Partonic Contributions to the Proton Spin in Lattice QCD

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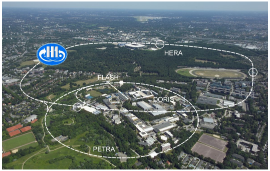
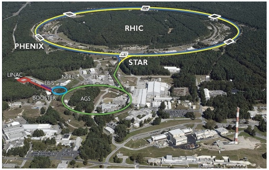
Outline

Ever since the EMC “Proton Spin Crisis” (1987), the proton spin structure has been one of the most important subjects in hadron physics.

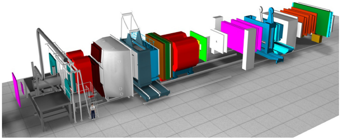
- HERMES at DESY, fixed target, 27.6 GeV e^+/e^- beam



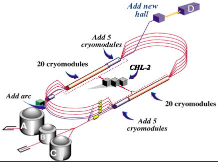
- PHENIX, STAR at RHIC, p/p collision, $\sqrt{s} = 200, 500$ GeV



- COMPASS at CERN, fixed target, 160 GeV μ^+ beam

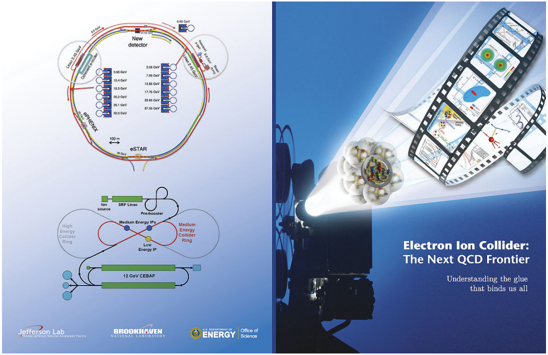


- JLab 12 GeV Upgrade, fixed target, 12 GeV e^- beam



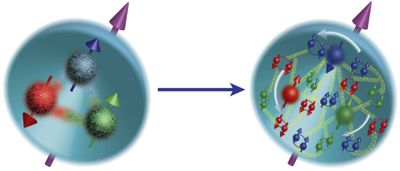
Electron Ion Collider (EIC)

- Highly polarized ($\sim$ 70%) beams
- Ion beams from deuteron to the heaviest nuclei
- Variable $\sqrt{s}$ from 20–100 GeV, upgradable to $\sim$150 GeV
- High collision luminosity $10^{33-34} cm^{-2} s^{-1}$
- Multiple interaction regions



Still in the white book, A. Accardi *et al.*, arXiv:1212.1701.

The proton is a bound state of quarks and gluons. For bound state gluons, there is no physically meaningful notion of spin or orbital angular momentum (OAM).



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However,

- In high energy scattering experiments, quarks and gluons inside the fast-moving proton are almost interaction-free, and the proton structure can be described by the Feynman parton model;
- With "free" partons one can ask more physically meaningful questions, e.g., the gluon spin and OAM, and come up with a naive spin sum rule:

$$S^z = \frac{1}{2} = \frac{1}{2}\Delta\Sigma(\mu) + \Delta G(\mu) + \Delta L_q^z(\mu) + \Delta L_g^z(\mu) , \quad (1)$$

- How to justify this sum rule in quantum field theory?

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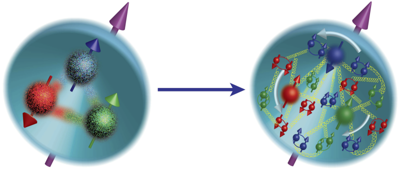

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Expression of the QCD angular momentum from textbooks:

$$\vec{J} = \int d^3x\, \psi^\dagger \frac{\vec{\Sigma}}{2} \psi + \int d^3x\, \psi^\dagger \vec{x} \times (-i\vec{\nabla}) \psi + \int d^3x\, \vec{E} \times \vec{A} + \int d^3x\, E^i\, \vec{x} \times \vec{\nabla} A^i \,. \quad (2)$$

- ☺ For each term there is a free-field theory expansion;
- ☺ This expression has the potential to obtain the spin sum rule in Eq. (1);
- ☹ Except for the quark spin, the other three terms are not gauge invariant.

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In recent years, parton physics, which emerges in the infinite momentum frame (IMF) of the hadron, has been formulated in light-front quantization. It can be understood as the proton being in the rest frame while the probe is moving at the speed of light.

In this formalism, quark and gluon spin (helicity) are given by QCD factorization theorems (A. Manohar, 1991):

$$\begin{aligned}\Delta\Sigma &= \int_0^1 dx \frac{1}{4\pi P^+}\int d\lambda e^{ix\lambda}\langle PS|\bar{\psi}(\lambda n)\gamma^+\gamma^5 \mathcal{P}e^{ig\int_0^\lambda A^+(\eta n)d\eta}\psi(0)|PS\rangle \\ &= \frac{1}{2P^+}\langle PS|\psi^\dagger(0)\Sigma^3\psi(0)|PS\rangle\ . \end{aligned} \tag{3}$$

$$\begin{aligned}\Delta G &= \int_0^1 dx \frac{i}{4\pi x(P^+)^2}\int d\lambda e^{ix\lambda}\langle PS|F^{+\alpha}(\lambda n)\mathcal{P}e^{ig\int_0^\lambda A^+(\eta n)d\eta}\tilde{F}^+_{\ \alpha}(0)|PS\rangle \\ &\overset{A^+=0}{=} \frac{1}{2P^+}\langle PS|\left(\vec{E}(0)\times\vec{A}(0)\right)^3\Big|_{A^+=0}|PS\rangle\ . \end{aligned} \tag{4}$$

- Light-cone coordinates are used in this formalism, i.e.,

$$\xi^+ = (\xi^0 + \xi^3)/\sqrt{2}, \quad \xi^- = (\xi^0 - \xi^3)/\sqrt{2}\,. \tag{5}$$

- In the light-cone gauge $A^+ = 0$, the quark and gluon spin (helicity) have the same free-field form as in Eq. (2);

- This leads to a definition of the parton spin and OAM as:

$$\begin{aligned}
\Delta\Sigma(\mu) &\equiv \langle PS|\psi^\dagger\Sigma^3\psi|PS\rangle\,, \\
\Delta G(\mu) &\equiv \langle PS|\,(\vec{E}\times\vec{A})^3\Big|_{A^+=0}|PS\rangle\,, \\
\Delta L_q(\mu) &\equiv \langle PS|\int d^3\xi\,\psi^\dagger\,\left(\vec{x}\times(-i\vec{\nabla})\right)^3\psi\Big|_{A^+=0}|PS\rangle/(2P^+)(2\pi)^3\delta^{(3)}(0)\,, \\
\Delta L_g(\mu) &\equiv \langle PS|\int d^3\xi\,E^i\,(\vec{x}\times\vec{\nabla})^3A^i\Big|_{A^+=0}|PS\rangle/(2P^+)(2\pi)^3\delta^{(3)}(0)\,.
\end{aligned} \tag{6}$$

How to measure parton spin and orbital angular momentum?

- $\Delta\Sigma$

  $\Delta q(x)$ can be extracted from the spin structure function $g_1(x, Q^2)$ measured in polarized DIS experiments (W. Vogelsang *et al.*, 2009),

$$\Delta\Sigma \approx \int_{0.001}^{1} dx\, \Sigma_{i\in q,\bar{q}} \Delta q_i(x, Q^2 = 10\text{GeV}^2) = 0.366^{+0.042}_{-0.062}\,. \quad (7)$$

- ΔG

  $\Delta g(x)$ can be extracted from the scaling violation of $g_1(x, Q^2)$ (E. C. Aschenauer *et al.*, 1304.0079),

$$\Delta G = \int_{0}^{1} dx \Delta g(x), \quad \int_{0.05}^{0.2} dx \Delta g(x, Q^2 = 10\text{GeV}^2) = 0.1^{+0.06}_{-0.07}\,. \quad (8)$$

- ΔL_q and ΔL_g
  - Can be measured through twist-three generalized parton distributions (GPD's) (Ji, Xiong, and Yuan, 2012, 2013)

$$\begin{aligned} \Delta L_q(x) &= \tilde{H}_q^{(3)}(x,0,0)\,, \\ x\Delta L_g(x) &= \tilde{H}_g^{(3)}(x,0,0)\,. \end{aligned} \tag{9}$$

    GPD's can be measured in deeply virtual Compton scattering (DVCS) processes (A. Belitsky *et al.*, 2001, 2002):
  - Can also be measured through a Wigner distribution [4] (Lorce and Pasquini, 2011; Lorce, Pasquini, Xiong, Yuan, 2011).

$$\begin{aligned} \Delta L_q(x) &= \int d^2\vec{b}_\perp d^2\vec{k}_\perp (\vec{b}_\perp \times \vec{k}_\perp)^3 W_{LC}^q(x,\vec{b}_\perp,\vec{k}_\perp)\,, \\ \Delta L_g(x) &= \int d^2\vec{b}_\perp d^2\vec{k}_\perp (\vec{b}_\perp \times \vec{k}_\perp)^3 W_{LC}^g(x,\vec{b}_\perp,\vec{k}_\perp)\,. \end{aligned} \tag{10}$$

    The Wigner distributions can be measured through hard exclusive processes (Ji, Xiong, and Yuan, 2013).

Why is a lattice calculation of these matrix elements difficult?

The lattice theory is the only practical non-perturbative approach to solve QCD, but no one knows how to use it to directly calculate the partonic contributions so far.

Lattice QCD is formulated in Euclidean space:

- The light-cone time $\xi^+ = (-i\tau + z)/\sqrt{2}$, where τ is real;
- The light-cone gauge $A^+ = (-iA^4 + A^3)/\sqrt{2} = 0$, where A^4 is real;
- A direct calculation of the light-cone matrix elements on the lattice is not possible.

What if we avoid using the light-front formalism?

Actually when the expression in Eq. (2) was first considered by Jaffe and Manohar, the parton spin and OAM were defined as proton matrix elements of the angular momentum operators at $P \equiv P_\infty \to \infty$ (Jaffe and Manohar, 1991):

$$
\begin{aligned}
\Delta\Sigma(\mu) &\equiv \langle P_\infty, S|\psi^\dagger \Sigma^3 \psi|P_\infty, S\rangle \,, \\
\Delta G(\mu) &\equiv \langle P_\infty, S|\,(\vec{E}\times\vec{A})^3\Big|_{A^0=0}|P_\infty, S\rangle \,, \\
\Delta L_q(\mu) &\equiv \langle P_\infty, S|\int d^3x\, \psi^\dagger \left(\vec{x}\times(-i\vec{\nabla})\right)^3 \psi\Big|_{A^0=0}|P_\infty, S\rangle/(2E)(2\pi)^3\delta^{(3)}(0) \,, \\
\Delta L_g(\mu) &\equiv \langle P_\infty, S|\int d^3x\, E^i\,(\vec{x}\times\vec{\nabla})^3 A^i\Big|_{A^0=0}|P_\infty, S\rangle/(2E)(2\pi)^3\delta^{(3)}(0) \,.
\end{aligned}
\tag{11}
$$

This definition is equivalent to that in the light-front formalism:

- The angular momentum operators have the same form;
- The temporal gauge $A^0 = 0$ under an infinite Lorentz boost Λ will be transformed into the light-cone gauge:

$$A^0 = \frac{1}{\sqrt{2}}(A^+ + A^-) = 0 \Rightarrow \frac{1}{\sqrt{2}}(\Lambda A^+ + \Lambda^{-1} A^-) = 0 \Rightarrow A'^+ = 0. \quad (12)$$

However, it is still not possible to directly calculate these matrix elements in lattice QCD ☹☹:

- Lattice QCD has a momentum cut-off π/a.
- The external momentum of physical states on the lattice are usually much smaller than π/a. One cannot calculate the Euclidean matrix elements at infinite momentum.

- The basic problem is that in lattice QCD we are only able to calculate the Euclidean matrix elements at finite momentum, but these matrix elements are not the physical results from high energy scattering experiments.

- Acknowledging this fact, can we find an approach to relate the finite momentum matrix elements to the physical results in the IMF (or light-front formalism)?

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2. A Large Momentum Effective Field Theory (LaMET) Approach
    - How does a LaMET work
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How does a LaMET work?

- Construct a Euclidean quasi-observable "O" that can be calculated in lattice QCD;
- The matrix elements of the observable should be dependent on the lattice spacing a and the momentum of the physical state P, i.e., $O(P, a)$;
- Extract the IMF (or light-front) result $O(\mu)$ from $O(P, a)$ as P becomes large through a factorization formula (or matching condition)

$$O(P, a) = Z(\mu/P)O(\mu) + \frac{c_2}{P^2} + \frac{c_4}{P^4} + \cdots . \quad (13)$$

With a LaMET, one can formulate many ways to calcualte parton spin and OAM on the lattice:

- The choice of the Euclidean quasi-observables is not unique;
- The expression defined in the $A^0 = 0$ gauge by Jaffe and Manohar is one of the choices that we can make;
- In the past few years, new expressions of the QCD angular momentum have been proposed (Chen *et al.*, 2008; Wakamatsu, 2010; Hatta, 2011) with the idea of

$$\vec{A} = \vec{A}_{\text{phys}} + \vec{A}_{\text{pure}} . \quad (14)$$

  $\vec{A}_{\text{phys}}$ and $\vec{A}_{\text{pure}}$ can be used to construct the quasi-observables.

- Out of the many choices, we are interested in the decomposition of $\vec{A} = \vec{A}_\perp + \vec{A}_\parallel$ proposed by Chen *et al.* because it offers more physical insights.

Outline

An alternative expression of the QCD angular momentum :

$$\vec{J} = \int d^3x\, \psi^\dagger \frac{\vec{\Sigma}}{2}\psi + \int d^3x\, \psi^\dagger \vec{x} \times \frac{1}{i}(\vec{\nabla} - ig\vec{A}_{\|})\psi + \int d^3x\, \vec{E}_a \times \vec{A}^a_\perp + \int d^3x\, E^i_a\, \vec{x} \times \vec{\nabla} A^{i,a}_\perp \,. \tag{18}$$

(X. S. Chen *et al.*, 2008)

Each term is gauge invariant ☺, but nonlocal (except for the quark spin). Take U(1) gauge theory as an example:

$$\vec{A} = \vec{A}_\perp + \vec{A}_{\|} \,, \tag{15}$$

where

$$\vec{\nabla} \cdot \vec{A}_\perp = 0, \qquad \vec{\nabla} \times \vec{A}_{\|} = 0. \tag{16}$$

Under gauge transformation $\vec{A} \to \vec{A} + \nabla\chi$,

$$\vec{A}'_\perp = \vec{A}_\perp, \qquad \vec{A}'_{\|} = \vec{A}_{\|} + \nabla\chi \,. \tag{17}$$

In non-Abelian theory, under a gauge transformation (R. Treat, 1972):

$$\vec{A}_\perp \to U(x)\vec{A}_\perp U^\dagger(x)\ , \qquad \vec{A}_\parallel \to U(x)\vec{A}_\parallel U^\dagger(x) + \frac{i}{g}\left(\vec{\nabla}U(x)\right)U^\dagger(x)\ , \quad (19)$$

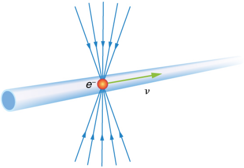
and thus each term is still gauge invariant.

However, the operators in this expression, except for quark spin, were not considered as physical observables at first (X. Ji, 2010):

- $\vec{E}_\perp$ and $\vec{E}_\parallel$ are generated from $\vec{A}_\perp$ and $\vec{A}_\parallel$ (plus the scalar potential), but there is no charge responding differently to $\vec{E}_\perp$ and $\vec{E}_\parallel$, i.e., $\vec{E}_\perp$ and $\vec{E}_\parallel$ cannot be separately probed as physical observables.
- The $\vec{A}_\perp$ and $\vec{A}_\parallel$ are not Lorentz covariant vectors. The decomposition of $\vec{A}(\vec{E})$ into $\vec{A}_\perp(\vec{E}_\perp)$ and $\vec{A}_\parallel(\vec{E}_\parallel)$ has to be done in a fixed frame. Physical observables should be frame independent.

Why is this expression still useful to us?

1. For a static charge, the electric field only has a longitudinal component: $\vec{E} = \vec{E}_{\|} = \vec{\nabla}\varphi$, $\vec{\nabla} \times \vec{E}_{\|} = 0$;
2. As the charge moves, the field lines contract in the transverse direction;



3. The moving charge generates $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}_{\perp}$, while the changing $\vec{A}_{\perp}$ in turn generates $\vec{E}_{\perp} = -(\partial/\partial t)\vec{A}_{\perp}$;
4. As the momentum goes to infinity ($\Lambda \to \infty$), $\vec{E}_{\perp} \gg \vec{E}_{\|}$, $\vec{E}_{\perp} \sim \vec{B}$, while $\vec{E}_{\perp}$ and $\vec{B}$ are both accounted by $\vec{A}_{\perp}$. The EM fields are almost like free radiation, i.e., Weizsäcker-Williams approximation (J. D. Jackson, CED).

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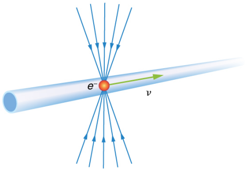

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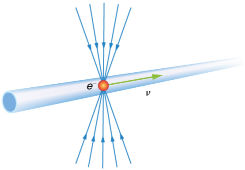

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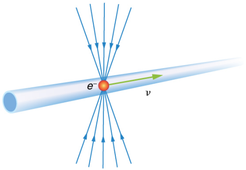

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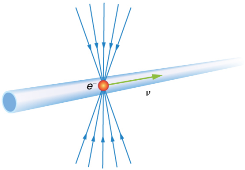

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Conlusion:

- In the IMF limit, $\vec{A}_\perp$ accounts for all the d.o.f. of the EM field, thus it becomes physically meaningful;
- If we boost the operators in the new expression to the IMF, each becomes physical observables, according to Weizsäcker-Williams approximation.

Next Question:

In the IMF limit, are these operators equivalent to the physical observables that are defined in the IMF (or light-front formalism)?

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IMF limit of the operators

In QED, with proper boundary conditions,

$$\vec{A}_{\parallel} = \vec{\nabla}\frac{1}{\nabla^2}\nabla\cdot\vec{A}\,, \quad \vec{A}_{\perp} = \vec{A} - \vec{\nabla}\frac{1}{\nabla^2}\nabla\cdot\vec{A}\,. \tag{20}$$

A four vector V^μ under a Lorentz boost (Λ) along the z direction transforms as

$$V^+ \to \Lambda V^+, \quad V^- \to \Lambda^{-1}V^-, \quad V^\perp \to V^\perp\,, \tag{21}$$

so in the limit of $\Lambda \to \infty$,

$$\vec{A}_{\perp} \to \vec{A} - \vec{\nabla}\frac{1}{\Lambda^2(\nabla^+)^2}\Lambda^2\nabla^+A^+ = \vec{A} - \vec{\nabla}\frac{1}{\nabla^+}A^+\,. \tag{22}$$

Therefore, we have a formal proof that the photon spin

$$\vec{E}\times\vec{A}_{\perp} \to \vec{E}\times\left(\vec{A} - \vec{\nabla}\frac{1}{\nabla^+}A^+\right)\,. \tag{23}$$

In non-Abelian gauge theories (R. Treat, 1972),

$$\partial^i A_{\parallel}^{j,a} - \partial^j A_{\parallel}^{i,a} - gf^{abc} A_{\parallel}^{i,b} A_{\parallel}^{j,c} = 0 \,, \tag{24}$$

$$\partial^i A_{\perp}^i = ig[A^i, A_{\perp}^i] \,. \tag{25}$$

In the IMF limit,

$$\partial^+ A_{\parallel}^{i,a} - gf^{abc} A^{+,b} A_{\parallel}^{i,c} = \partial^i A^{+,a} \,. \tag{26}$$

We prove that (Ji, Zhang, and Zhao, 2013):

(a) In the IMF limit,

$$\left[\vec{E}^a(0)\times\vec{A}^a_\perp(0)\right]^3 \to \left[\vec{E}^a(0)\times\left(\vec{A}^a(0)-\frac{1}{\nabla^+}(\vec{\nabla}A^{+,b})\mathcal{L}^{ba}(\xi^-,0)\right)\right]^3 . \tag{27}$$

(b) In momentum space, after integration by parts, this is proved to be exactly the gauge-invariant gluon helicity operator from factorization theorems

$$\begin{aligned} S_g^3 &= \int_0^1 dx\frac{i}{xP^+}\int\frac{d\lambda}{2\pi}e^{ix\lambda}F^{+\alpha}(\lambda n)\mathcal{P}e^{ig\int_0^\lambda A^+(\eta n)d\eta}\tilde{F}^+_{\ \alpha}(0) \\ &\overset{A^+=0}{=} \left(\vec{E}(0)\times\vec{A}(0)\right)^3 . \end{aligned} \tag{28}$$

(c) This proof in turn justifies that light-cone gauge is a natural choice in the IMF, where $\vec{E}\times\vec{A}$ becomes physically meaningful.

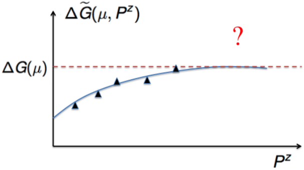
Outline

At finite momentum,

$$S^z = \frac{1}{2} = \frac{1}{2}\Delta\widetilde{\Sigma}(\mu, P^z) + \Delta\widetilde{G}(\mu, P^z) + \Delta\widetilde{L}_q^z(\mu, P^z) + \Delta\widetilde{L}_g^z(\mu, P^z) \,. \tag{30}$$

Based on our proof, we should have

$$\begin{aligned}
\lim_{P^z\to\infty} \Delta\widetilde{\Sigma}(\mu, P^z) &= \Delta\Sigma(\mu) \,, \\
\lim_{P^z\to\infty} \Delta\widetilde{G}(\mu, P^z) &= \Delta G(\mu) \,, \\
\lim_{P^z\to\infty} \Delta\widetilde{L}_q(\mu, P^z) &= \Delta L_q(\mu) \,, \\
\lim_{P^z\to\infty} \Delta\widetilde{L}_g(\mu, P^z) &= \Delta L_g(\mu) \,.
\end{aligned} \tag{29}$$



If we extrapolate from the results at different finite momentum P^z, maybe we can obtain the physical result in the IMF.

Unfortunately, life is not so easy:

Taking the IMF limit is subtle, consider

$$\int \frac{d^4k}{k^2(k-P)^2(\vec{k}+\vec{P})^2} \,,$$

where $P^\mu = (P^0, 0, 0, P^z)$ and $P^2 = 0$.

- At finite P^z, the integral is finite but will include logarithm of P^z which is the only scale, so the result cannot be analytically conitnued to $P^z \to \infty$;
- If $P^z \to \infty$ limit is taken before integration, the UV behavior of this integral is changed;

The two limiting procedures are not exchangeable. And for lattice QCD, one can only choose the first procedure and do calculation at finite P^z.

How do we relate the finite momentum result to the IMF?

The solution is found in a large momentum EFT where a factorization formula or matching condition can provide such connection:

$$
\begin{aligned}
\Delta\widetilde{\Sigma}(\mu, P^z) &= \Delta\Sigma(\mu)\ , \\
\Delta\widetilde{G}(\mu, P^z) &= z_{qg}\Delta\Sigma(\mu) + z_{gg}\Delta G(\mu) + O(\frac{M^2}{P_z^2})\ , \\
\Delta\widetilde{L}_q(\mu, P^z) &= P_{qq}\Delta L_q(\mu) + P_{gq}\Delta L_g(\mu) + p_{qq}\Delta\Sigma(\mu) + p_{gq}\Delta G(\mu) + O(\frac{M^2}{P_z^2})\ , \\
\Delta\widetilde{L}_g(\mu, P^z) &= P_{qg}\Delta L_q(\mu) + P_{gg}\Delta L_g(\mu) + p_{qg}\Delta\Sigma(\mu) + p_{gg}\Delta G(\mu) + O(\frac{M^2}{P_z^2})\ .
\end{aligned}
\tag{31}
$$

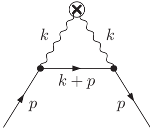
To obtain the IMF (or light-cone) results, one thus has to determine the coefficients $z's$, p's and P's, and invert the above matrix.

How can a factorization formula exist?

Both $\Delta\widetilde{G}$ ($\Delta\widetilde{L}_q$ and $\Delta\widetilde{L}_g$), as calculated at finite P^z in lattice QCD, and ΔG (ΔL_q and ΔL_g), as formulated in IMF and measured in high energy scattering experiments, have short range (perturbative) and long range (nonperturbative) contributions.

The IMF limit only affects the UV behavior, so long range physics of the theory must be the same for both at finite P^z or IMF. Therefore, the difference between the two is just a perturbatively calculable quantity, which lays the foundation of the factorization formula.

One-loop example (Ji, Zhang, and Zhao, 2013)



- At large but finite $\vec{p}$, taking the loop momentum $k \to \infty$ first

$$\langle p,s| \left(\vec{E}\times\vec{A}_\perp\right)^3 |p,s\rangle = \frac{\alpha_S C_F}{4\pi}\left[\frac{5}{3\epsilon}+\frac{5}{3}\ln\frac{\mu^2}{m^2}+\frac{4}{3}\ln\frac{4\vec{p}^2}{m^2}-\frac{1}{9}\right]u^\dagger\Sigma^3 u + O\left(\frac{m^2}{\vec{p}^2}\right), \tag{32}$$

- Taking $\vec{p}^2 \to \infty$ first,

$$\langle p,s| \left(\vec{E}\times\vec{A}_\perp\right)^3 |p,s\rangle = \frac{\alpha_S C_F}{4\pi}\left(\frac{3}{\epsilon}+3\ln\frac{\mu^2}{m^2}+7\right)u^\dagger\Sigma^3 u\,, \tag{33}$$

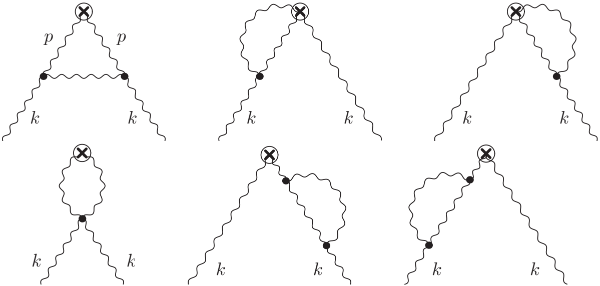
- The light-cone gauge result,

$$\langle p,s| \left(\vec{E}\times\vec{A}\right)^3 |p,s\rangle \overset{A^+=0}{=} \frac{\alpha_S C_F}{4\pi}\left(\frac{3}{\epsilon}+3\ln\frac{\mu^2}{m^2}+7\right)u^\dagger\Sigma^3 u\,. \tag{34}$$

Calculating the matching coefficients at leading logarithm

Let's take z_{qg} and z_{gg} as an example:

At tree level, $\Delta\widetilde{G}_{\text{tree}} = \Delta G_{\text{tree}}$. At one loop level, for the gluonic matrix elements of $\vec{E} \times \vec{A}_\perp$ we calculate Feynman diagrams:



Using dimensional regularization, for massless onshell quarks and gluons,

$$\Delta\widetilde{G}_{\text{1-loop}} = \frac{\alpha_S(\mu)}{4\pi}\left(\frac{5}{3\epsilon_{UV}} + \frac{4}{3}\ln\frac{P_z^2}{\mu^2} - \frac{3}{\epsilon_{IR}} + R_1\right) C_F \Delta\Sigma_{\text{tree}} + \frac{\alpha_S(\mu)}{4\pi}\left[\left(4 - \frac{2n_f}{3}\right)\frac{1}{\epsilon_{UV}} + 7\ln\frac{P_z^2}{\mu^2} - \left(11 - \frac{2n_f}{3}\right)\frac{1}{\epsilon_{IR}} + R_2\right]\Delta G_{\text{tree}} \; ,$$

where R_1 and R_2 are constant numbers. We already know that (Ji, Tang, Hoodbhoy, 1996):

$$\Delta G_{\text{1-loop}} = \frac{\alpha_S(\mu)}{4\pi}\left(\frac{3}{\epsilon_{UV}} - \frac{3}{\epsilon_{IR}}\right) C_F \Delta\Sigma_{\text{tree}} + \frac{\alpha_S(\mu)}{4\pi}\left[\left(11 - \frac{2n_f}{3}\right)\frac{1}{\epsilon_{UV}} - \left(11 - \frac{2n_f}{3}\right)\frac{1}{\epsilon_{IR}}\right]\Delta G_{\text{tree}} \; .$$

In the $\overline{\text{MS}}$ scheme, we subtract the $1/\epsilon_{UV}$'s, replace $1/\epsilon_{IR}$'s with $\Delta G_{\text{1-loop}}$, and have

$$\Delta\widetilde{G}_{\text{1-loop}} = \frac{\alpha_S(\mu)C_F}{4\pi}\cdot\left(\frac{4}{3}\ln\frac{P_z^2}{\mu^2}+R_1\right)\Delta\Sigma_{\text{tree}}+\frac{\alpha_S(\mu)}{4\pi}\cdot\left(7\ln\frac{P_z^2}{\mu^2}+R_2\right)\Delta G_{\text{tree}}+\Delta G_{\text{1-loop}}\ . \tag{35}$$

Therefore,

$$\begin{aligned} z_{qg}(\mu/P^z) &= \frac{\alpha_S(\mu)C_F}{4\pi}\cdot\left(\frac{4}{3}\ln\frac{P_z^2}{\mu^2}+R_1\right)\ , \\ z_{gg}(\mu/P^z) &= 1+\frac{\alpha_S(\mu)}{4\pi}\cdot\left(7\ln\frac{P_z^2}{\mu^2}+R_2\right)\ . \end{aligned} \tag{36}$$

- Because the IR divergence is the same for both ΔG and $\Delta\widetilde{G}$, we have the matching coefficients z_{qg} and z_{gg} being only dependent on μ/P^z.
- However, we still need a rigorous proof that this factorization is valid to all orders in perturbation theory.

Complete results of the matching coefficients at leading logarithm

$$\begin{aligned}
P_{qq} &= 1+\frac{\alpha_S(\mu)C_F}{4\pi}\cdot\left(-2\ln\frac{P_z^2}{\mu^2}+R_3\right)\,, & P_{gq} &= 0\,,\\
P_{qg} &= \frac{\alpha_S(\mu)C_F}{4\pi}\cdot\left(2\ln\frac{P_z^2}{\mu^2}-R_3\right)\,, & P_{gg} &= 0\,,\\
p_{qq} &= \frac{\alpha_S(\mu)C_F}{4\pi}\cdot\left(-\frac{2}{3}\ln\frac{P_z^2}{\mu^2}+R_4\right)\,, & p_{gq} &= 0\,,\\
p_{qg} &= \frac{\alpha_S(\mu)C_F}{4\pi}\cdot\left(-\frac{2}{3}\ln\frac{P_z^2}{\mu^2}-R_1-R_4\right)\,, & p_{gg} &= \frac{\alpha_S(\mu)C_F}{4\pi}\cdot\left(-7\ln\frac{P_z^2}{\mu^2}-R_2\right)\,.
\end{aligned} \tag{37}$$

- All the R's are finite constants that depend on the regularization scheme;
- For all the matching coefficients, we have implicitly included the contributions at $O(M^2/P_z^2)$.

Milestone:

Now we have shown how to relate $\Delta\widetilde{G}(\mu, P^z)$, $\Delta\widetilde{L}_q^z(\mu, P^z)$, and $\Delta\widetilde{L}_g^z(\mu, P^z)$ at large finite P^z to corresponding physical matrix elements in the IMF (or light-front formalism) at leading logarithm!

Ultimate goal: extracting physical results from lattice calculation

- To extract physical results from lattice matrix elements we need a lattice renormalization, which is performed by lattice perturbation theory (S. Capitani, 2003).

- We will match the finite matrix elements in lattice cut-off scheme to the physical results in the IMF (or light-front formalism) $\overline{\text{MS}}$ scheme.
  - The leading logarithms in the matching coefficients will be the same as what we have obtained because UV divergence is regularization scheme independent;
  - The finite constants will be different because they are not scheme independent.

- After all the lattice simulation is done, we can extract out the parton spin and OAM and compare with experimental results!

Summary

- We have matched the matrix elements of certain Euclidean operators at finite momentum to the physical parton spin and OAM in the IMF (or light-front formalism) through a LaMET at leading logarithm;
- We have also justified that the light-cone gauge $A^+ = 0$ is the natural choice in the IMF, where free-field expressions such as $\vec{E} \times \vec{A}$ acquire physical significance.
- With the LaMET there are many ways to calculate the proton spin content in lattice QCD. We choose the expression of QCD angular momentum by Chen *et. al.* for it offers more physical insights in the IMF limit.