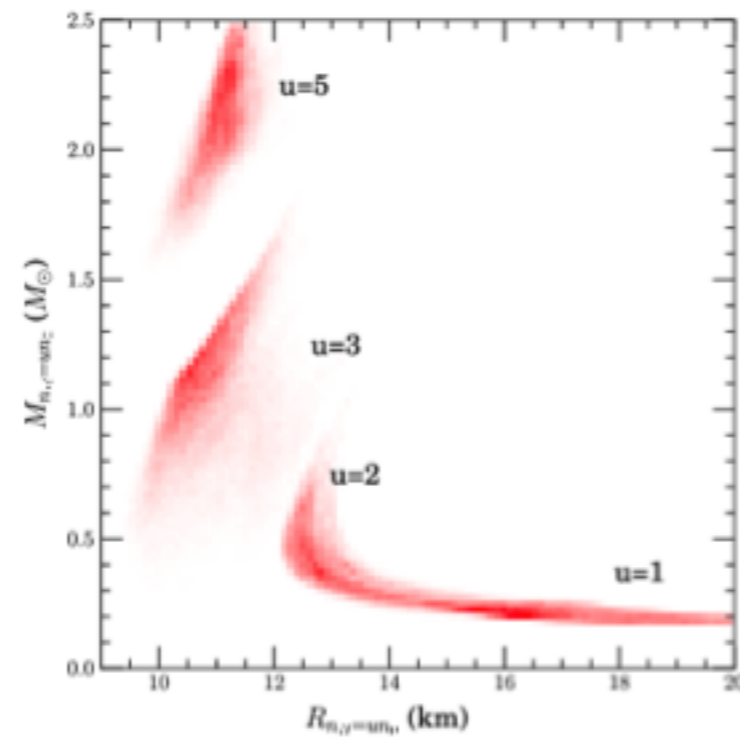
## A More Detailed Look



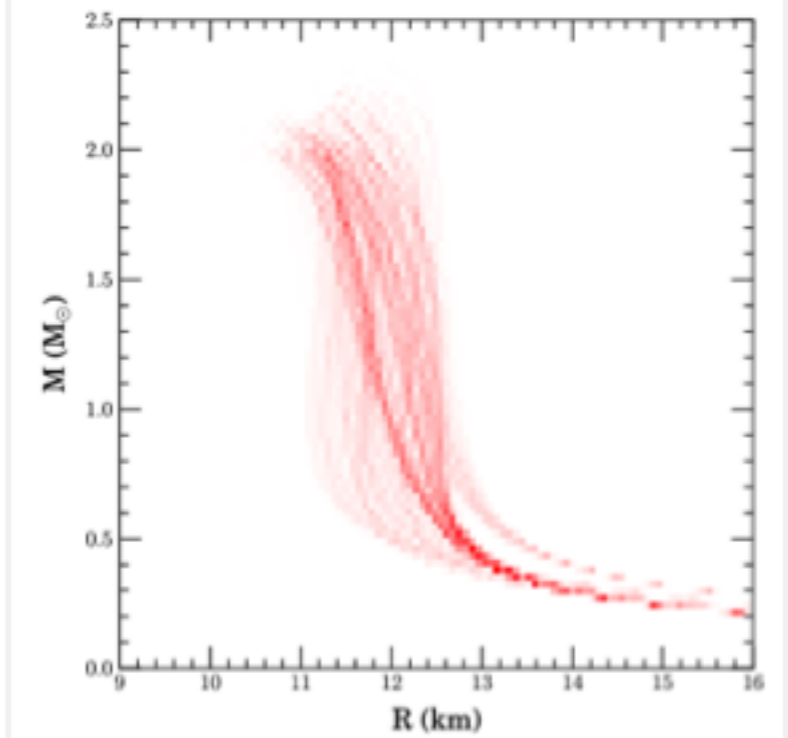
Model A from Steiner, Gandolfi, Fattoyev and Newton (2014)



Model A  
 $L = 32 \pm 0.6$  MeV



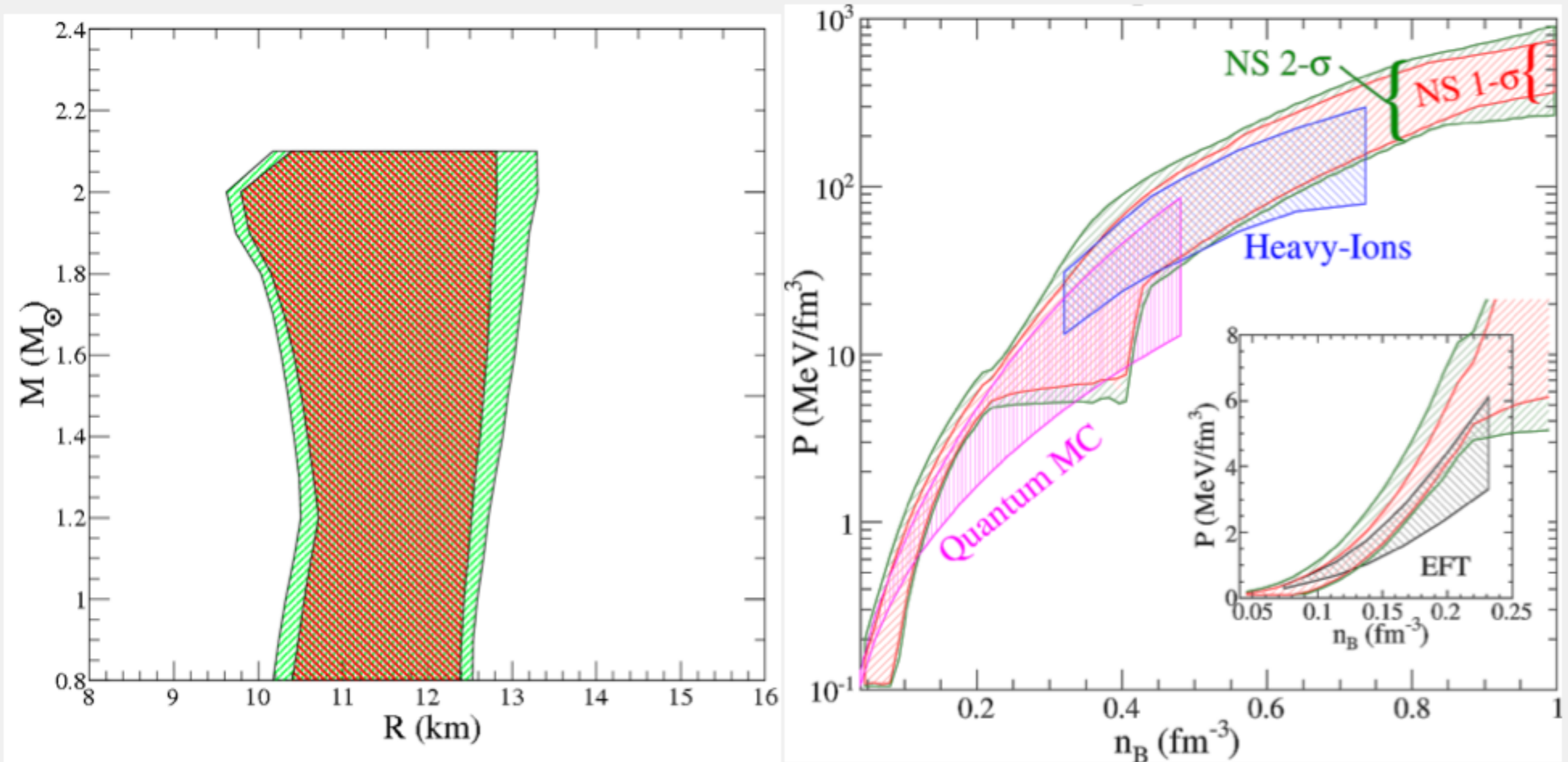
Model C from Steiner, Gandolfi, Fattoyev and Newton (2014)



Model A  
 $L = 62 \pm 1$  MeV

# The M-R curve and the EOS of Dense Matter

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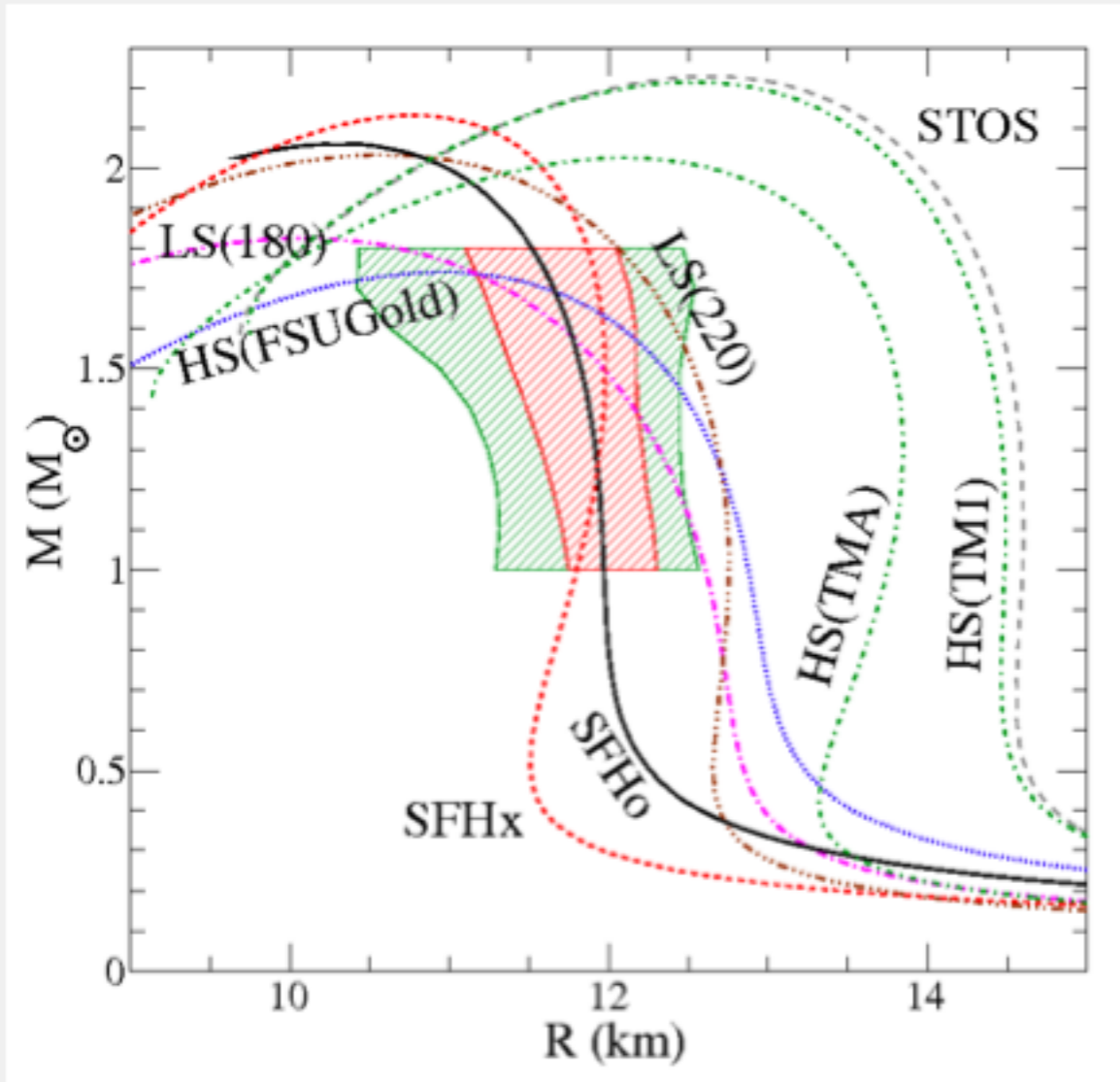
Steiner, Lattimer, and Brown (2013); red and green outlines 68% and 95% regions

- Full Bayesian MCMC sampling of the likelihood (times prior)
- Radius of a 1.4 solar mass neutron star is 10.4 - 12.9 km
- Note the uncertainty in the EOS at a few times saturation
- No assumption that pressure is correlated between low and high-densities
- Some bursts/models give larger radii, inconsistent with QLMXB results [Suleimanov et al.](#)

# Self-Consistency?!

- Take e.g. a model of r-process from NS-NS mergers
- The nucleon-nucleon interaction's impact is extensive
  - The original supernova
  - Neutron star structure
  - Mass ejected in the merger
  - The nuclear masses, beta-decay, and neutron capture rates
  - The neutrino spectra
- More self-consistency can lead to advances
- Until then, we have to be careful about how we choose our models

# EOS Tables for Simulations



Steiner, Hempel, and Fischer (2013)

- Limited number of supernova EOSs which satisfy  $M - R$  constraints and nuclear structure constraints
- Smaller radii may increase amount of ejecta
- Maximum mass is also an important parameter  
Does merger nucleosynthesis constrain the maximum mass?

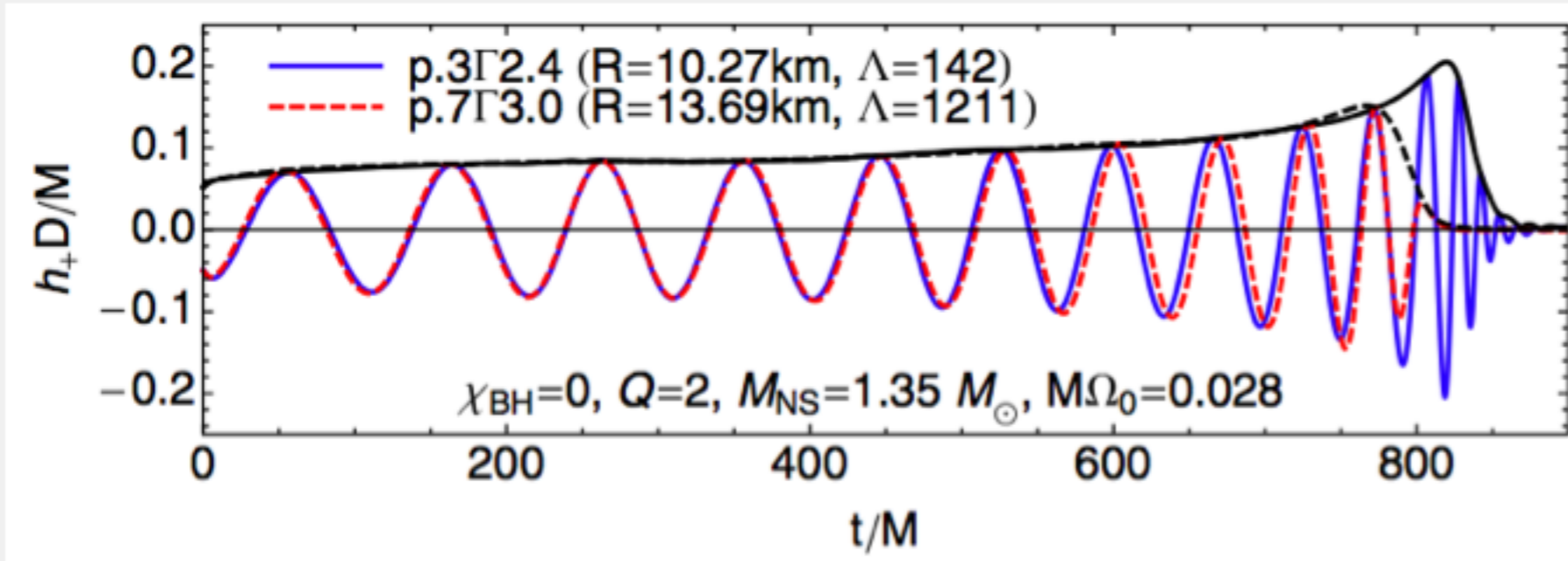
$$\mathcal{L} = \bar{\Psi} \left[ i\bar{\partial} - g_{\omega}\phi - \frac{1}{2}g_{\rho}\vec{\rho} \cdot \vec{\tau} - M + g_{\sigma}\sigma - \frac{1}{2}e(1 + \tau_3)A \right] \Psi + \frac{1}{2}(\partial_{\mu}\sigma)^2$$

$$- V(\sigma) - \frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \frac{1}{2}m_{\omega}^2\omega^{\mu}\omega_{\mu} - \frac{1}{4}\vec{B}_{\mu\nu} \cdot \vec{B}^{\mu\nu} + \frac{1}{2}m_{\rho}^2\vec{\rho}^{\mu} \cdot \vec{\rho}_{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

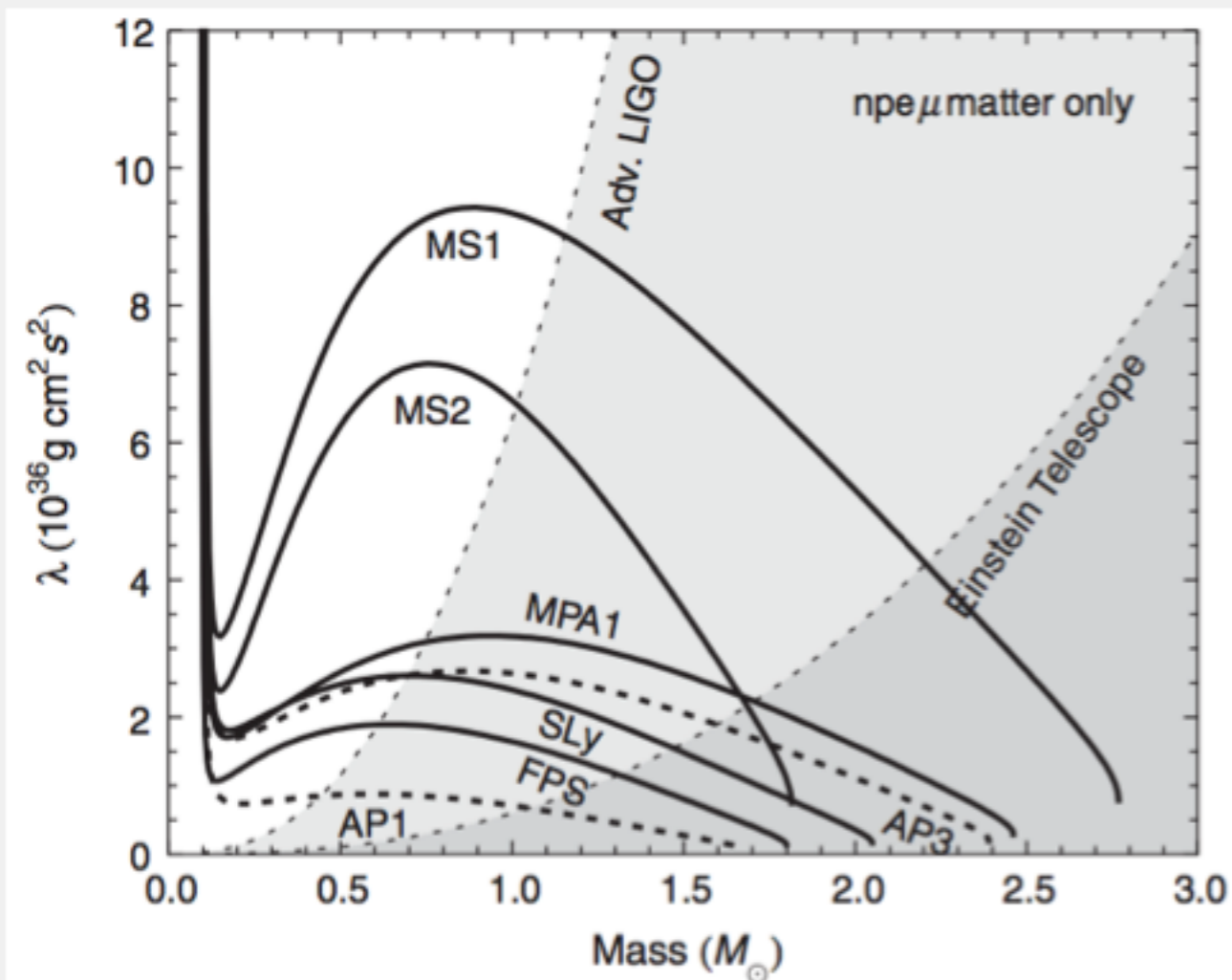
$$+ \frac{\zeta}{24}g_{\omega}^4(\omega^{\mu}\omega_{\mu})^2 + \frac{\xi}{24}g_{\rho}^4(\vec{\rho}^{\mu} \cdot \vec{\rho}_{\mu})^2 + g_{\rho}^2 f(\sigma, \omega_{\mu}\omega^{\mu})\vec{\rho}^{\mu} \cdot \vec{\rho}_{\mu},$$

- Binding energies and charge radii of  $^{208}\text{Pb}$  and  $^{90}\text{Zr}$  within 2% of experiment
- Fix  $K$  to  $240 \pm 20$  MeV
- Match  $M$ - $R$  curves from SLB '2010

# Neutron Star Tidal Deformabilities



Lackey et al. (2014)



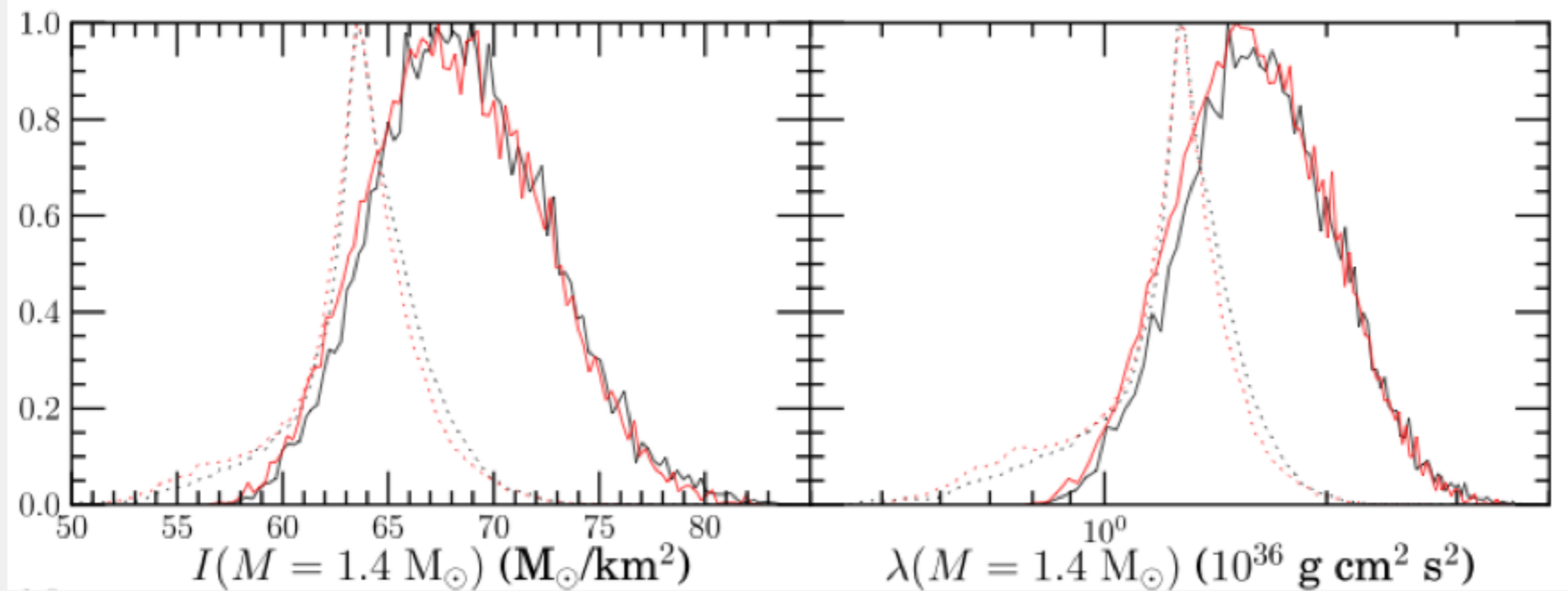
Hinderer, Lackey, Lang, and Read (2010)

- Gravitational wave signal from an NS merger measures tidal deformability  $\lambda$
- Point masses early on; deformation near 400 Hz
- Easier to detect larger tidal deformations



# Tidal Deformabilities

- Current neutron star mass and radius observations suggest tidal deformabilities are small

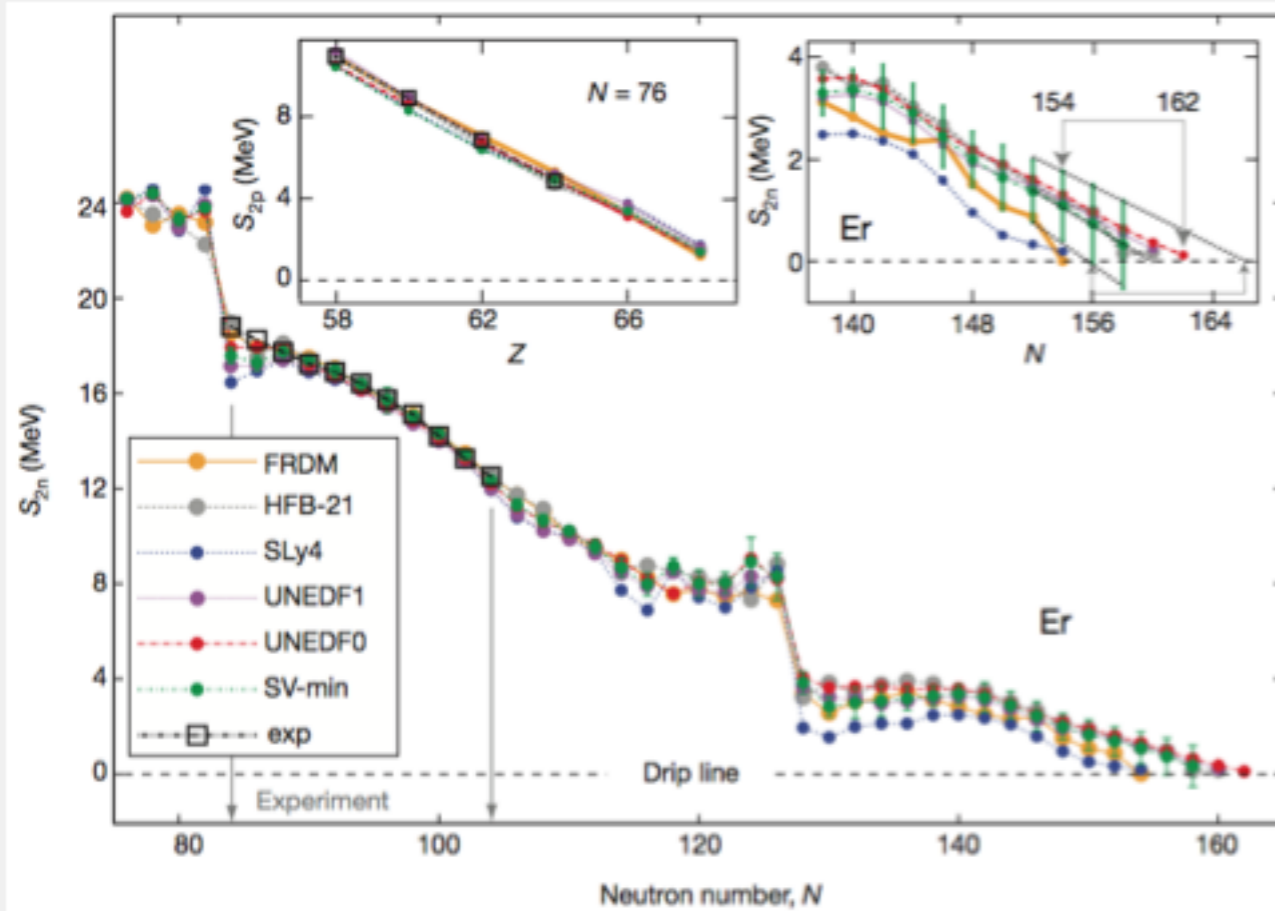


Steiner, Gandfi, Fattoyev, and Newton (2014)

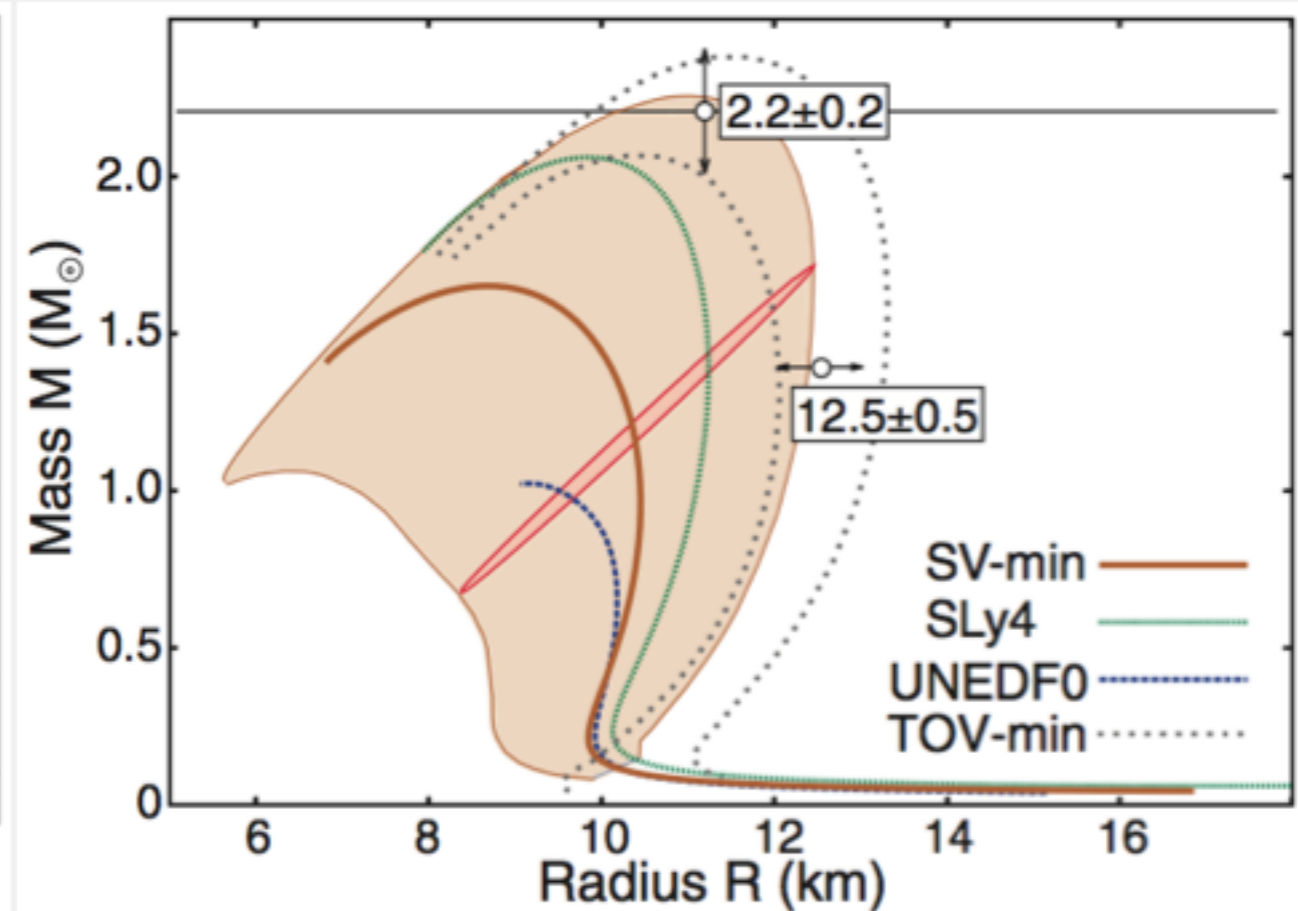
- LIGO sensitivity improved by
  - multiple measurements  
[del Pozzo, et al \(2013\)](#)
  - and by directly simulating the EOS  
[Lackey et al. \(2014\)](#)
  - Using information beyond 400 Hz  
[Hinderer, Lackey, Lang, and Read \(2010\)](#); [Damour, Nagar, and Villain \(2012\)](#)
- For large enough signal-to-noise, LIGO may have smaller systematics

# Summary

- Currently available neutron star mass and radius observations constrain the universal neutron star  $M - R$  curve
  - Neutron star radii are likely between 10.4 and 12.9 km  
Ranges from 9-15 km are possible, but improbable
  - We now have constraints on the EOS
  - $1 < \lambda < 3 \times 10^{36} \text{ g cm}^2 \text{ s}^2$
- Constrain the nucleon-nucleon interaction and QCD.
  - (41)  $43 \text{ MeV} < L < 67$  (83) MeV
- Published 1000s of EOS parameterizations of cold EOSs
- Need more EOS tables and better nuclear physics input, but where should improvements be focused?
- There is a bright future ahead: LIGO, Chandra, NICER, Athena+, LOFT, and more



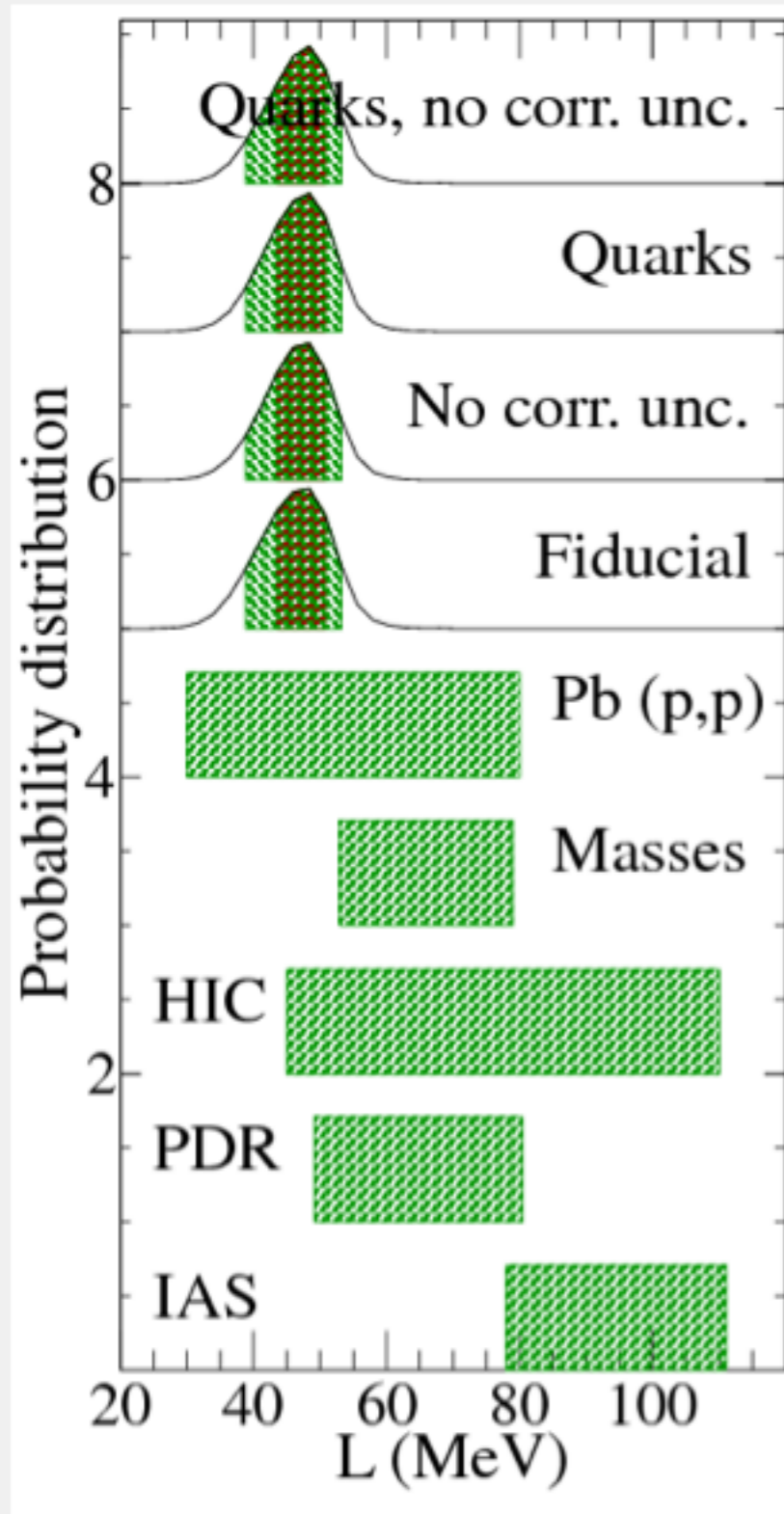
Erler et al. (2012)



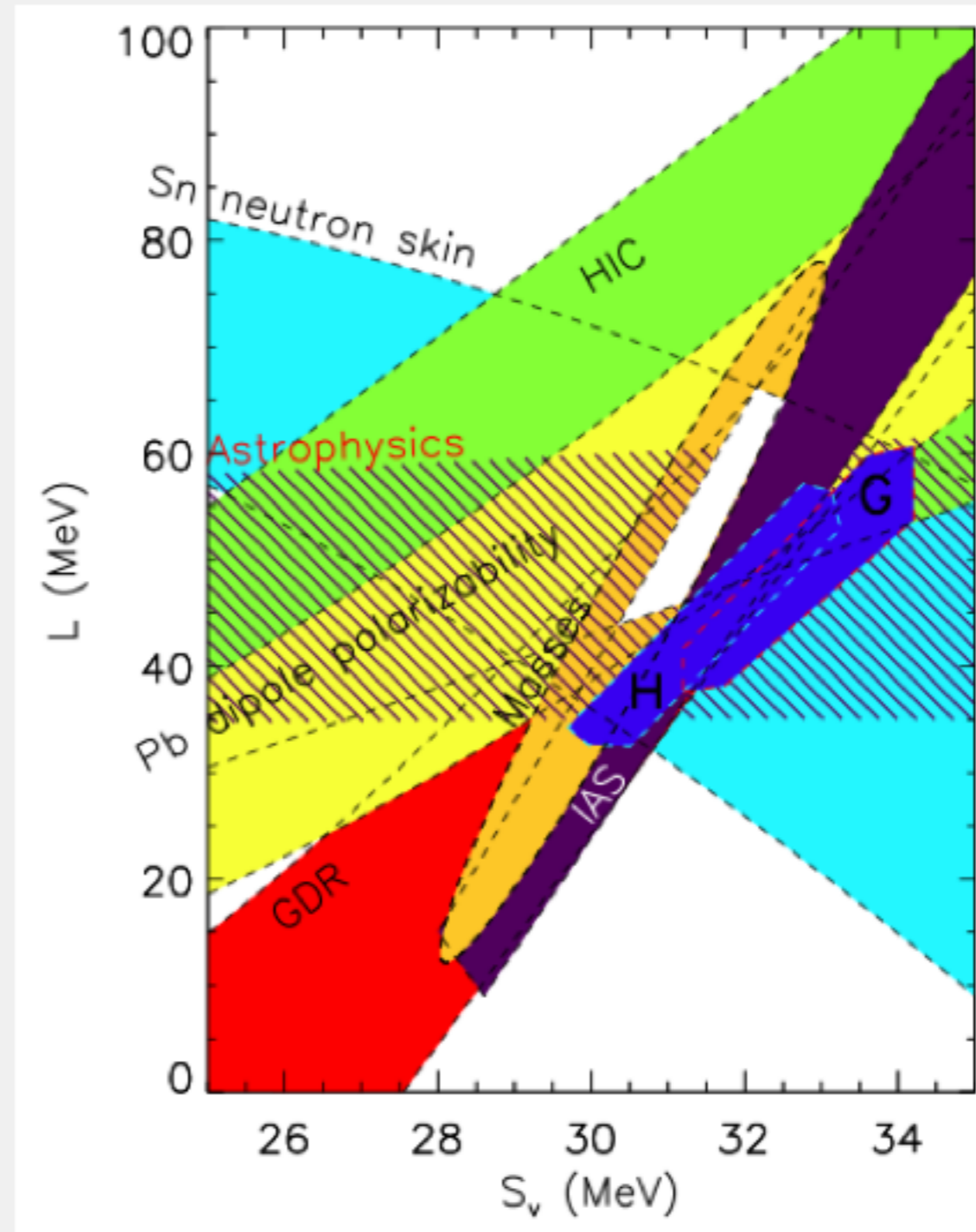
Erler et al. (2013)

- It might be possible to constrain the neutron drip line from neutron radii
- But the connection between these observables has not been fully explored
- In particular:
  - The isovector part of the Skyrme functional is incomplete  
[Sheikh et al. \(2014\)](#)
  - Current work assumes low- and high-density matter are correlated

# Neutron Star Constraints on L



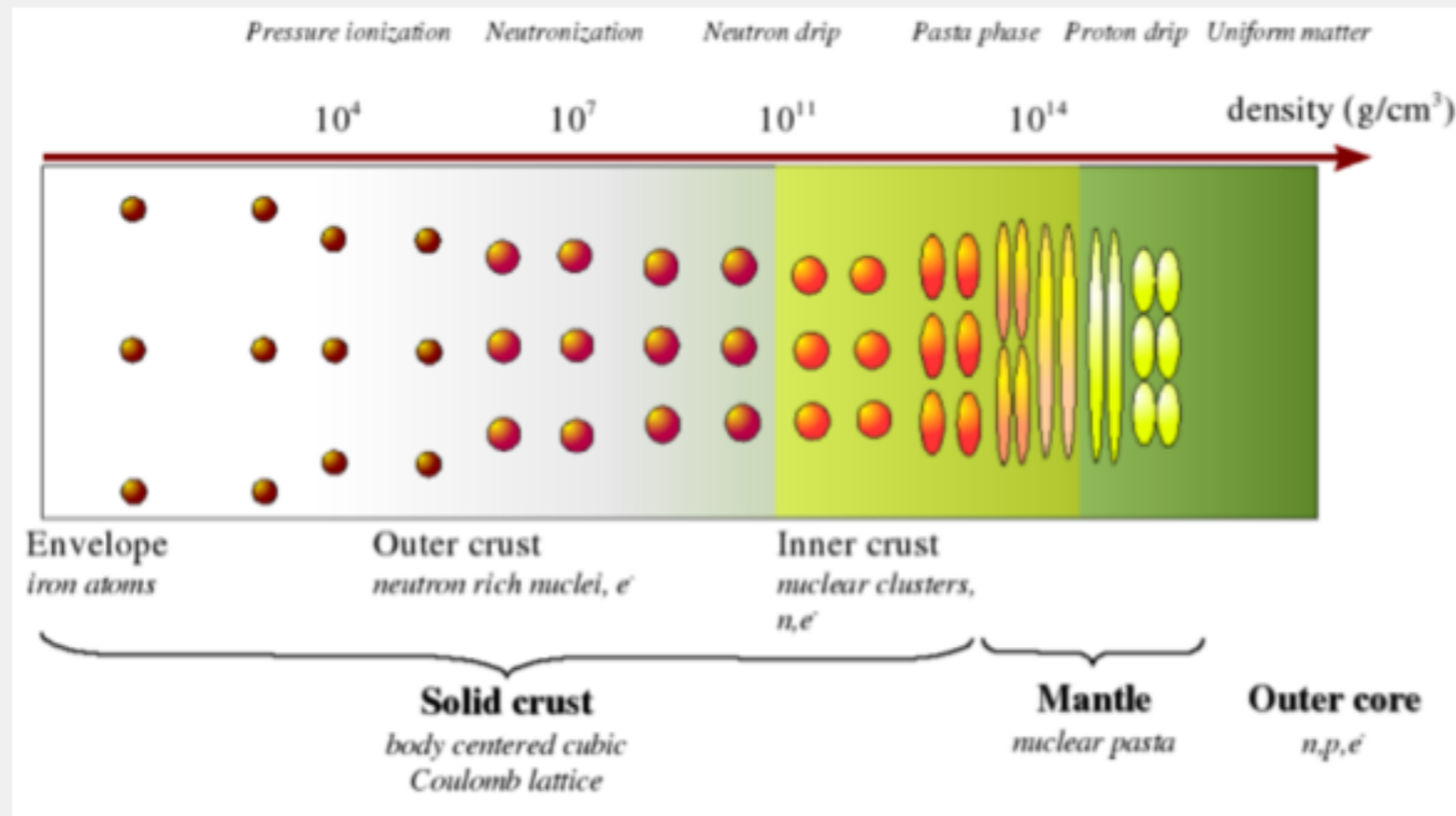
Steiner and Gandolfi (2012)  
(IAS results have since come down)



Lattimer and Steiner (2014)

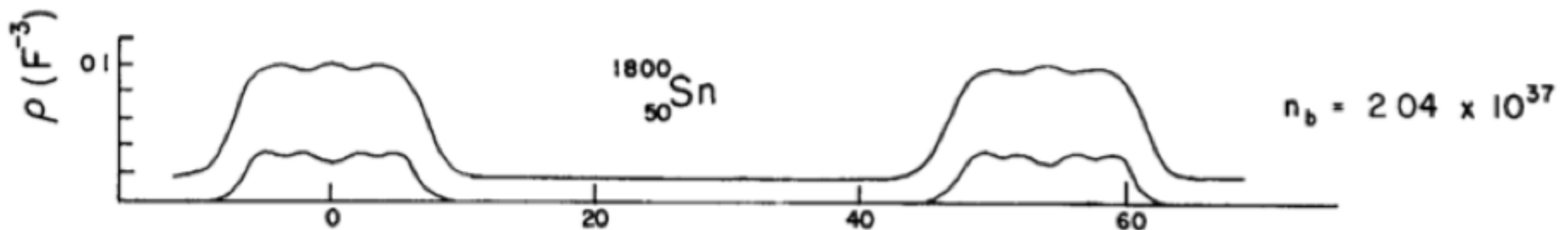
- Neutron stars strongly constrain  $L$
- We also found  $R_n - R_p < 0.2$  fm  
Confirmed by MAMI data

# Structure of Matter in the Neutron Star Crust



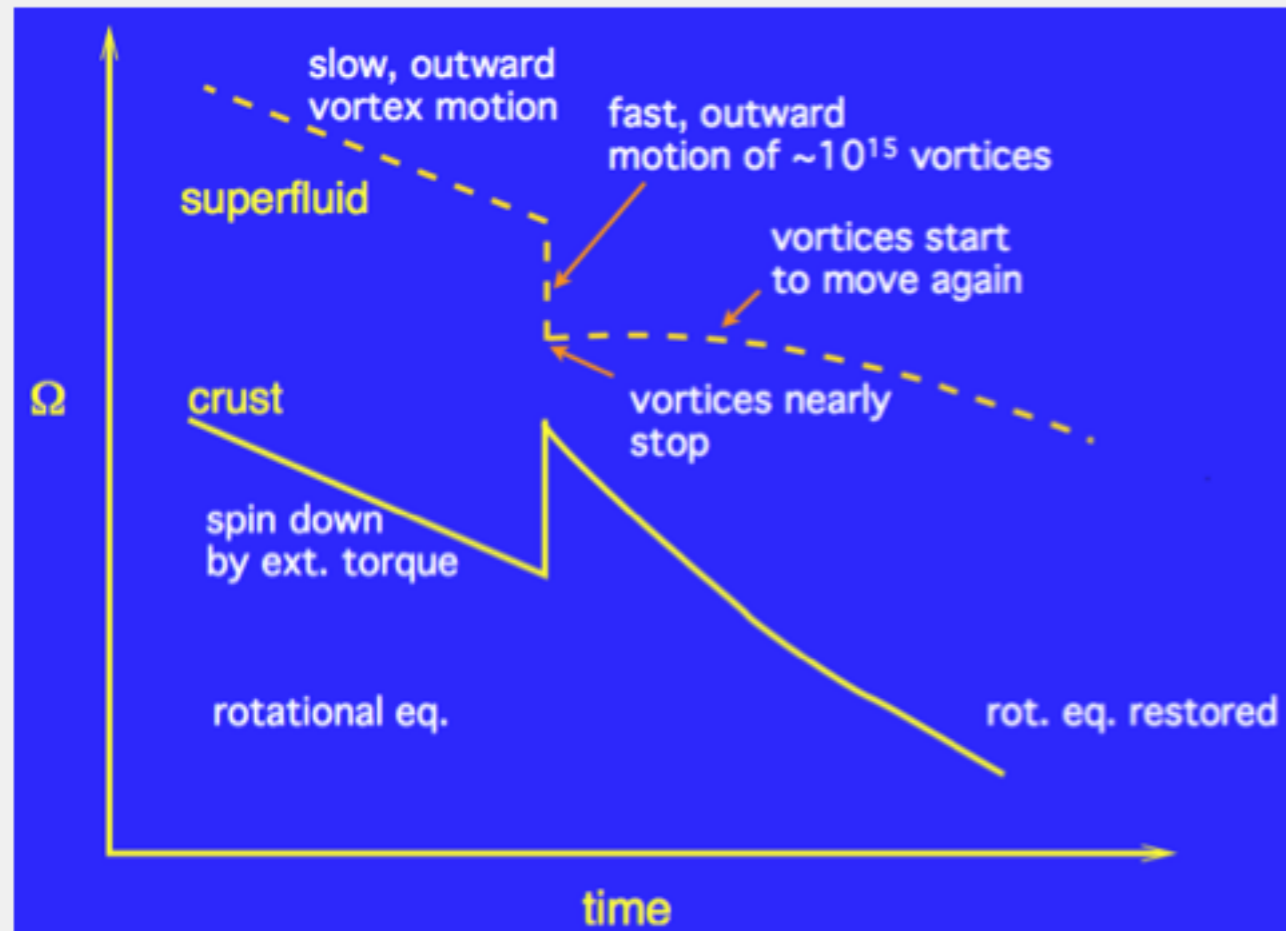
Picture from N. Chamel

- Neutron-rich nuclei
- Sea of superfluid neutrons
- Crust-core transition



Negele and Vautherin (1973!)

# Pulsar Glitch Mechanism



Picture from B. Link

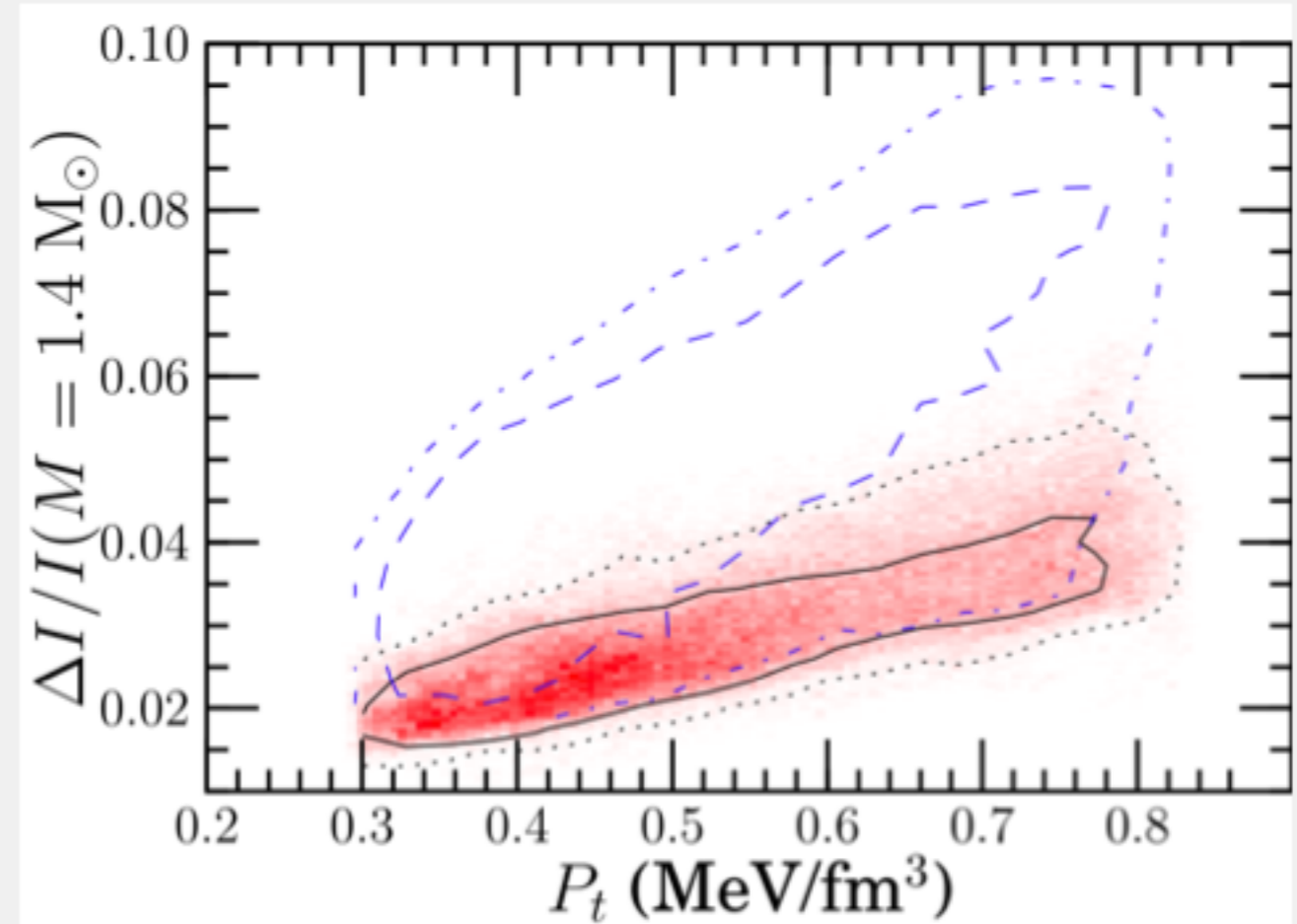
- Superfluid component, decoupled from rotation at the surface
- Natural to associate the superfluid component with the superfluid neutrons in the crust
- What is the mechanism for the sudden change?

- Superfluid vortices pinned to the lattice
- Neutron star spins down, vortices bend creating tension, eventually they must shift lattice sites
- Quasi-free neutrons are entrained with the lattice

Chamel 2012, Chamel et al. 2013

# Is There Enough Superfluid in the Crust?

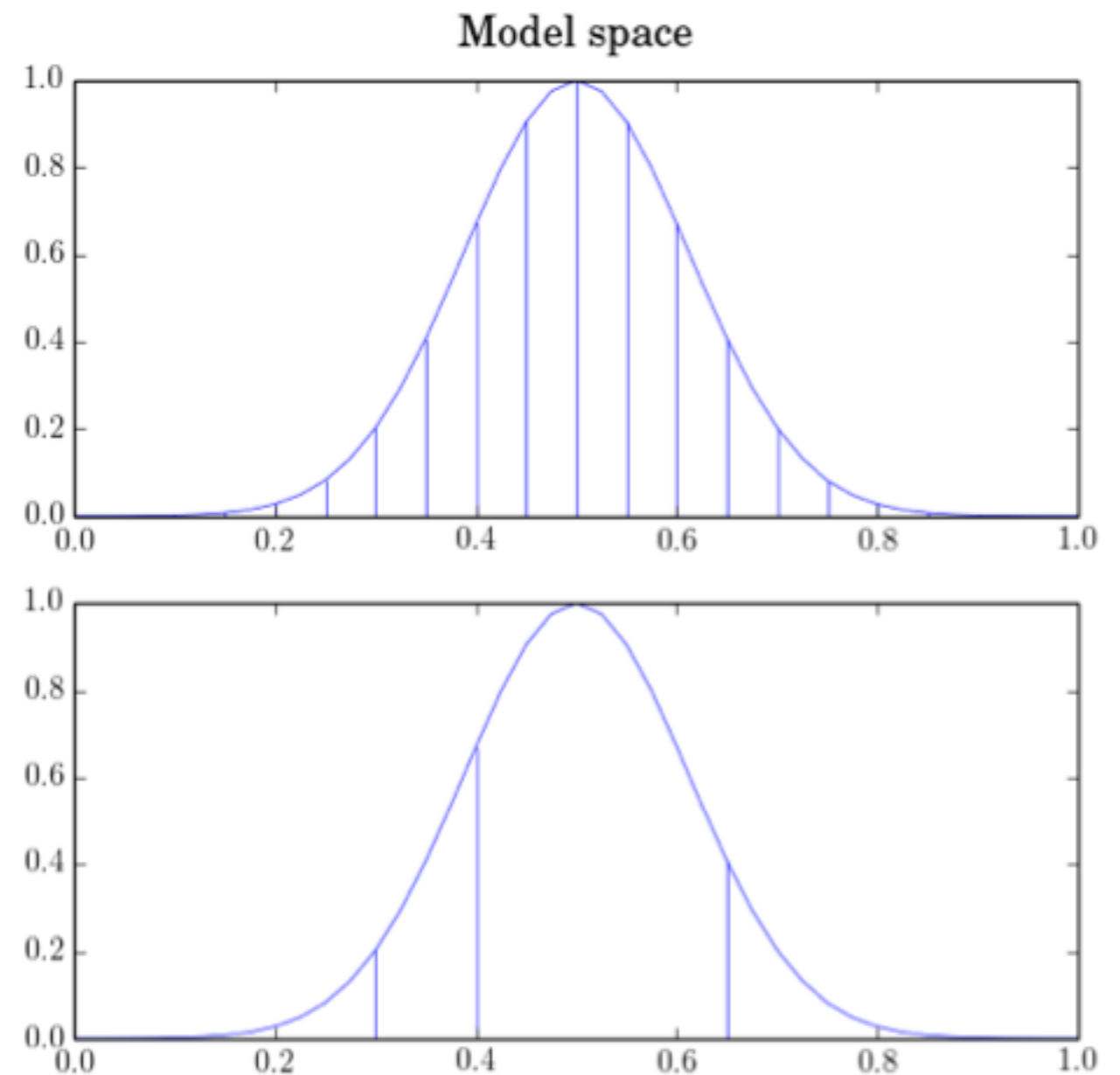
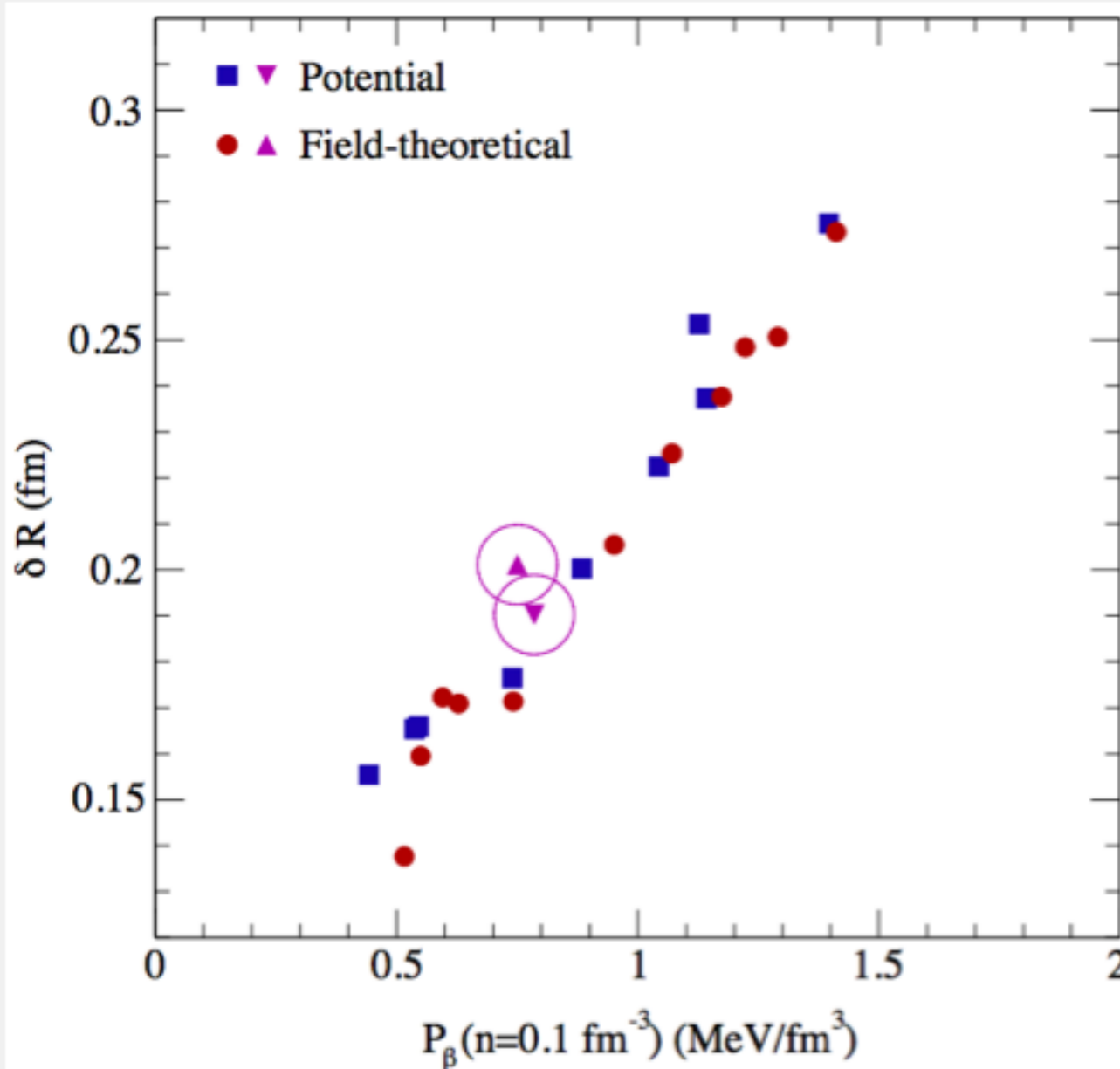
- We require 1.6% of  $I$  to explain glitches in Vela  
[Link, Epstein, and Lattimer \(1999\)](#)
- Entrainment: 75-85% of otherwise superfluid neutrons 'connected' to the lattice  
[N. Chamel \(2012\)](#)
- Current  $M$  and  $R$  observations suggest there is not enough  $I$  in the crust  
[See Andersson et al. \(2012\)](#)
- Unless the systematics force much larger neutron star radii and  $P_t$  is large



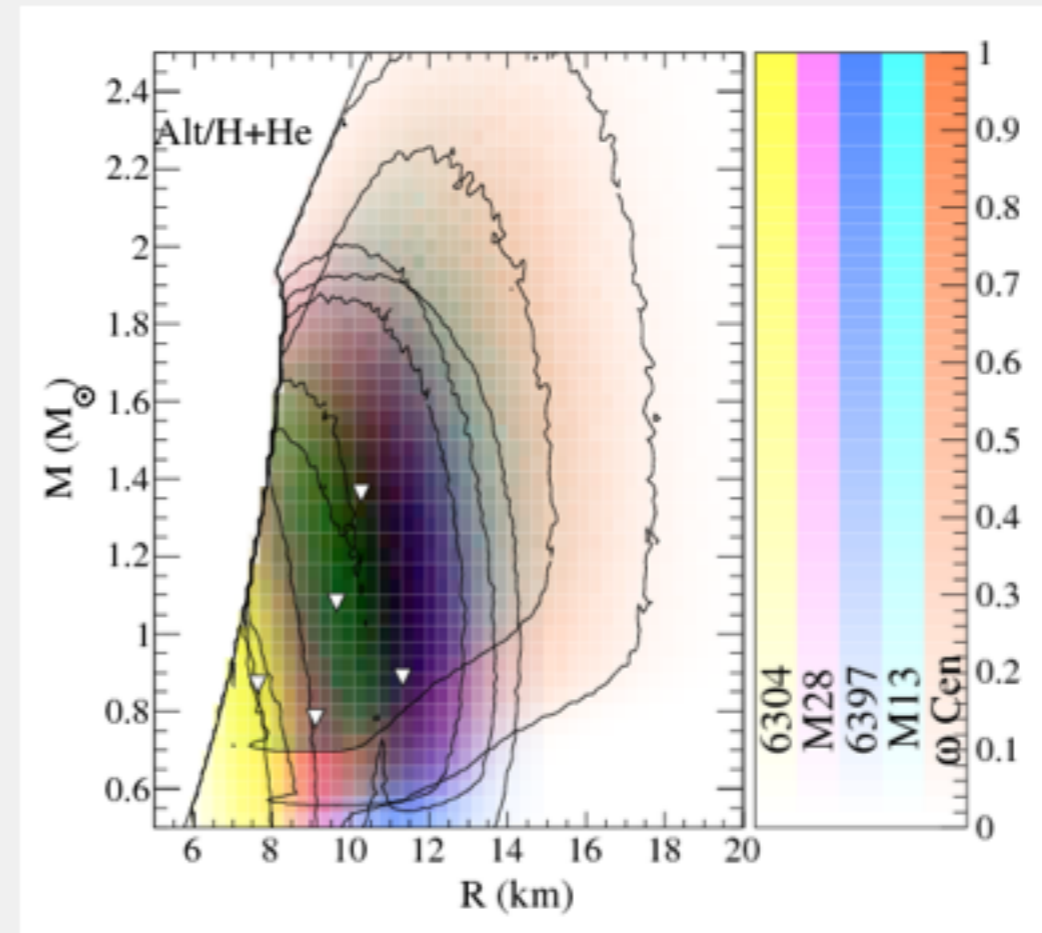
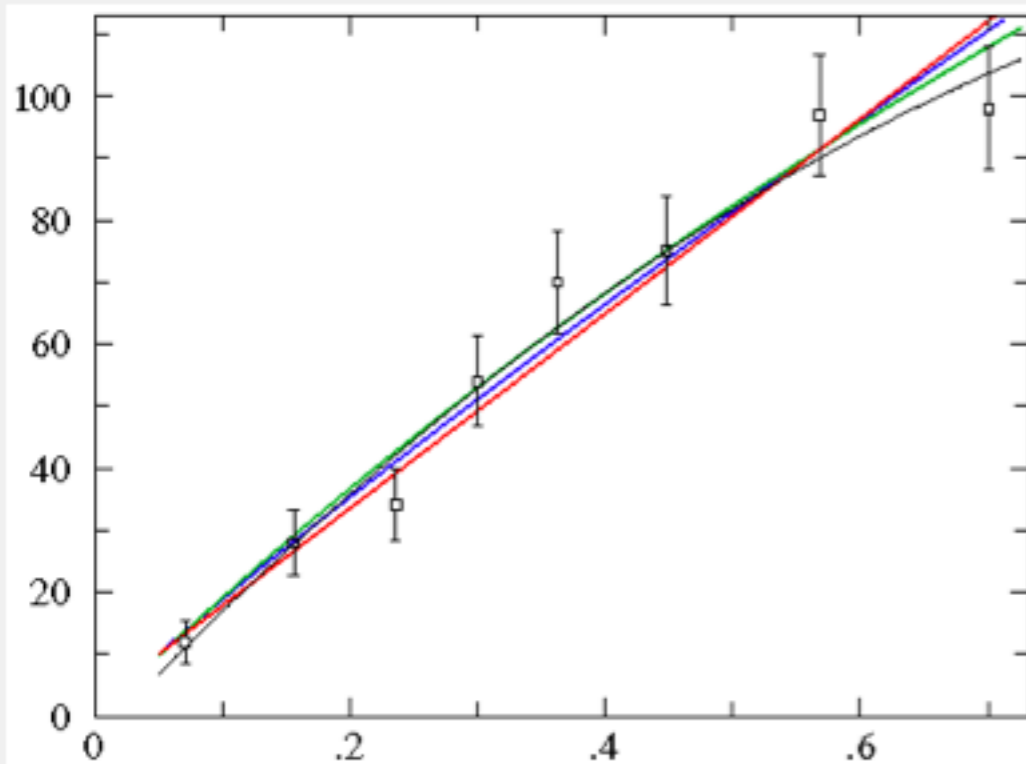
Steiner et al. (2014); black and red are with  $M$  &  $R$  observations, blue contours are with  $I = 70 M_{\odot} \text{ km}^2$

# Purpose of a Model

- Model comparison  
*Model A vs. Model B?*
- Parameter estimation  
*However, parameters don't always have a clear physical meaning*
- Predictions  
*Produce a probability distribution*
  - Distribution over parameter space and also over model space

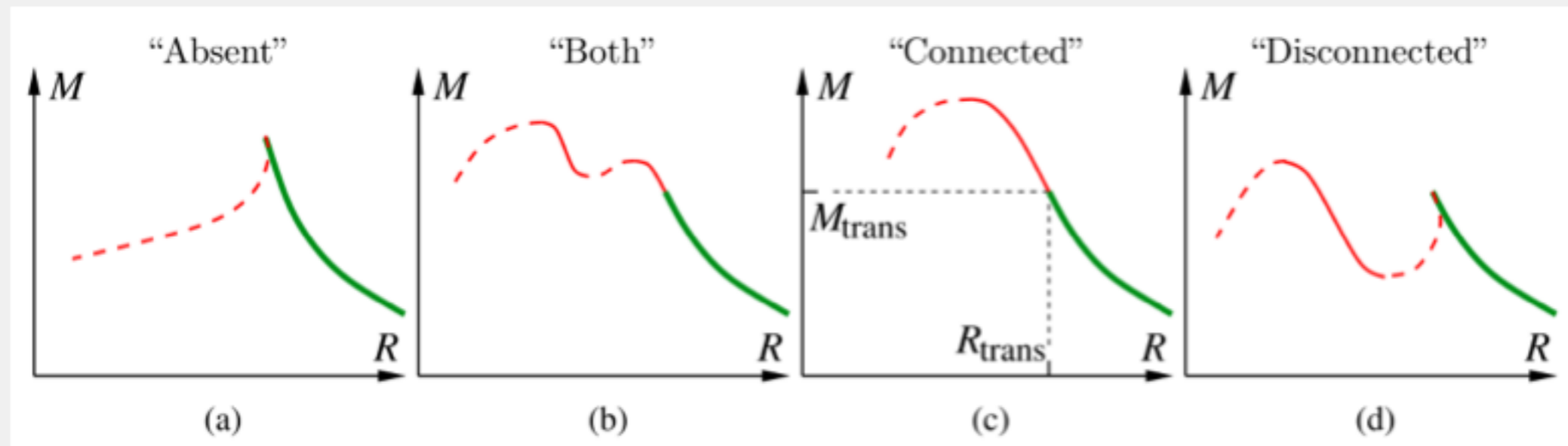




$\chi^2$  fits

Lattimer and Steiner (2014)

- Straightforward when there's no uncertainty in the horizontal axis
- When there is: e.g. "Deming regression"
- M-R curve is not a function (central pressure is a better "independent" variable)



Alford, Han, and Prakash (2014)

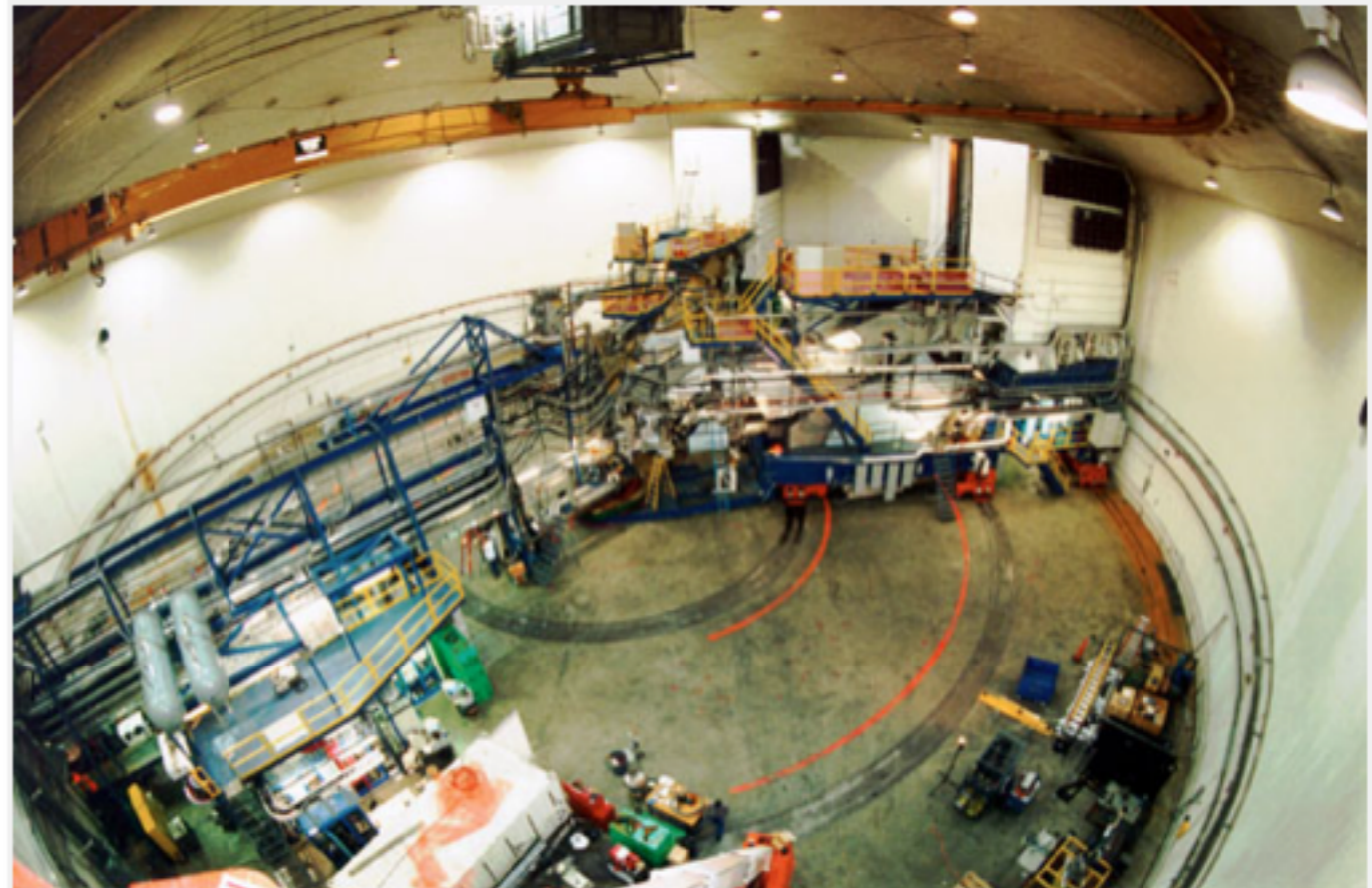
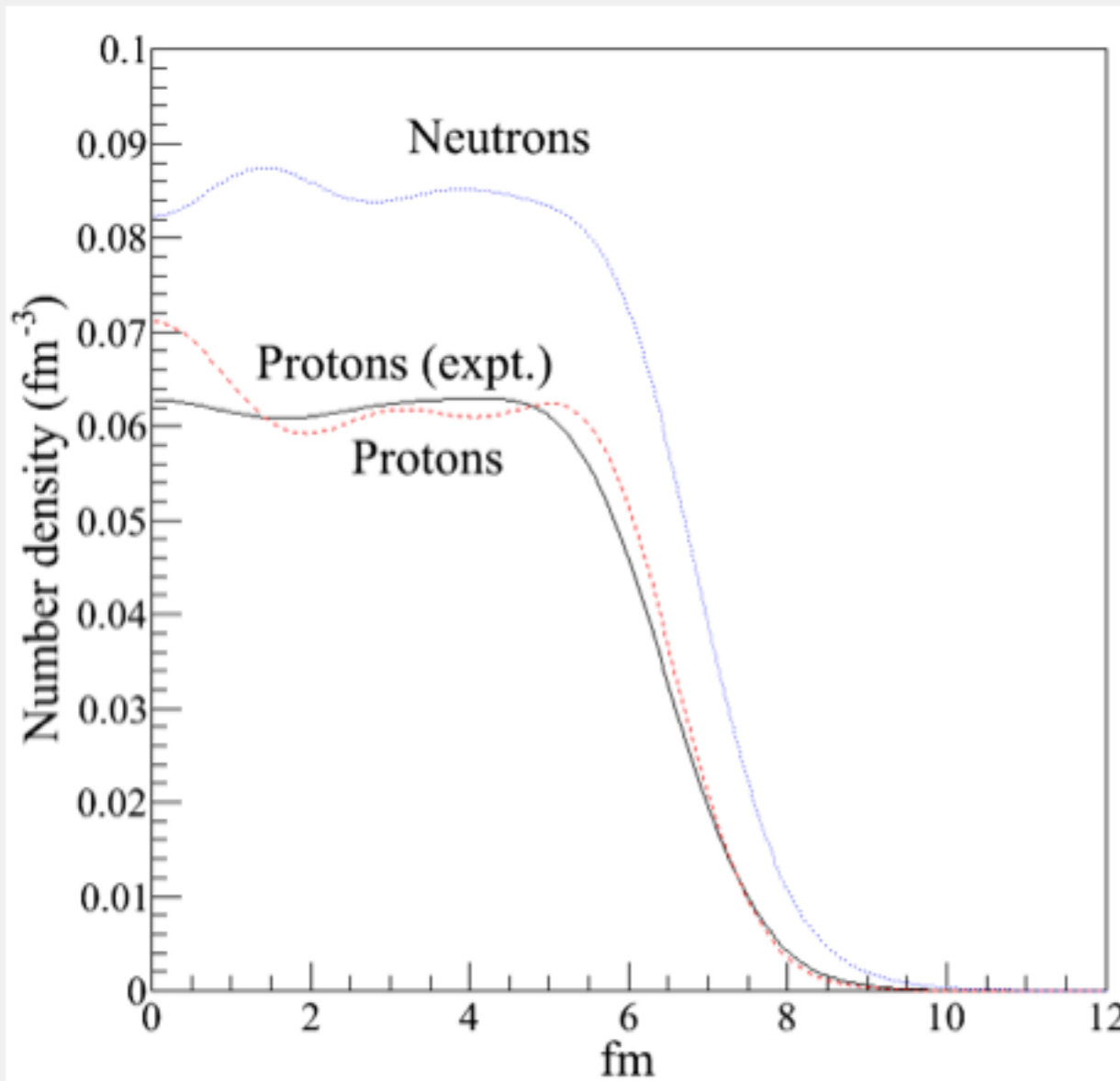
# The Neutron Skin Thickness of Lead

- Lead-208: 82 protons, 126 neutrons

$$R_n^2 \equiv \int r^2 n_n(r) d^3 r \quad R_p^2 \equiv \int r^2 n_p(r) d^3 r$$

- Neutron radii are hard to measure, use parity-violating electron scattering
- Weak charge of neutron  $\gg$  weak charge of proton, i.e.

$$|-1| \gg 1 - 4 \sin^2 \theta_W$$

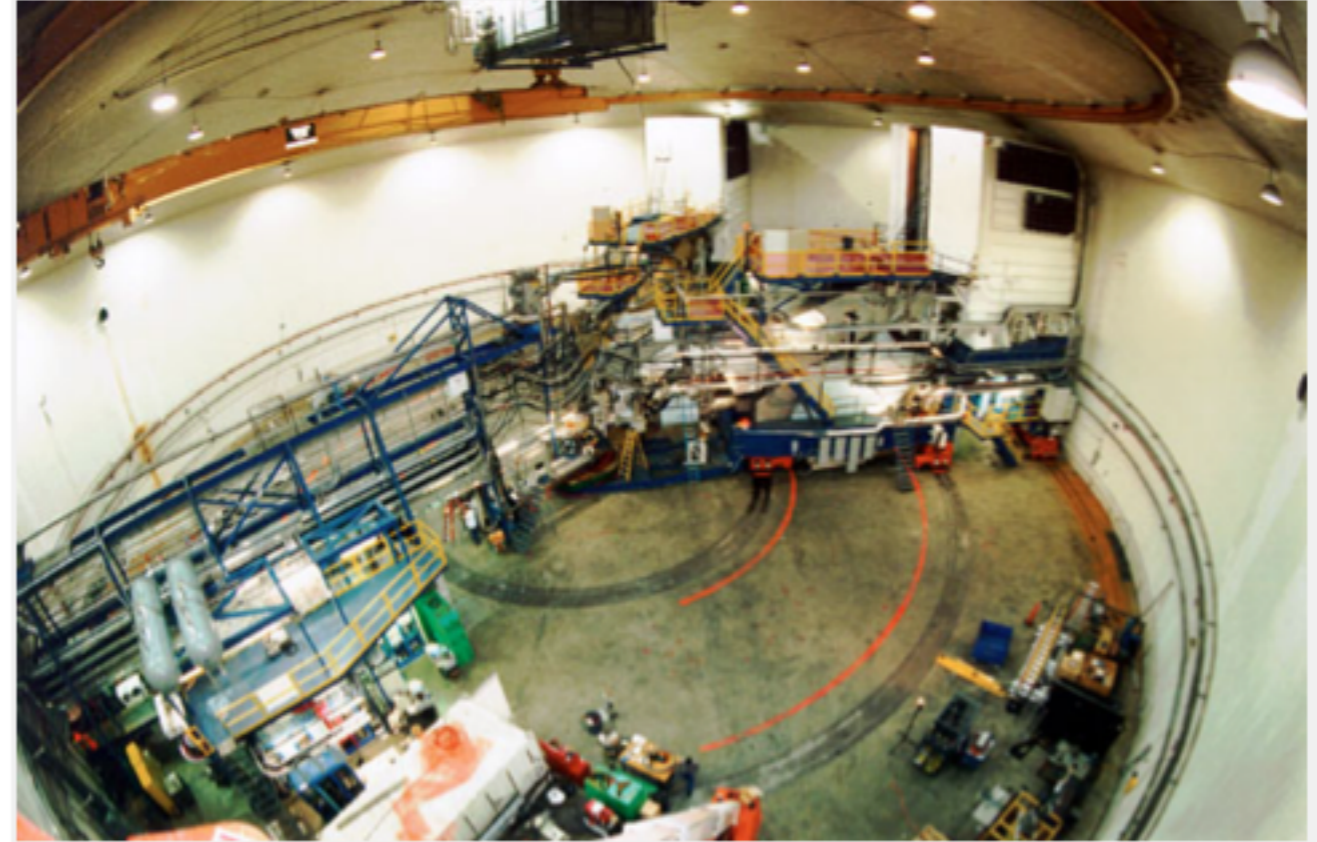
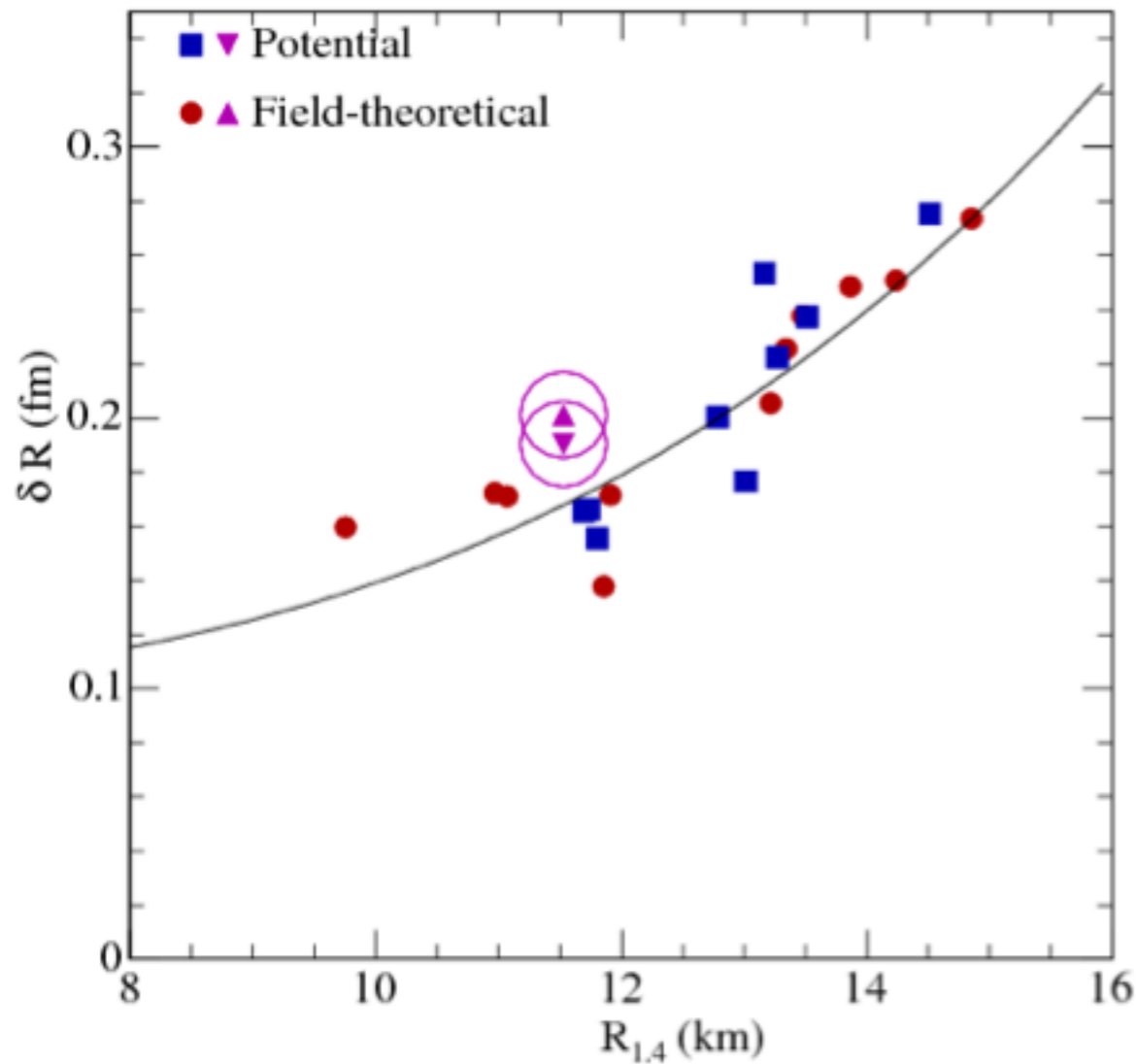


Jefferson Lab's Hall A

Measured  $R_n - R_p = 0.33 \pm 0.16$  fm

# The Neutron Skin Thickness of Lead

- The quantity  $\delta R \equiv R_n - R_p$  is related to  $L$  as are neutron star radii



Jefferson Lab's Hall A: Measuring  $R_n$

Steiner, Prakash, Lattimer, and Ellis (2005),  
based on Horowitz and Piekarewicz (2001)

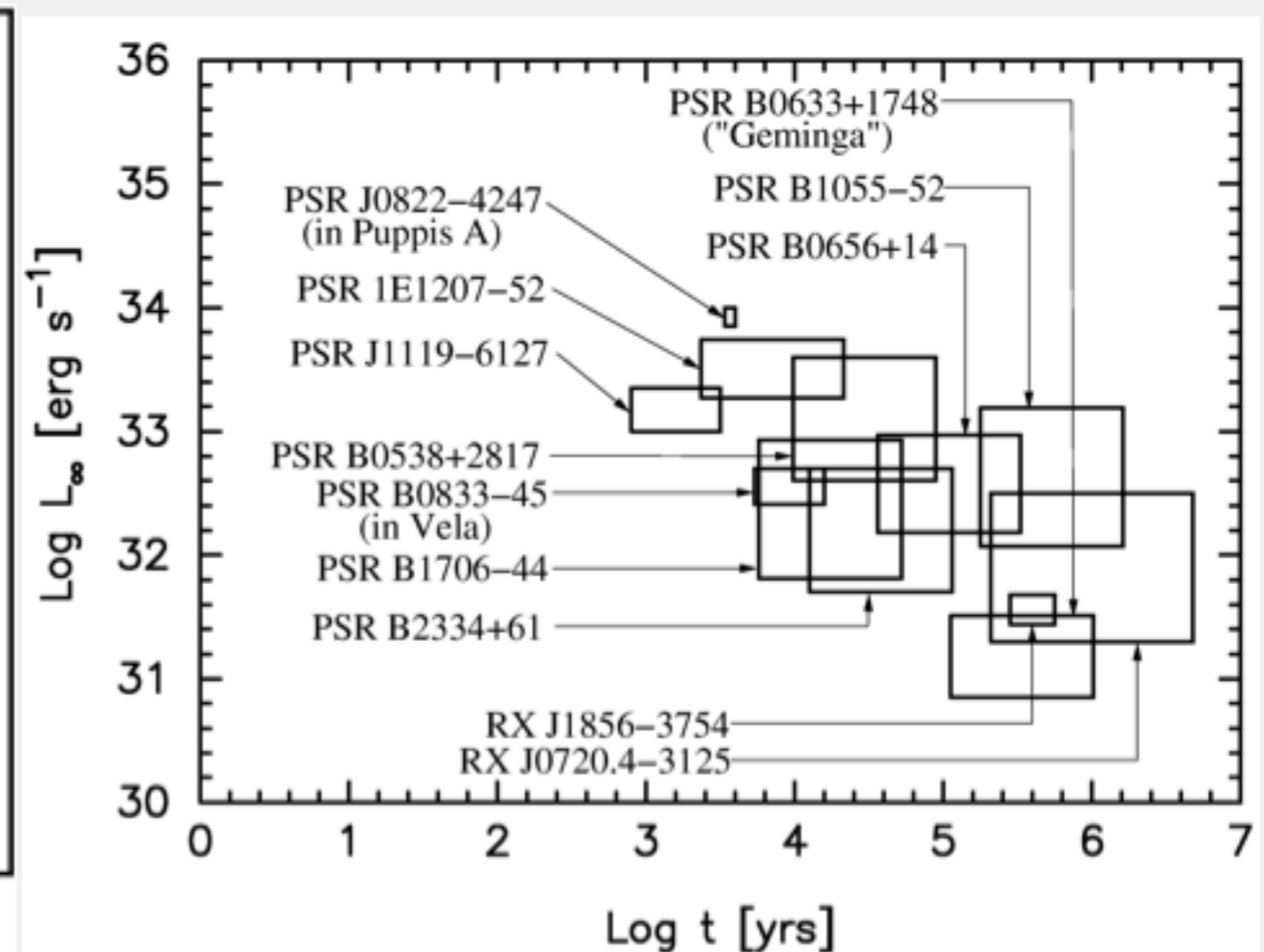
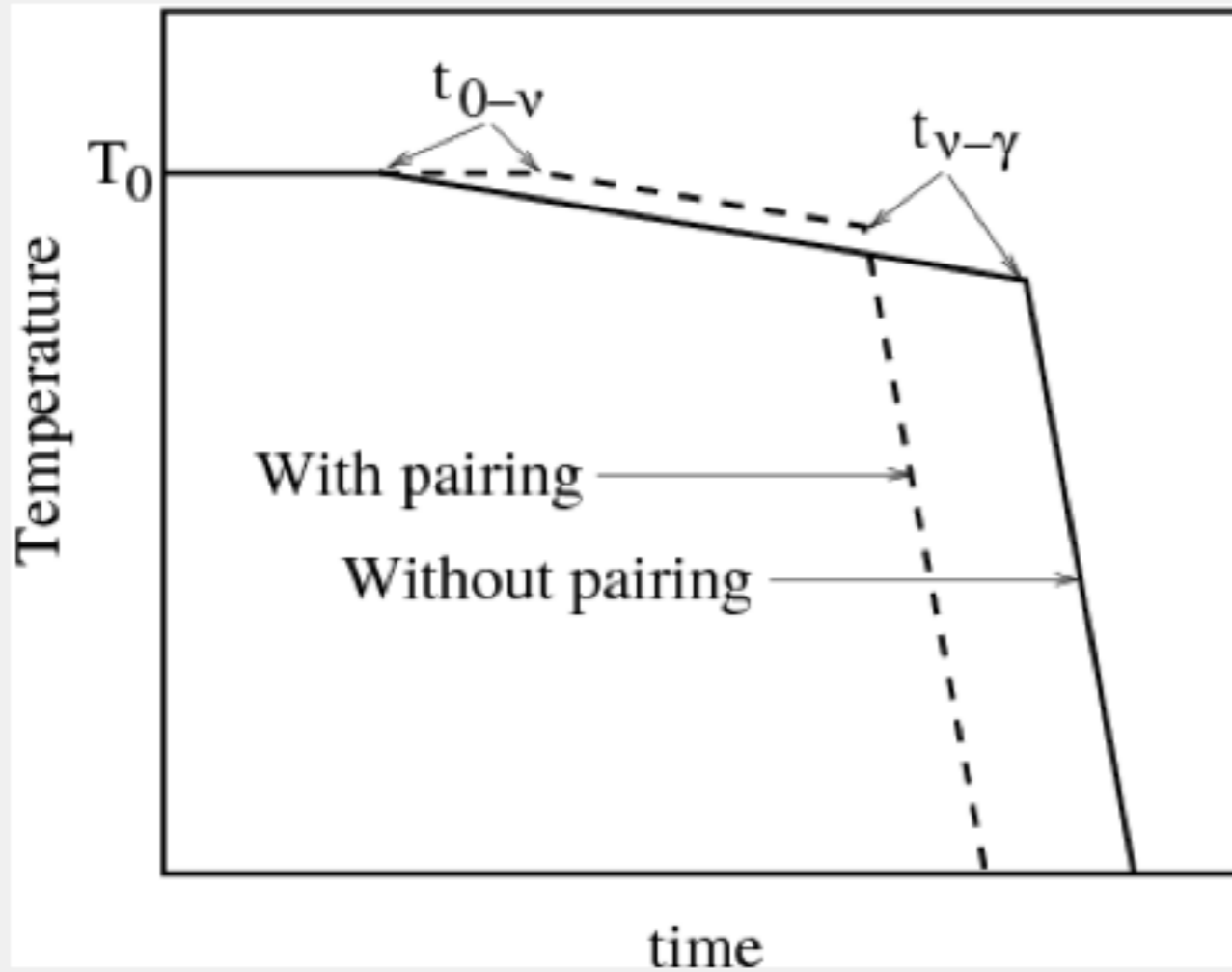
- We find  $\delta R < 0.2$  fm (68%) from neutron star observations

# Thermal Emission from Isolated Neutron Stars

- After  $\sim 10$  years, the star is isothermal  $\Rightarrow$  one temperature =  $T$

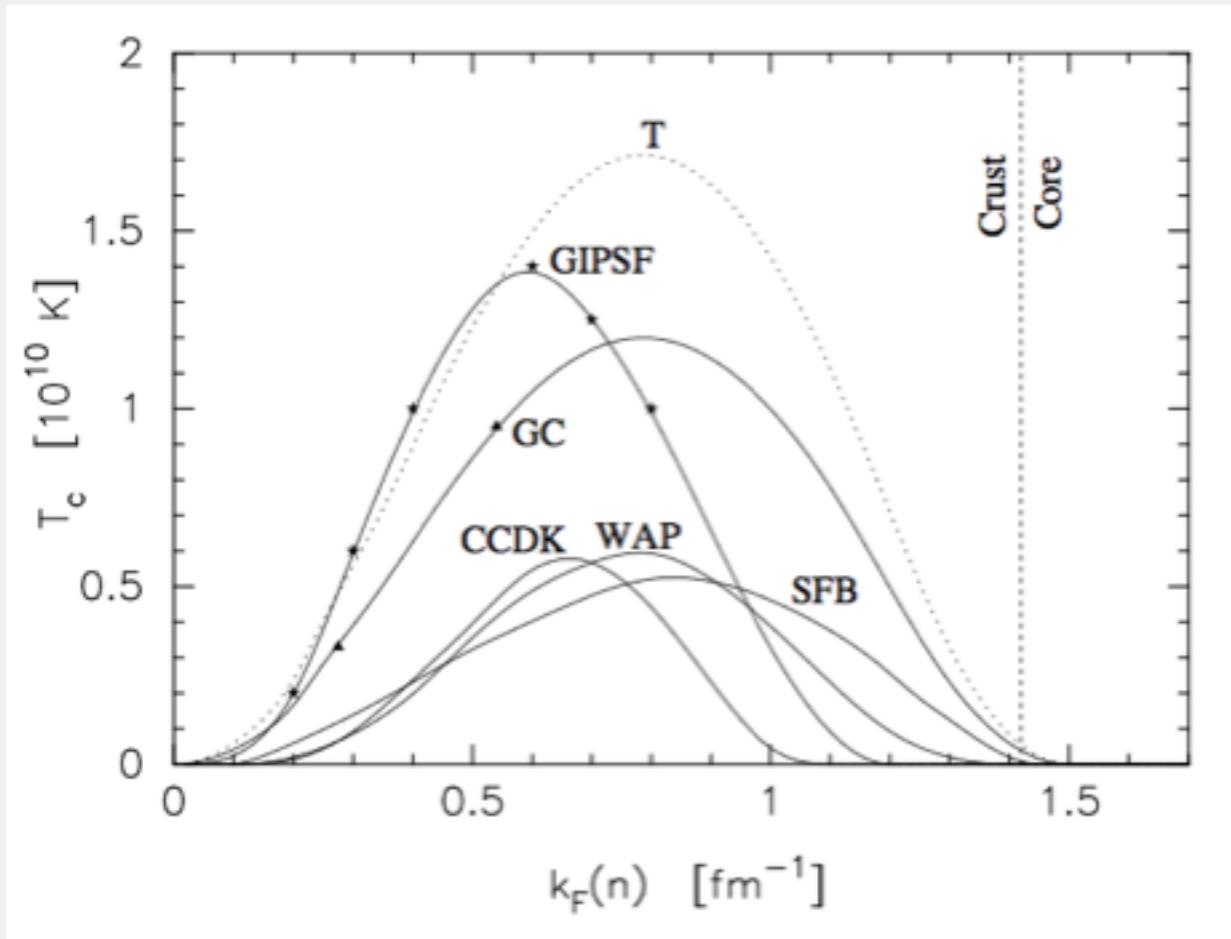
$$C_V \frac{dT}{dt} = L_\nu + L_\gamma, \quad L_\gamma \sim T^{2+4\alpha}, \quad L_\nu \sim T^8 \text{ (Modified Urca)}, \quad C_V \sim CT$$

- Age assumed from spin-down age or associated with a supernova remnant



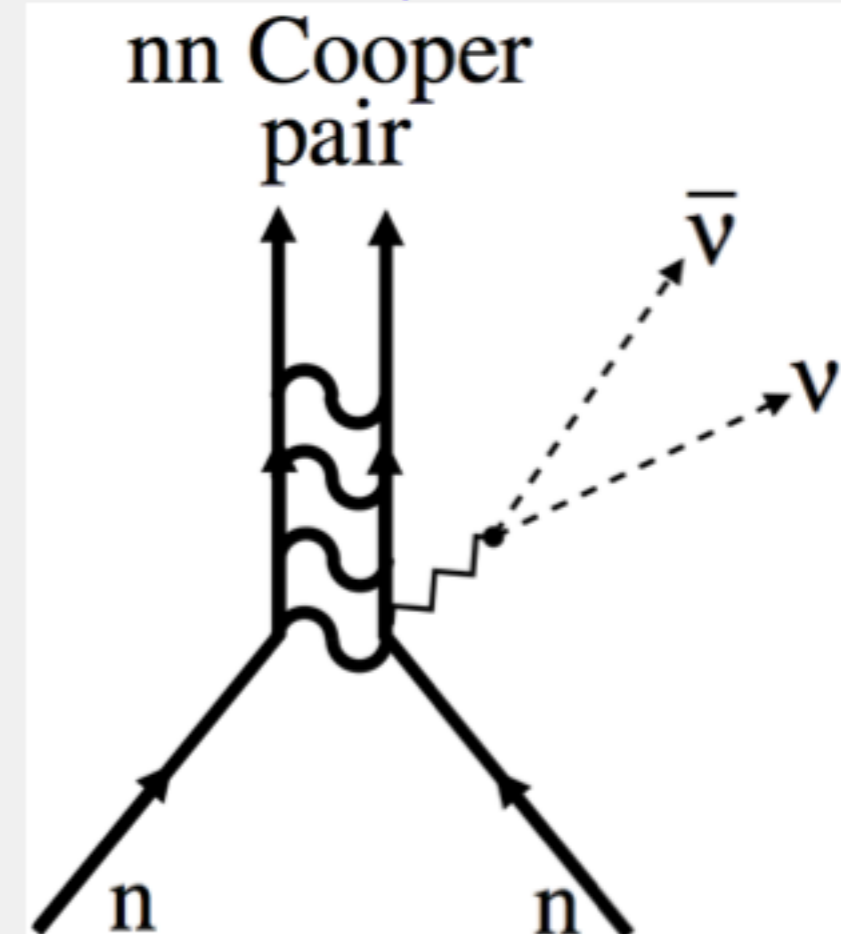
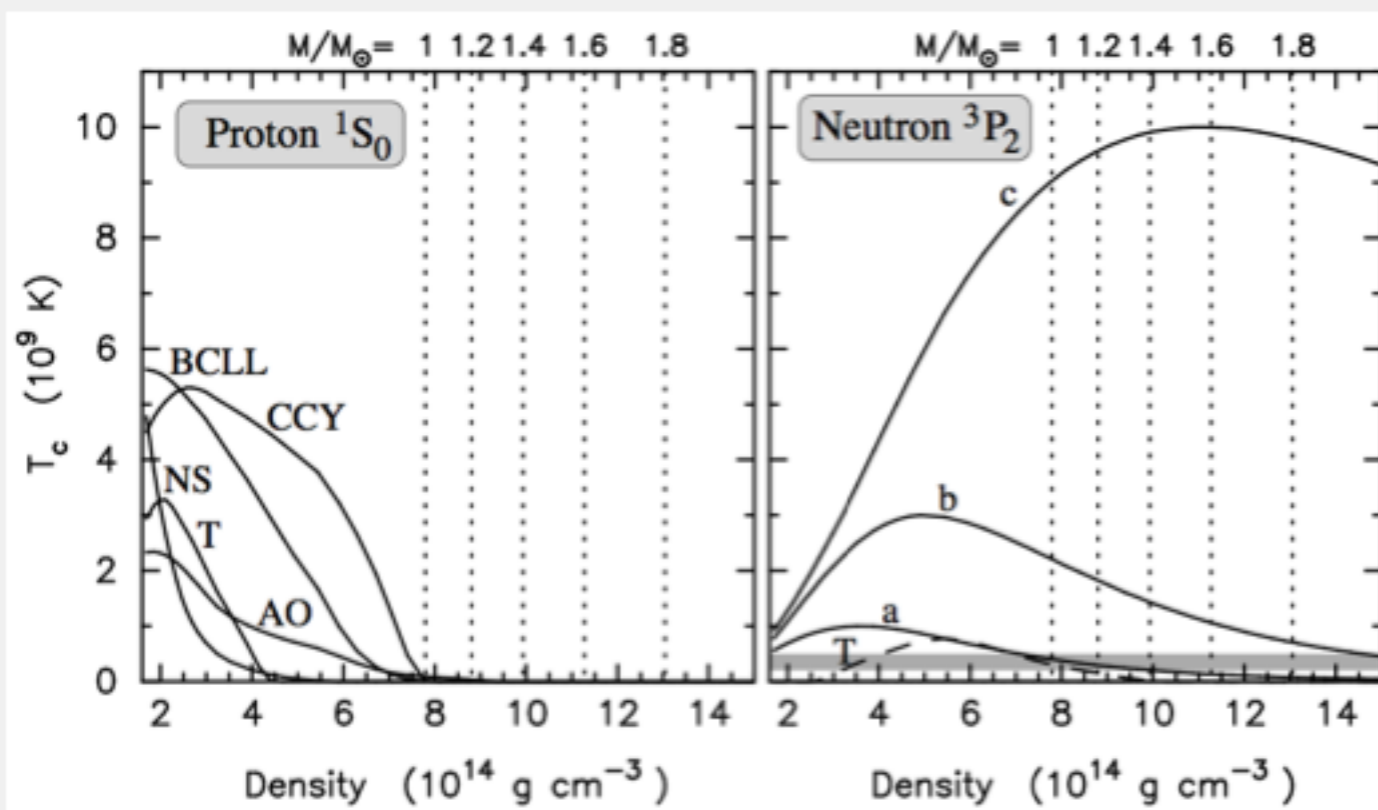
# Neutron Star Superfluidity

(See our review at 1302.6626)



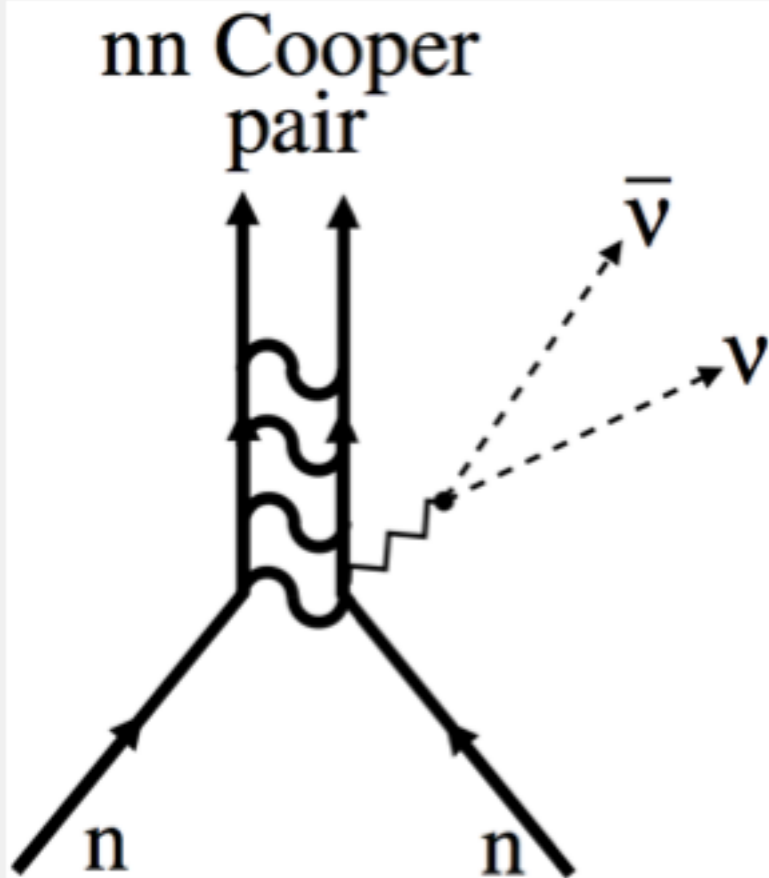
- $^1S_0$  gap increases with increasing density, but drops off at higher densities because of n-n repulsion
- Superfluidity can block cooling processes
- ...but it opens up new ways of cooling

Steiner and Reddy (2009)

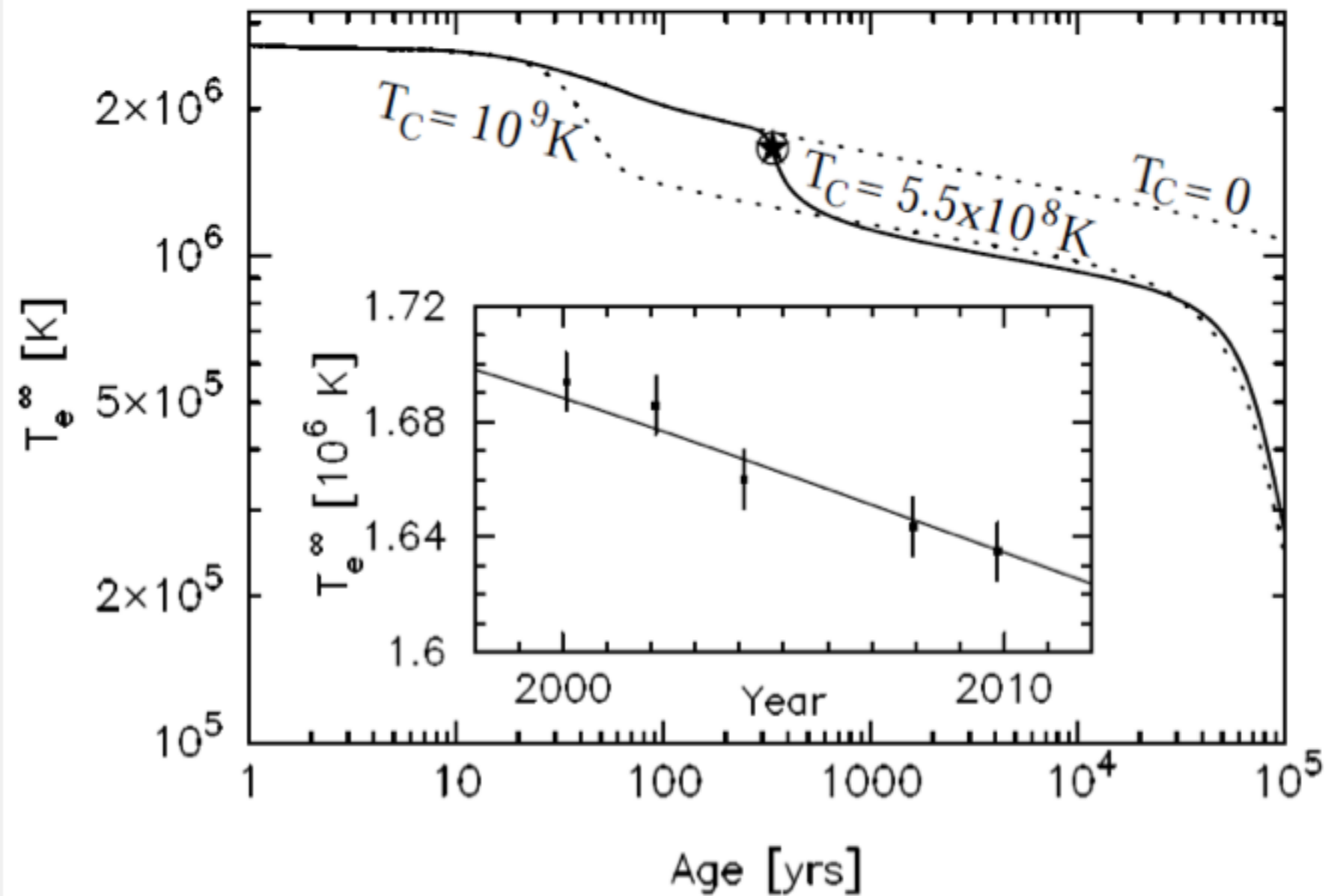


# Detecting Neutron Star Superfluidity

- The large slope is only well reproduced by the neutron triplet superfluid transition and associated emissivity
- Cas A requires a very particular triplet gap  $\Delta(T = 0) \propto T_C$



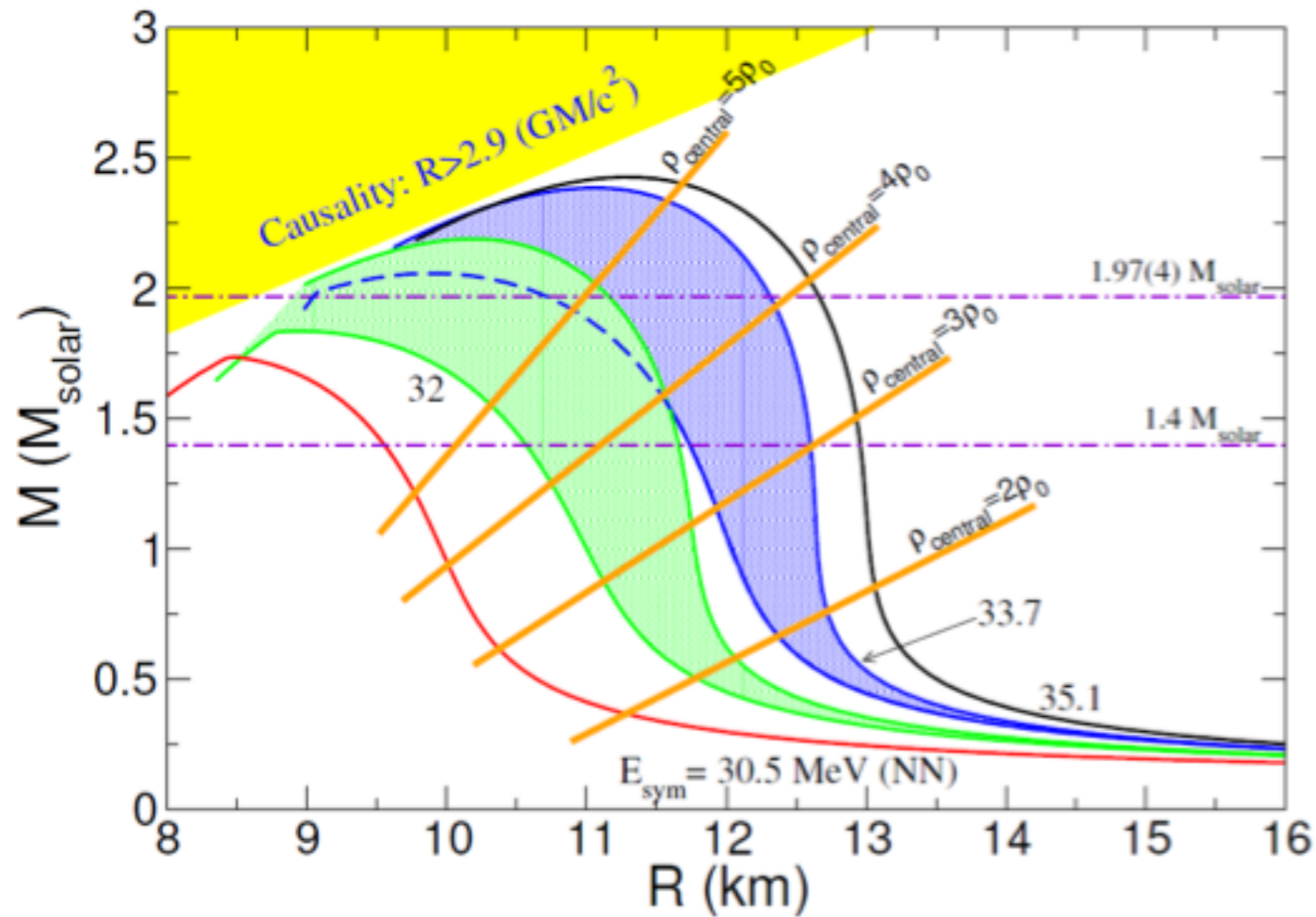
- If you form a Cooper pair, you gain energy



# PRE X-ray bursts

- van Paradijs et al. pioneer the idea, it's rarely used until Özel writes several papers starting in 2007ish, getting small radii
- We demonstrate that photosphere radii are large at touchdown, add qLMXB data, use some nuclear physics, and get  $\sim 11$  km radii.
- Suleimanov gets larger radii (14 km) for a long burst in XTE J1701, and claims other PRE X-ray data is poisoned by accretion ([Suleimanov et al. 2011](#))
- Yet the larger radius is somewhat inconsistent with qLMXB radii ([Steiner et al. 2012](#))
- Becomes clear that there may be (at least) two types of PRE X-ray bursts, which have different properties. Also some variation in normalization between bursts.  
([Work by G. Zhang](#))
- Güver et al. do a systematic analysis of several sources and show that the fit of XTE J1701 is poor, but good for other sources ([Güver et al. 2012a and 2012b](#))
- Guillot et al. revisit qLMXB measurements, still find small radii ([Guillot et al. 2013](#))
- Lattimer and I re-revisit them, still finding smaller radii, but larger uncertainties  
([Lattimer et al. 2013](#))
- Work with Suleimanov finds XTE J1701 is complicated by a boundary layer (possibly explaining the poor fit?) ([Retvinsev et al. 2013](#))
- Status: Larger ( $\sim 14$  km) radii are not preferred and result in poorer fits, unless you presume something has gone terribly wrong in qLMXBs. Nevertheless, PRE X-ray bursts are not well-understood and much work remains.

# Connection to Nuclear Three-Body Forces



Colored regions denote different three-body forces

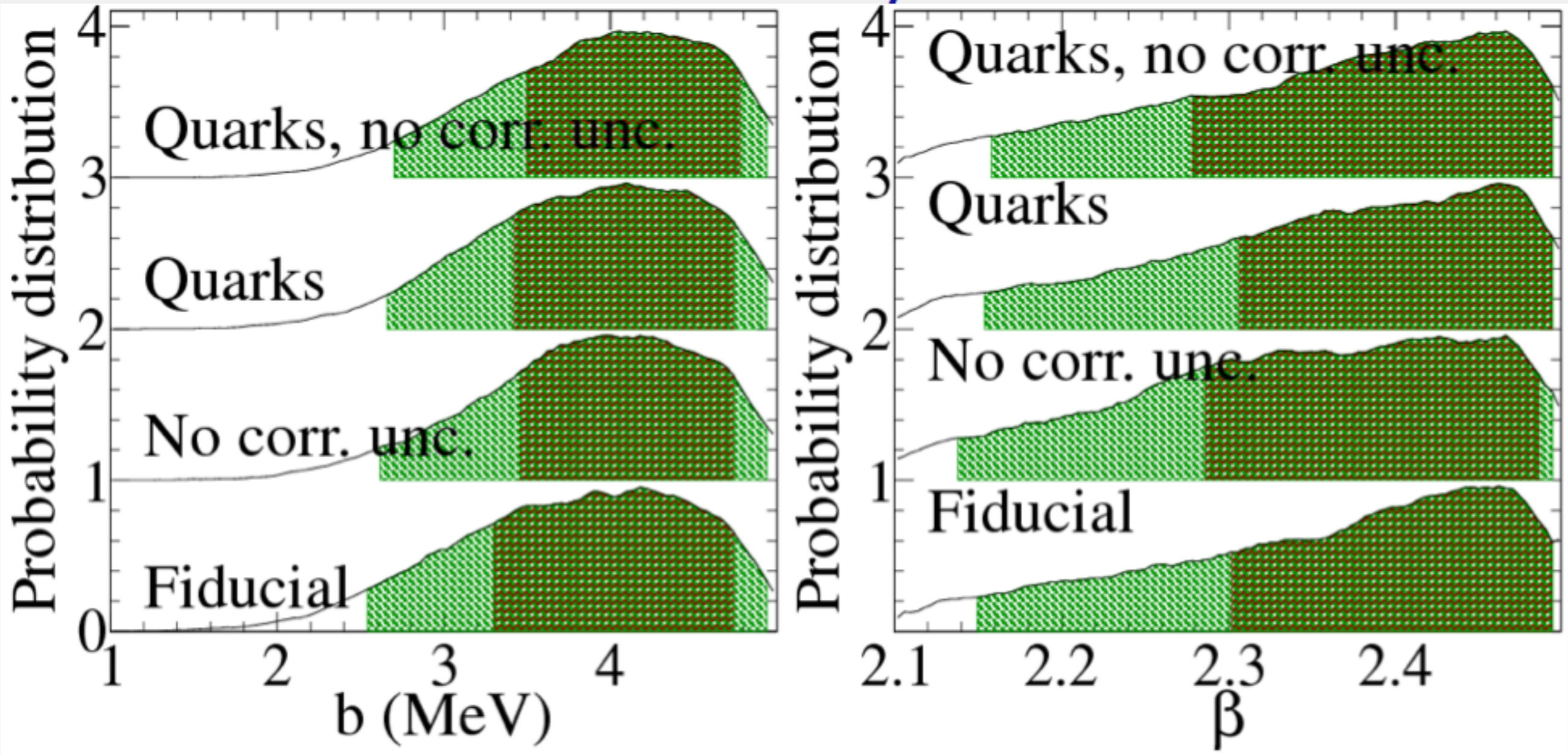
$$E_{\text{neut}} = a \left( \frac{n}{n_0} \right)^\alpha + b \left( \frac{n}{n_0} \right)^\beta$$

Gandolfi, Carlson, and Reddy (2012)

- Three-nucleon interactions are important for nuclei and neutron star radii
- Quantum Monte Carlo (AFDMC) + Stellar structure
- How do neutron star observations constrain  $b$  and  $\beta$ ?

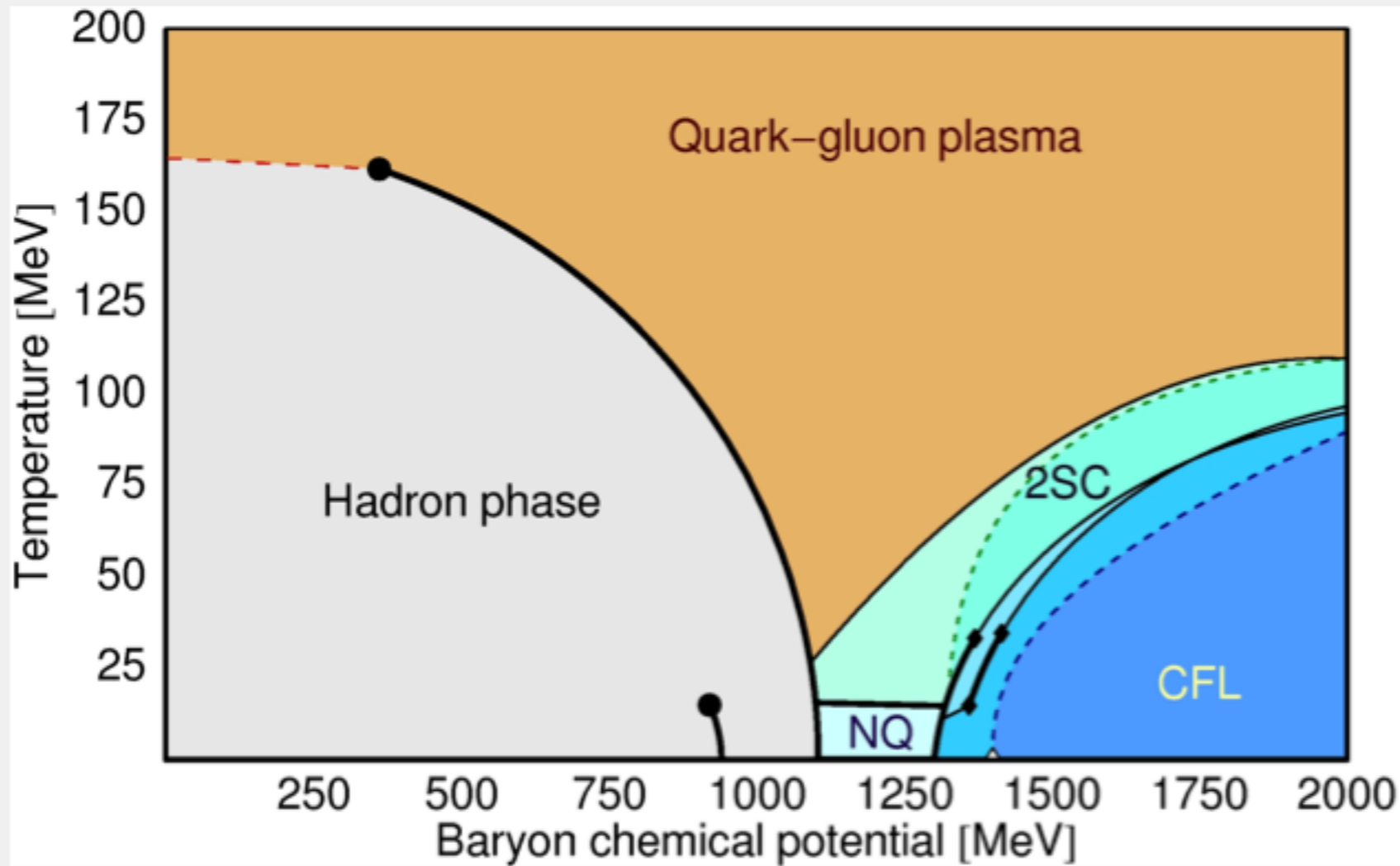


# Constraints on Three-Body Force Parameters



Steiner and Gandolfi (2012)

- Values of  $a$  and  $\alpha$  are unconstrained, but constraints on  $b$  and  $\beta$
- Left and right plot boundaries exhaust expected range
- Neutron star radii are indirectly constraining nuclear three-body forces
- Limitation: if hyperons (or other strong phase transition) happens at a sufficiently low density, then results are modified



QCD phase diagram, Ruster et al. (2006)

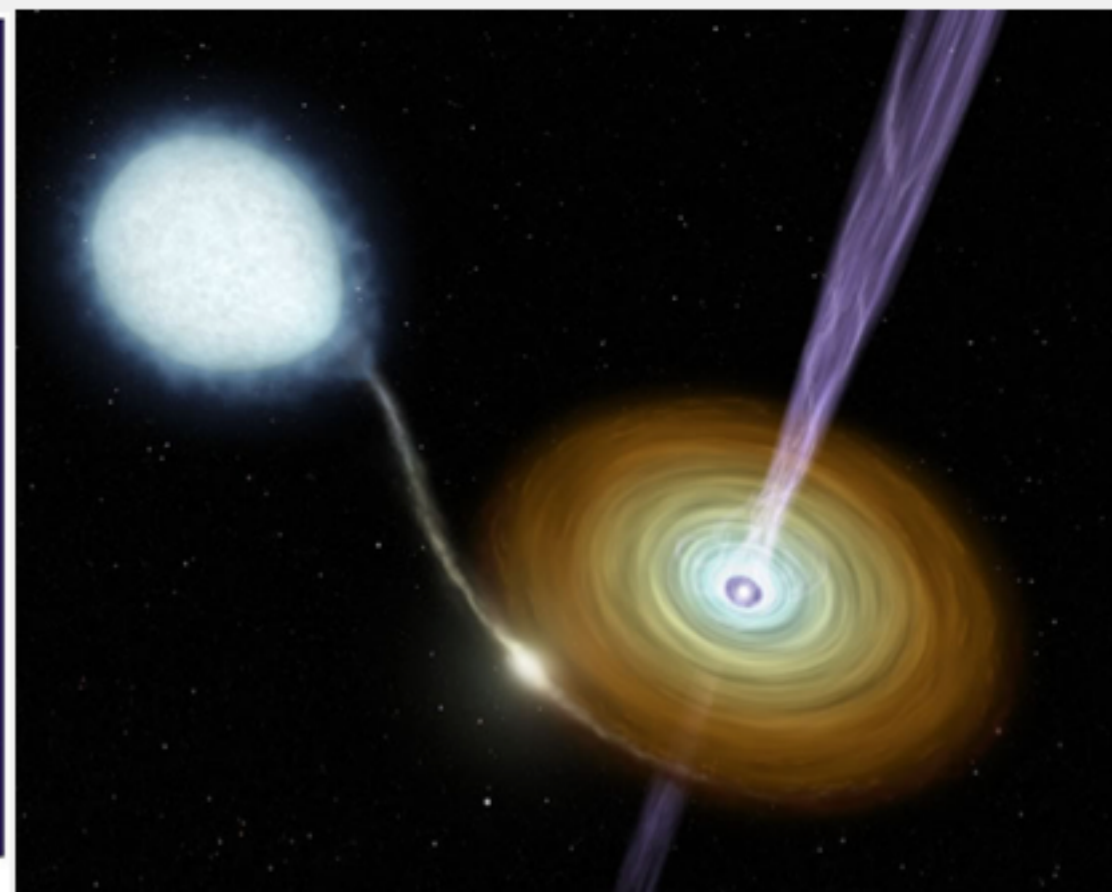
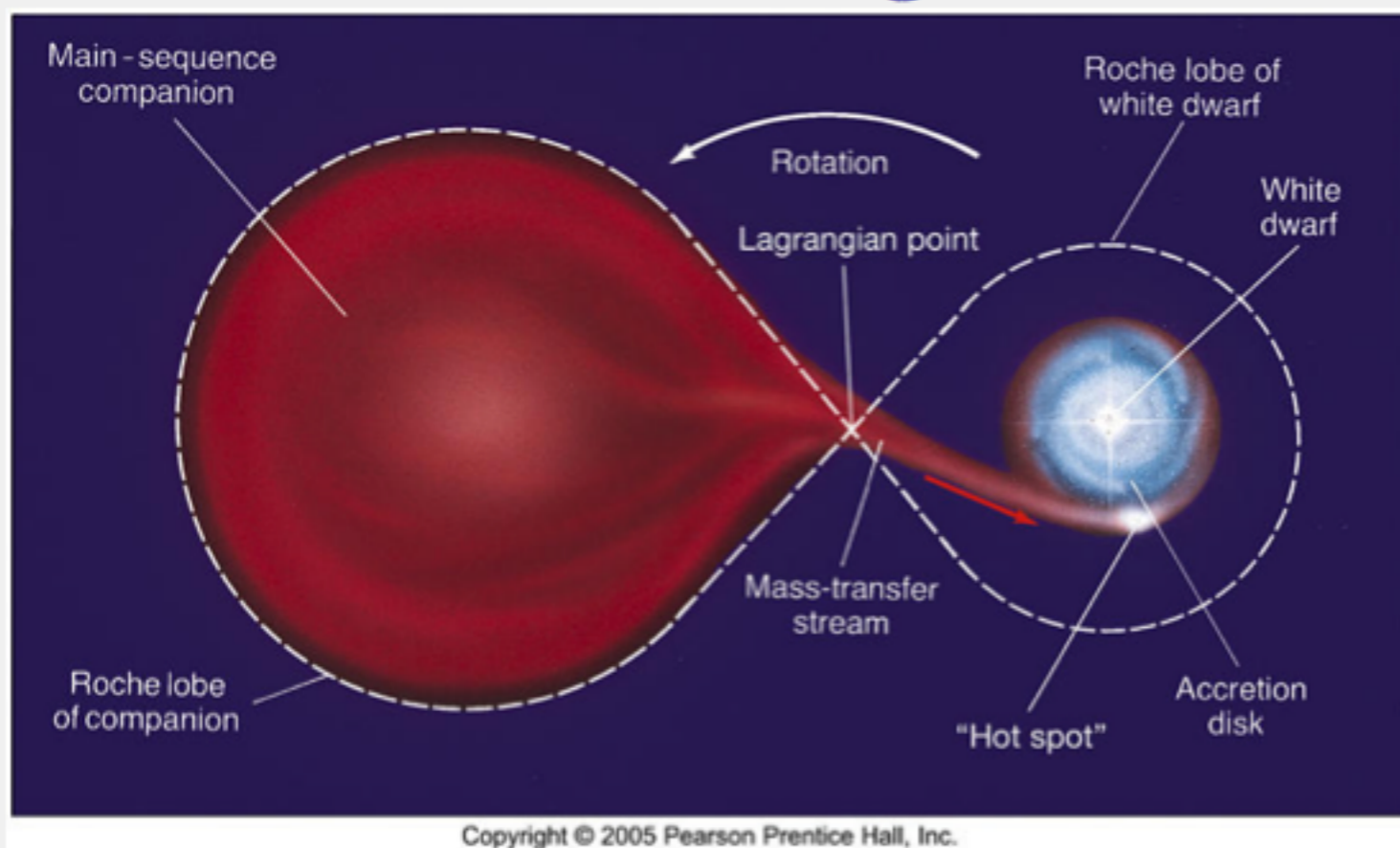
Frontiers of nuclear science (2007):

- What is the nature of neutron stars and dense matter?
- What is the nucleon-nucleon interaction?
- What is the origin of the elements?

Also,

- How do neutron stars merge?
- How do mergers generate gravitational waves?

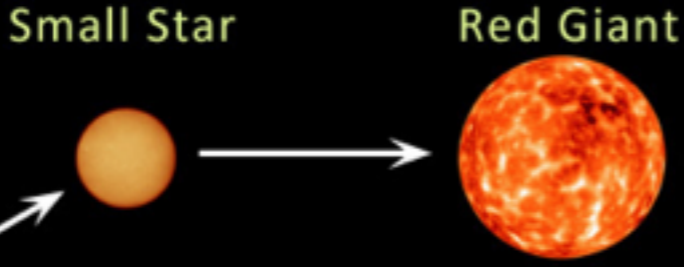
# Accreting Neutron Stars: LMXBs



- Most stars have companions: neutron stars can have main-sequence, "normal star", companions
- Accretion heats the crust and is episodic
- At high enough density, H and He are unstable to thermonuclear explosions, i.e. X-ray bursts

# Stellar Evolution

## EVOLUTION OF STARS



0-8 solar masses

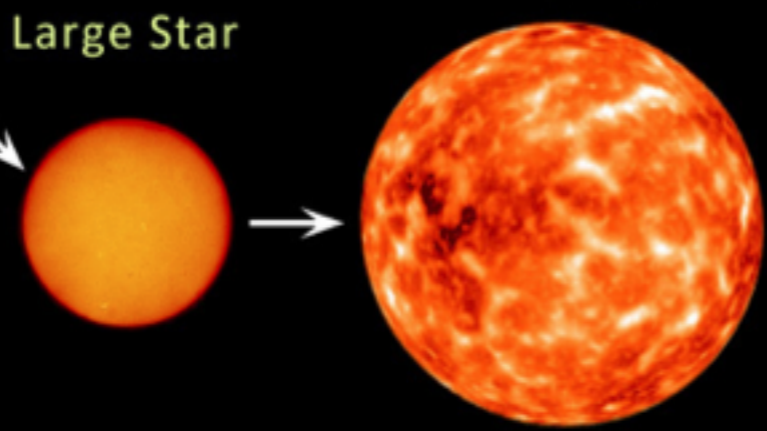
Planetary Nebula



White Dwarf



8-20 solar masses

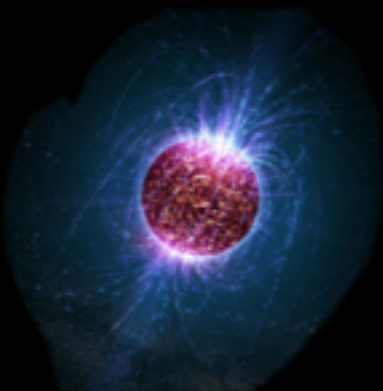


>20 solar masses

Supernova



Neutron Star

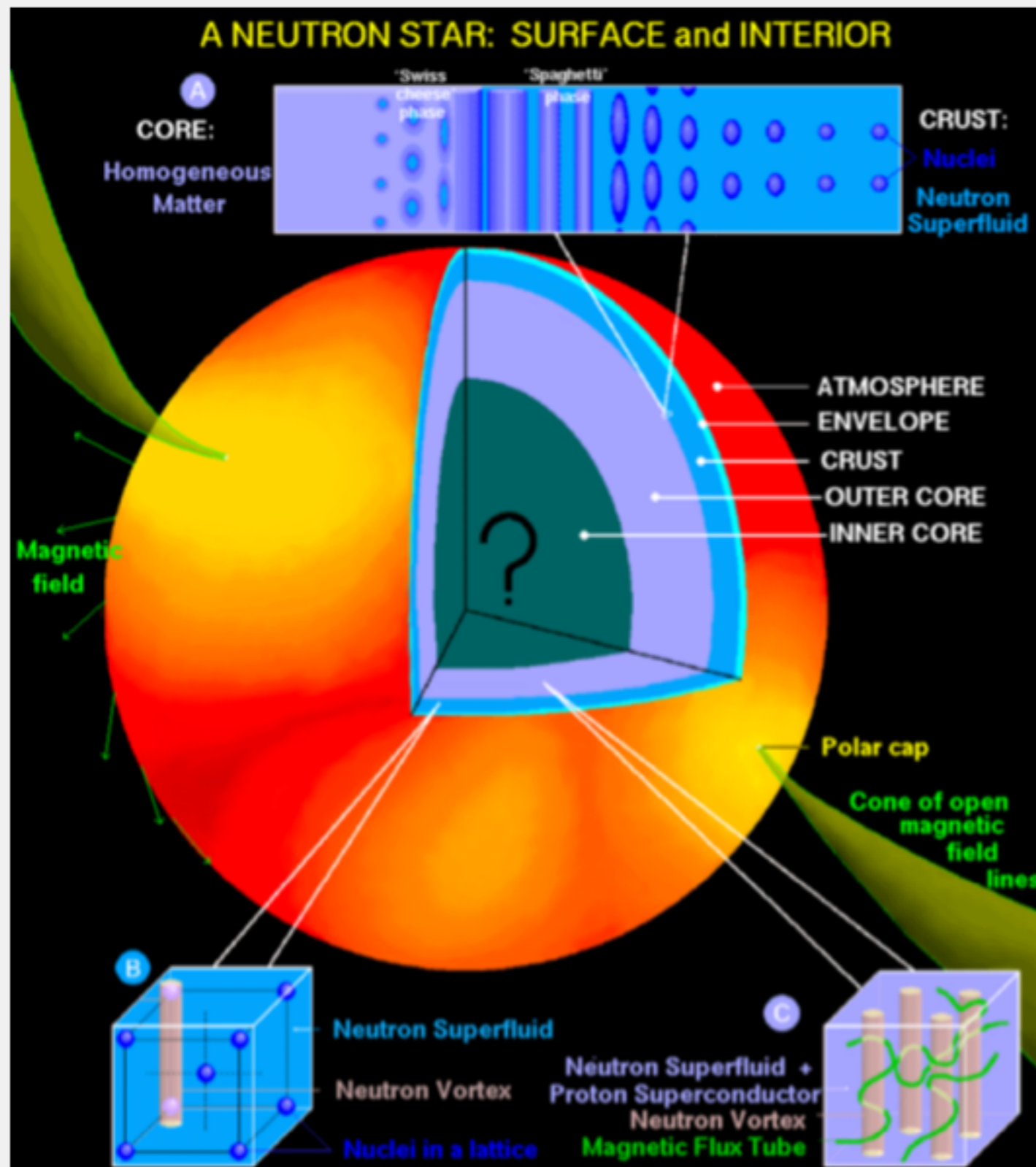


Black Hole



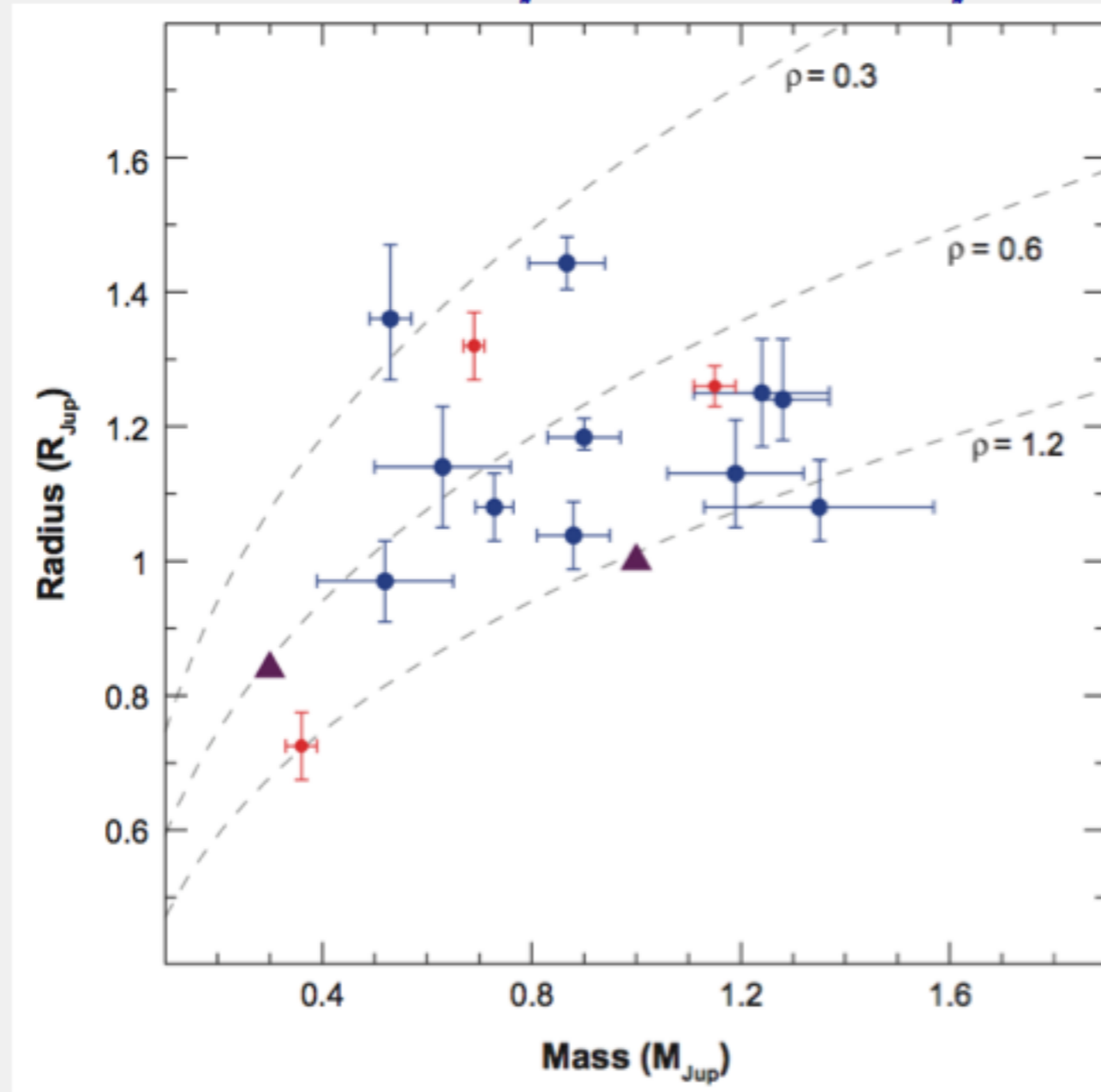
IMAGES NOT TO SCALE

# Neutron Star Composition



- Outer crust: of neutron-rich nuclei
- Inner crust: neutron-rich nuclei embedded in a sea of quasi-free superfluid neutrons
- Outer core: fluid of neutrons, protons, and electrons
- Inner core: hyperons, Bose condensates, deconfined quark matter
- **What are the correct degrees of freedom for the effective field theory which describes dense matter?**

# Planetary Diversity



Udry et al. (2007)

- Varying composition, thus varying radius for a fixed mass