How Universal are the Super-Efimov States?

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Efimov

3D 3bosons L=0

Resonant s-wave

$$E_n = E_0 e^{-2\pi n/s_0}$$

$$-\frac{d^2}{d\rho^2} - \frac{s_0^2 + 1/4}{\rho^2}$$

$$s_0 \approx 1.00624$$

V. Efimov, (1970)
Physics Letters B 33: 563–564

Super-Efimov

2D 3fermions L=1

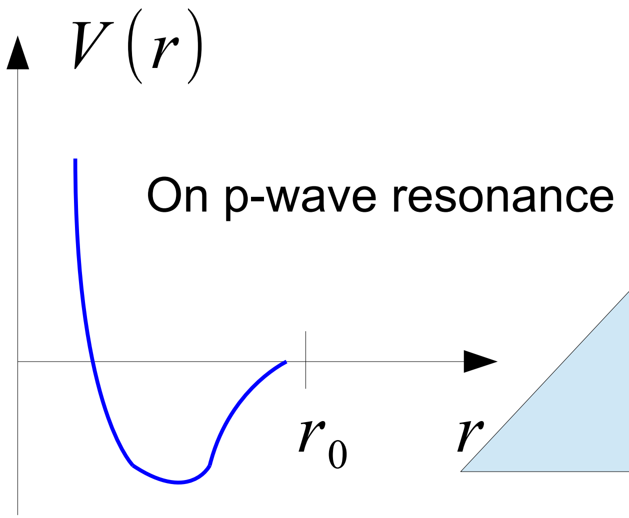
Resonant p-wave

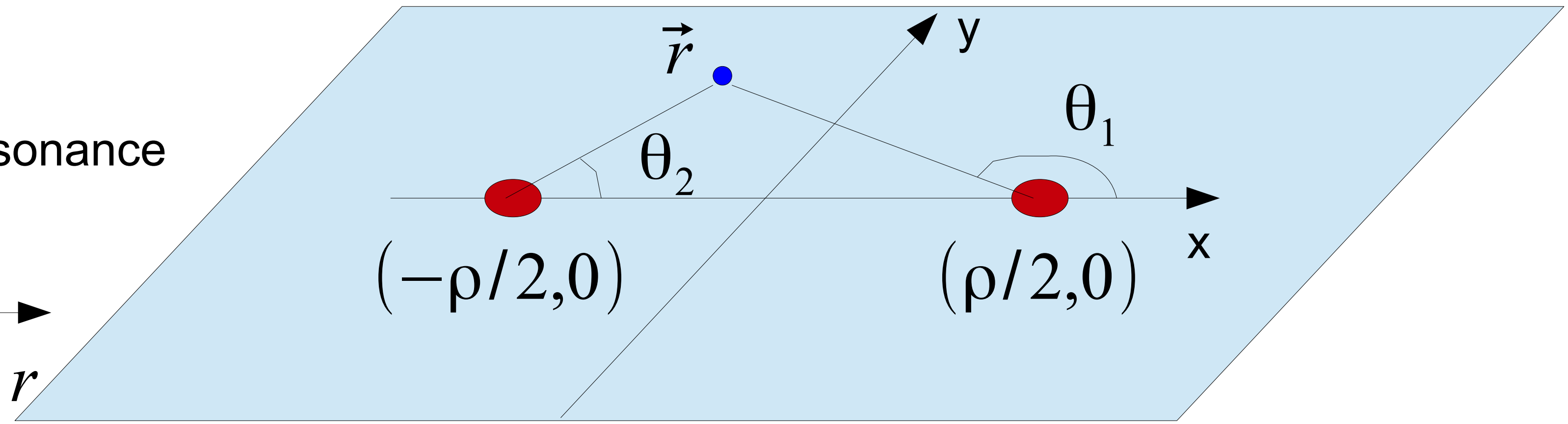
$$E_n \sim \exp\left(-2 e^{3\pi n/4 + \theta}\right)$$

$$-\frac{d^2}{d\rho^2} - \frac{1}{4\rho^2} - \frac{16/9 + 1/4}{\rho^2 \ln^2 \rho}$$

Field theory and separable potential
Calculation by Y. Nishida, S. Moroz,
and D.T. Son PRL **110**, 235301 (2013)

Born Approximation





$$\Psi(\vec{\rho},\vec{r}) = f(\vec{\rho})\,\phi(\vec{\rho},\vec{r})$$

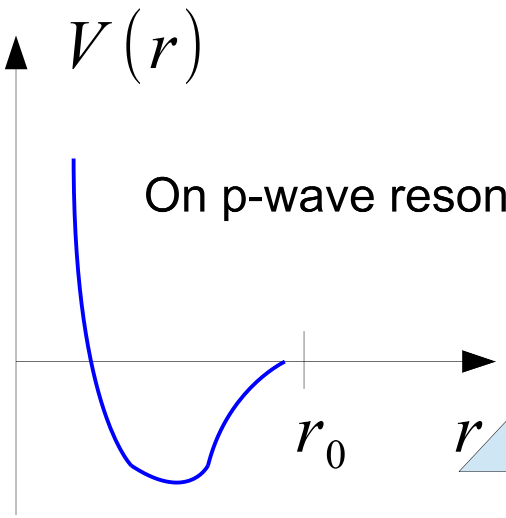
$$\left[-\frac{\nabla_r^2}{2\mathrm{m}} + V(|\vec{r}-\vec{\rho}/2|) + V(|\vec{r}+\vec{\rho}/2|)\right]\phi(\vec{\rho},\vec{r}) = -\frac{\kappa^2}{2\mathrm{m}}\phi(\vec{\rho},\vec{r})$$

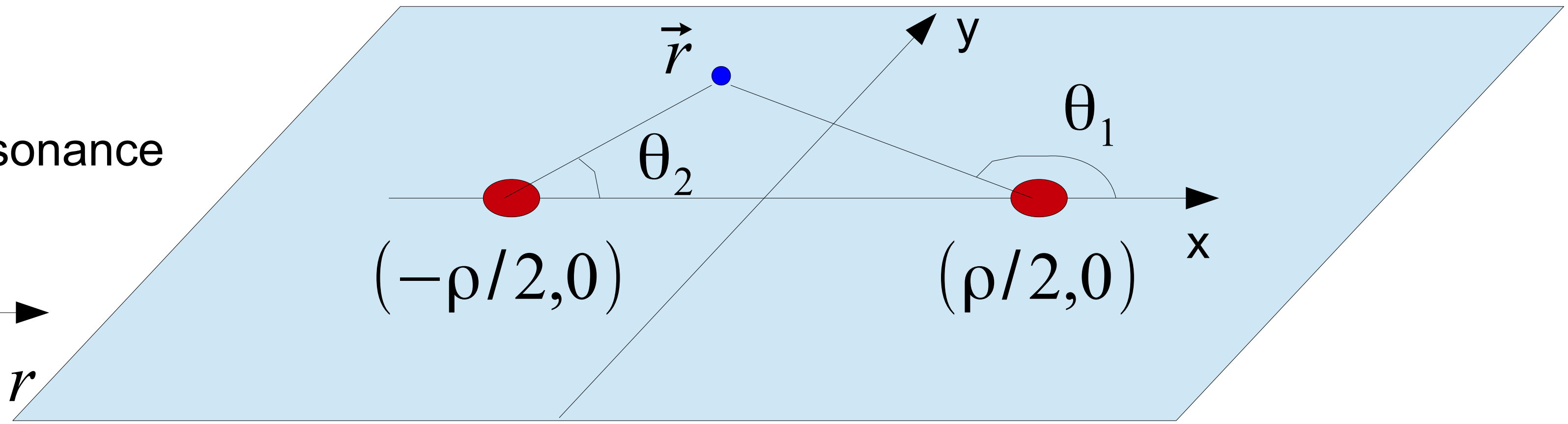
Ansatz:

$$\phi(\vec{\rho},\vec{r}) = K_1(\kappa|\vec{r}-\vec{\rho}/2|)\left(a_1 e^{-i\theta_1} + a_2 e^{i\theta_1}\right) + K_1(\kappa|\vec{r}+\vec{\rho}/2|)\left(b_1 e^{-i\theta_2} + b_2 e^{i\theta_2}\right)$$

For Efimov states, see D. Petrov, arXiv:1206.5752v2

Matching with Asymptotic Behavior





$$r_0 < s = |\vec{r} - \vec{\rho}/2|, |\vec{r} + \vec{\rho}/2| \ll 1/\kappa$$

$$\phi(\vec{\rho}, \vec{r}) \sim [\cot\delta(k)] J_1(ks) + Y_1(ks)$$

Analytic continuition $k \rightarrow i\kappa$

$$\cot\delta(i\kappa) = \frac{1}{\kappa^2 a^2} + \frac{2}{\pi}\ln(\kappa r_{eff}) + i$$

Low Energy Solution

$$\phi(\vec{\rho},\vec{r}) = K_1(\kappa|\vec{r}-\vec{\rho}/2|)\sin\theta_1 + K_1(\kappa|\vec{r}+\vec{\rho}/2|)\sin\theta_2$$

$$\kappa^2 \approx \frac{4}{\rho^2 \ln(\rho/r_{eff})}$$

Require $\Psi(\vec{\rho},\vec{r}) = -\Psi(-\vec{\rho},\vec{r})$

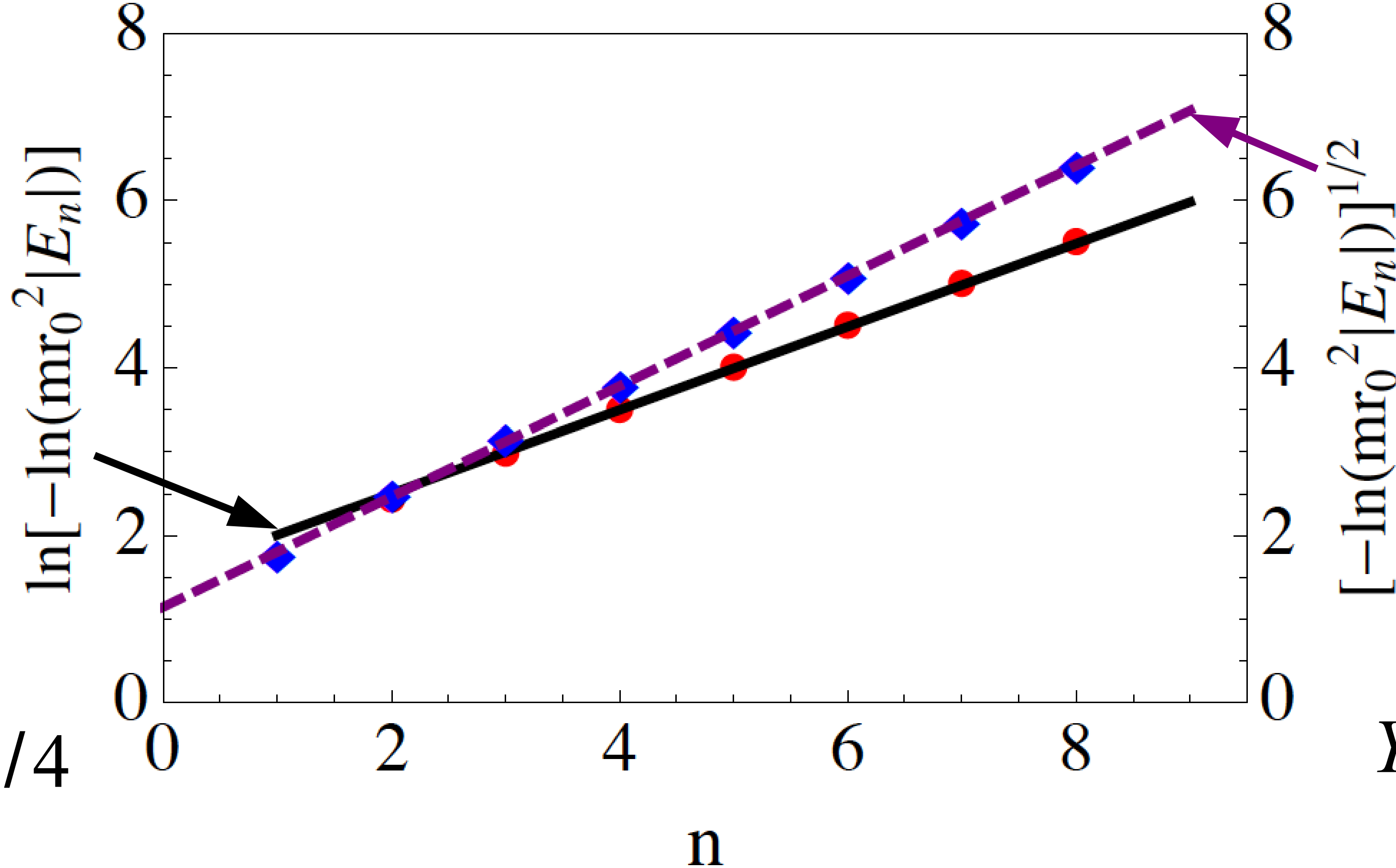
Since $\phi(\vec{\rho},\vec{r}) = -\phi(-\vec{\rho},\vec{r})$

Seek $f(\vec{\rho}) = f(-\vec{\rho})$ Lowest order s-wave $f(\vec{\rho}) = f(\rho)$

$$\left[-\frac{d^2}{d\rho^2} - \frac{d}{\rho\, d\rho} - \frac{2M/m}{\rho^2 \ln(\rho/r_{eff})}\right] f(\rho) = MEf(\rho)$$

Effective attractive potential

Bound States



$s_0^2 = 40 - 1/4$

$Y = 10$

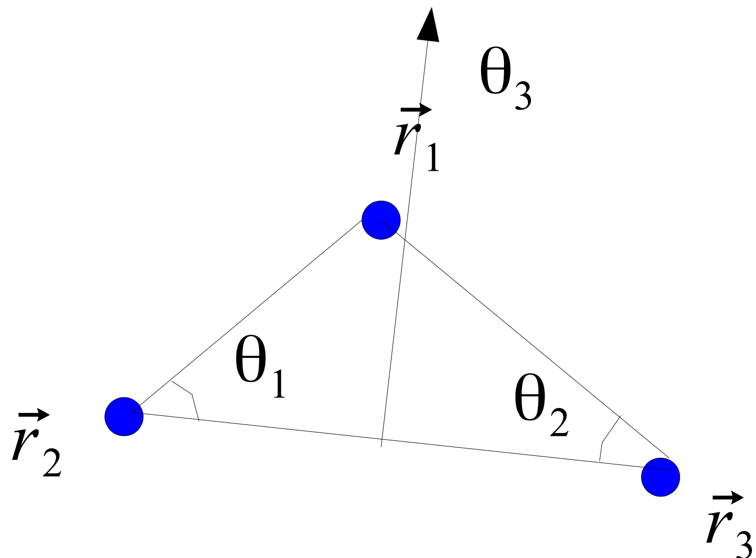
Effective potential: $V_{eff}(\rho) = -\frac{s_0^2 + 1/4}{\rho^2 \ln^2 \rho}$ | $V_{eff}(\rho) = -\frac{Y}{\rho^2 \ln \rho}$

WKB Approx: $E_n \sim \exp\left(-2\, e^{\pi n / s_0 + \theta}\right)$ | $E_n \sim e^{-(n\pi)^2 / 2Y}$

Intermission

- The big mass ratio limit may not in the same universal class as the equal mass case.
- Transition when M/m is tuned?
- Equally interesting: $E_n \sim e^{-(n\pi)^2/2Y}$

Hyperspherical Formalism (three identical fermions)



Separate out the COM, the configuration is determined in terms of

"one" hyperradius: $$\rho = \left[\frac{2}{3} \sum_{i<j} \left(\vec{r}_i - \vec{r}_j \right)^2 \right]^{1/2}$$

and "three" hyperradius: $$\Omega = \left(\theta_{1,} \theta_{2,} \theta_3 \right)$$

E. Nielsen, D.V. Fedorov, A.S. Jensen, E. Garrido,
Physics Reports **347,** 373 (2001)

The Shrodinger Equation

$$\Psi(\rho,\Omega)=\sum_n \rho^{-3/2} f_n(\rho)\,\Phi_n(\rho,\Omega)$$

$$\left[\Lambda_\Omega^2+m\rho^2\sum_{i<j}V\left(\left|\vec{r}_i-\vec{r}_j\right|\right)\right]\Phi_n(\rho,\Omega)=\lambda_n(\rho)\,\Phi_n(\rho,\Omega)$$

$$\left[-\frac{d^2}{d\rho^2}+\frac{\lambda_n(\rho)+3/4}{\rho^2}-mE\right]f_n(\rho)=\sum_{n'}\left[P_{nn'}\frac{d}{d\rho}+Q_{nn'}\right]f_{n'}(\rho)$$

$$P_{nn'}=\left\langle\Phi_n\right|\frac{\partial}{\partial\rho}\left|\Phi_{n'}\right\rangle_\Omega \qquad Q_{nn'}=\left\langle\Phi_n\right|\frac{\partial^2}{\partial\rho^2}\left|\Phi_{n'}\right\rangle_\Omega$$

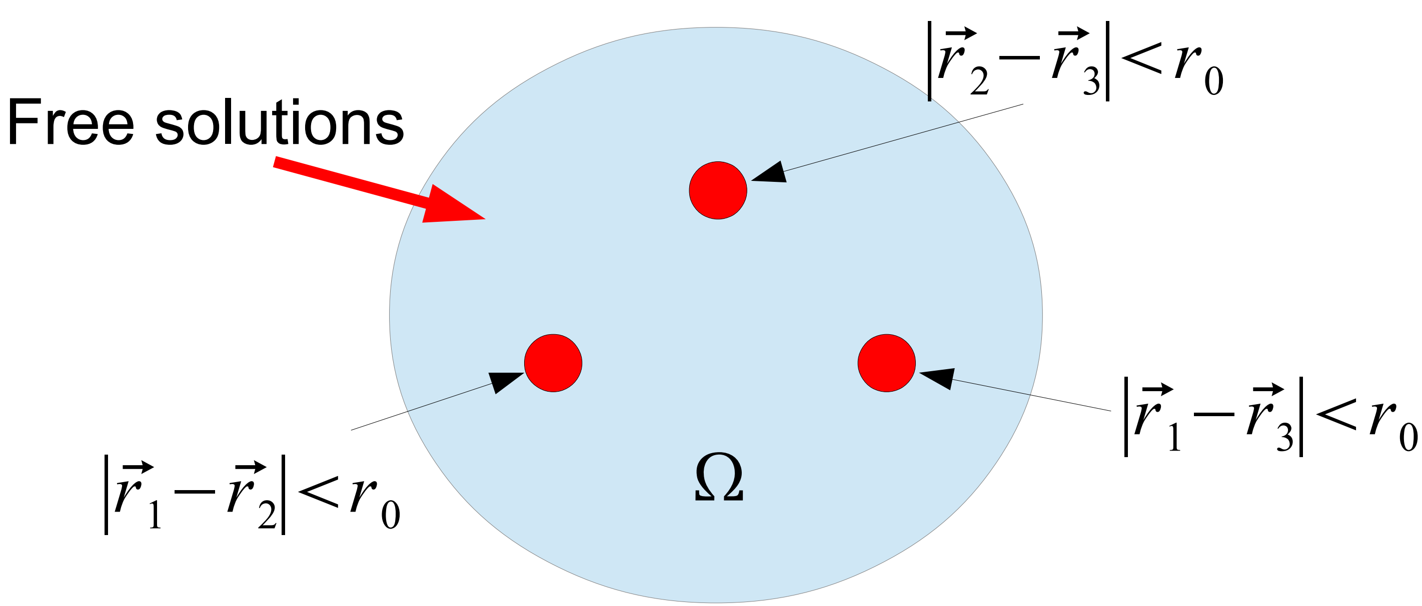
Adiabatic potential: $V_{adiabatic}(\rho)=\dfrac{\lambda_n(\rho)+1}{\rho^2}$

Effective potential: $V_{eff}(\rho)=\dfrac{\lambda_n(\rho)+1}{\rho^2}+Q_{nn}$

Asymptotic solution at $\rho \to \infty$

Schrodinger Equation → Faddeev Equation

E. Nielsen, D.V. Fedorov, A.S. Jensen, E. Garrido, Physics Reports **347,** 373 (2001)



$$\left[\Lambda_\Omega^2 + m\rho^2 \sum_{i<j} V\left(|\vec{r}_i - \vec{r}_j|\right)\right]\Phi_n(\rho,\Omega) = \lambda_n(\rho)\,\Phi_n(\rho,\Omega)$$

$$[\ln \Phi_n^{>}(\rho,\Omega)]'_{\partial\Omega} = [\ln \Phi_n^{<}(\rho,\Omega)]'_{\partial\Omega} = \sum_m c_m (r_0/\rho)^m$$

Asymptotic solution at $\rho \to \infty$

For L=1 states, if one keeps only the leading order of

$$[\ln \Phi_n^{<}(\rho, \Omega)]'_{\partial\Omega}$$

The lowest solution is

$$\lambda_0(\rho) + 1 \sim -\frac{16/9}{\ln^2(\rho/r_{eff})}$$

C. Gao and ZY, arXiv:1401.0965
A.G. Volosniev, D.V. Fedorov,
A.S. Jensen, N.T. Zinner, arXiv:1312.6535

$$\left[-\frac{d^2}{d\rho^2} - \frac{1}{4\rho^2} - \frac{16/9}{\rho^2 \ln^2(\rho/r_{eff})} - mE\right] f_0(\rho) = 0$$

$$E_n \sim \exp\left(-2 e^{\pi n/s_0 + \theta}\right)$$

$$s_0 = \sqrt{16/9 - 1/4}$$

vs

$$s_0 = \sqrt{16/9}$$ Nishida, et al

Asymptotic solution at $\rho \to \infty$

By systematic asymptotic expansion, one must keep up to the next leading order of

$$[\ln \Phi_n^{<}(\rho, \Omega)]'_{\partial\Omega}$$

The solution becomes

$$\lambda_0(\rho) + 1 \sim -\frac{Y}{\ln(\rho / r_{eff})}$$

Involving correlated movement of the third particle

The new parameter: $$Y = -1 - \frac{\int_0^\infty dr\, r^3 V(r) u_0^2(r)}{\lim_{r\to\infty} r^2 u_0^2(r)}$$

Two-body zero energy wavefunction

Can't be parameterized by a finite number of parameters from phase shift.

C. Gao and *ZY*, arXiv:1401.0965

$$Y = \frac{\int_0^\infty dr\, r \left[\frac{\partial (r u_0(r))}{\partial r}\right]^2}{\lim_{r\to\infty} r^2 u_0^2(r)} > 0$$

Y. Castin, private communication
A.G. Volosniev, D.V. Fedorov, A.S. Jensen, and N.T. Zinner, arXiv:1312.6535

TABLE I: The parameter Y calculated from Eq. (25) for different model potentials at from the first to the third p-wave resonance as the strength of the attractive potentials increases.

Resonance	Square well	Gaussian	Hardcore+$1/r^6$
1st	0.63	0.48	1.43
2nd	4.74	1.63	2.10
3rd	12.15	2.77	2.49

Within the adiabatic approximation:

$$V_{adiabatic}(\rho) = -\frac{Y}{\rho^2 \ln(\rho/r_{eff})}$$

$$E_n \sim e^{-(n\pi)^2/2Y}$$

Corrections

$$\left[-\frac{d^2}{d\rho^2}+\frac{\lambda_n(\rho)+3/4}{\rho^2}-mE\right]f_n(\rho)=\sum_{n'}\left[P_{nn'}\frac{d}{d\rho}+Q_{nn'}\right]f_{n'}(\rho)$$

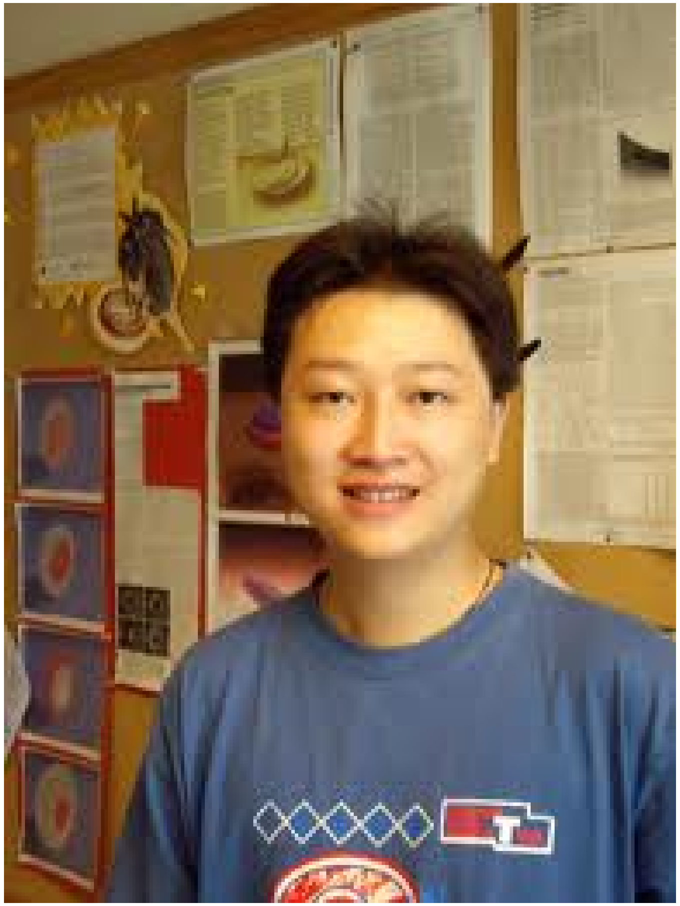
$$P_{00}=0 \qquad Q_{00}\sim-\frac{Y}{\rho^2\ln(\rho/r_{eff})}$$

A.G. Volosniev, D.V. Fedorov, A.S. Jensen, and N.T. Zinner, arXiv:1312.6535

$$V_{eff}(\rho)=\frac{\lambda_0(\rho)+1}{\rho^2}-Q_{00}(\rho)\sim\frac{C}{\rho^2\ln^2(\rho/r_{eff})}$$

$$?\,C\,,\,?\,P_{nn'}\,,Q_{nn'}$$

Numerical Attack



WANG Jia

To be continued ...

Thank you